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Steady-state vibration of a viscoelastic cylinder cover subjected to moving loads

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Abstract

The dynamic steady-state response of a viscoelastic cylinder cover subjected to circumferentially moving constant point and distributed loads is studied using a 1D Pasternak-type foundation model. The cover material is modeled according to the generalized Maxwell model as an incompressible frequency-dependent viscoelastic material spanning a wide relaxation spectrum. The vibration response of the cover for a moving twin point load is obtained using a modal expansion approach. On the basis of the solution, additional moving load cases are derived. In the case of a single moving point load, representing a load resultant due to rolling contact, numerical calculations show that regardless of the viscoelastic damping in the model, the critical load speed for the system can be well estimated by a resonance condition. In the vicinity of the critical speed, an incipient traveling wave arises behind the moving load. The viscoelastic cover stiffens for increasing excitation frequencies, thus, the cover response divides into two separate mode branches, of which the low-mode branch is dominant. A method to suppress the traveling wave vibrations in the cover at supercritical speeds using a moving twin point load, adjusted according to a dominant resonating mode, is presented. Using a distributed moving load, it is shown that depending on the wavelength, a traveling wave generated at the leading edge of the load may be reinforced at the trailing edge, the lift-off point, of the load. The developed model offers a fast and reliable way for practitioners to estimate the critical speeds of rolling contact machines with viscoelastic covers.

Keywords: cylinder cover, viscoelastic, moving load, modal expansion, traveling wave

1. Introduction

Viscoelastic cylinder covers are used increasingly in industrial rolling contact applications, for example, in calenders and size presses of paper machines. As the production speeds of such machines are pushed to new limits, the cylinder covers become plagued with previously avoided dynamic phenomena. In a recent paper, by using a 2D finite element (FE) plane strain model, we found that at high rolling speeds, a Rayleigh wave resonance takes place in the viscoelastic cover of a cylinder in rolling contact (Karttunen and von Hertzen, 2013). With this in mind, an elastic 1D analytical cover model was developed and it was shown through a comparison between the 1D analytical model and a 2D FE model that the 2D plane strain problem can be reduced to 1D using a Pasternak-type foundation model (Karttunen and von Hertzen, 2014). The purpose of

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the present paper is to develop the 1D analytical model further by describing the cylinder cover material as a frequency-dependent viscoelastic polymer that spans a wide relaxation spectrum. It is found that the viscoelasticity has a significant effect on the system behavior, and in comparison to earlier models, the current model offers a more efficient and realistic way to estimate the critical speeds of production machines with viscoelastic coatings.

In addition, an experimentally illustrated method (Chatterjee et al., 1999), which can be used to minimize the high power dissipation induced by traveling waves in rotating tires and covered cylinders, is studied and elucidated by the aid of a moving twin load. Vehicle tires are known to suffer from similar traveling waves, tire standing waves (Soedel, 1975; Padovan, 1975; Soedel, 2004), as the Rayleigh wave resonance brings about in cylinder covers.

Viscoelastic circular structures under moving radial loads or in rolling contact have been studied by several authors over the years. Padovan (1976, 1977) investigated the dynamic responses of a viscoelastic ring on a foundation model and a viscoelastic laminated shell subjected to circumferentially moving loads. Rings on viscoelastic foundations under moving loads have also been studied later by Kazempour and Padovan (1990) and by Metrikine and Tochilin (2000). Formulations for dynamic viscoelastic rolling contact problems have been provided by Oden and Lin (1986), Padovan and Paramadilok (1985), Padovan (1987a,b), Qiu (2009), and Zehil and Gavin (2013). In some of the mentioned cases, viscoelasticity is accounted for in a general form in the model description (e.g., Padovan, 1977; Qiu, 2009). However, in dynamic simulations, the general trend has been to consider only simple, frequency-independent viscoelastic material models as a way to incorporate a source of damping into the system at hand. The importance of using a frequency-dependent viscoelastic material in a moving load problem was well-illustrated by Hu and Huang (1999), who studied the dynamic response of a sandwich ring with a viscoelastic core subjected to a traveling circumferential load using a modal expansion approach. Due to the moving load, the response of the system becomes polyharmonic, that is, it is composed of multiple frequencies and, thus, the viscoelastic material properties have to span a wide frequency range.

In this paper, we study the dynamic response of a viscoelastic cylinder cover subjected to moving loads using a 1D Pasternak-type foundation model. The frequency-dependent viscoelastic cover behavior is modeled according to the generalized Maxwell model. The loading cases include a circumferentially moving constant radial i) point load, ii) twin point load, iii) distributed load, and the generalization of the latter two for multiple loads. The steady-state vibration response of the cover in the case of the moving twin point load is first obtained through the use of a modal expansion approach and a frequency response function. The solution is then extended to the other loading cases. The viscoelastic response of the cover for a moving point load reveals considerable differences in comparison to the damped elastic case studied in (Karttunen and von Hertzen, 2014).

2. Theory

2.1. Problem formulation

A schematic setup of the studied system is presented in Fig. 1. The model consists of a non-rotating rigid cylinder with a viscoelastic cover subjected to a circumferentially moving radial twin point load. A rotating cylinder was investigated in (Karttunen and von Hertzen, 2014), and the effect of rotation on the traveling waves was shown to be negligible for realistic material parameters. The cover is modeled as a Pasternak-type foundation consisting of a shear layer attached to a spring foundation. The equation of motion in a coordinate system fixed to the non-rotating cylinder in terms of the radial displacement u of the shear layer in Fig. 1 reads

Figure 1: Analytical model for a cylinder with a viscoelastic cover subjected to a circumferentially moving twin point load. The cover is modeled as a shear layer attached to a spring foundation and u is the radial displacement of the shear layer.

$$u_{tt} + \frac{E^*}{\rho h^2} u - \frac{\kappa G^*}{\rho R_e^2} u_{\theta\theta} = P(\theta, t) , \quad (1)$$

where the subscripts t and θ stand for differentiation with respect to time and angle θ , respectively. The term containing $u_{\theta\theta}$ couples the adjacent material elements to each other. The density and thickness of the cover are ρ and h , respectively. The radius $R_e = R_c + h/2$, where R_c is the cylinder radius, is used to determine the effective width of a material element. The shape factor κ of the shear layer is similar to that of a Timoshenko shear beam, for details, see (Karttunen and von Hertzen, 2014). The complex tension-compression modulus of the viscoelastic cover is given by

$$E^* = E_s + jE_l = E_s(1 + j\eta_E) , \quad (2)$$

where E_s and E_l are the storage and loss moduli, respectively, and $\eta_E = E_l/E_s$ is the tension-compression loss factor. The complex shear modulus is

$$G^* = G_s + jG_l = G_s(1 + j\eta_G) , \quad (3)$$

where G_s and G_l are the storage and loss moduli for shear, respectively, and $\eta_G = G_l/G_s$ is the shear loss factor. The polymeric cover material used in this study is assumed to be incompressible within the frequency range of interest, that is, within the “rubbery range”. Thus, the Poisson loss factor becomes zero and Poisson ratio ν is real and equal to $1/2$ (Pritz, 2007). The complex moduli are then related by the equation

$$G^* = \frac{E^*}{2(1 + \nu)} , \quad (4)$$

so that the relation $\eta_E = \eta_G (= \eta)$ holds for the loss factors. The polymeric material used in this study and its frequency-dependent viscoelastic behavior will be discussed in detail in Section 2.3. For the moving constant twin point load in Fig. 1, we have

$$P(\theta, t) = \frac{P_0}{2} \delta(\theta - \Omega t) + \frac{P_0}{2} \delta[\theta - (\Omega t - \Delta)] , \quad (5)$$

where P_0 is the load amplitude, Ω is the constant angular velocity of the load and Δ is the angle between the point loads. Finally, the requirement of continuity of the displacement and slope leads to

$$u(0, t) = u(2\pi, t) \quad \text{and} \quad u_\theta(0, t) = u_\theta(2\pi, t) . \quad (6)$$

Note that Eq. (1), a viscoelastic generalization of the corresponding elastic equation, is strictly speaking a non-equation (Crandall, 1970), since it is written in time-domain, while Eqs. (2) and (3) are valid in frequency-domain and well-defined only for purely harmonic motion. However, this does not impose a problem here because it turns out that the excitations studied in this paper lead to polyharmonic loads, which are cut down to purely harmonic loads in the modal equations of motion for which the analytical solution may be obtained by the aid of a frequency response function. Next, the steady-state response of the viscoelastic cover is derived for different moving load cases, starting from the circumferentially moving twin point load.

2.2. Steady-state vibration response of the cover

Each undamped natural mode $\sin(n\theta)$ or $\cos(n\theta)$ of the cover, where n is the mode number, consists of n full waves on the cover circumference with the exception of $n = 0$, which is the breathing mode. As n increases, the wavelength in a mode decreases. The solution for the moving twin load problem, with the employment of a modal expansion, can be found as

$$u(\theta, t) = \sum_{n=1}^{\infty} [c_n(t) \sin(n\theta) + d_n(t) \cos(n\theta)] + d_0(t) . \quad (7)$$

To solve the modal expansion coefficients c_n , d_n and d_0 , Eq. (7) is substituted into Eq. (1), which leads to the equations

$$\ddot{c}_n + \omega_n^2(1 + j\eta)c_n = \frac{P_0}{\pi} \cos\left(\frac{n\Delta}{2}\right) \sin\left[n\left(\Omega t - \frac{\Delta}{2}\right)\right] , \quad (8)$$

$$\ddot{d}_n + \omega_n^2(1 + j\eta)d_n = \frac{P_0}{\pi} \cos\left(\frac{n\Delta}{2}\right) \cos\left[n\left(\Omega t - \frac{\Delta}{2}\right)\right] , \quad (9)$$

$$\ddot{d}_0 + \omega_0^2(1 + j\eta)d_0 = \frac{P_0}{2\pi} , \quad (10)$$

where the notation

$$\omega_n = \sqrt{\frac{1}{\rho} \left(\frac{E_s}{h^2} + \frac{\kappa G_s}{R_e^2} n^2 \right)} \quad (11)$$

has been used. In the following, we utilize Eqs. (8)–(11) in two different situations, namely, in the solution of the steady-state vibrations and in determining the natural frequencies of the free vibrations of the system. In the former case, the moduli and loss factor are evaluated at the driving frequency $n\Omega$ of each mode (see Section 2.3). In the latter case, the storage moduli are to be evaluated at the natural frequency ω_n itself, which leads to a nonlinear Eq. (11) for the natural frequency ω_n . Note also that according to Eqs. (25), (26) and (29) the loss factor $\eta(n\Omega)$ for the breathing mode $n = 0$ under the moving load is zero, that is, $\eta(0) = 0$. The reason for this is that the breathing mode is excited statically by the moving load. Therefore, the steady-state solution of Eq. (10), to be included in the complete steady-state solution, is

$$d_0 = \frac{P_0}{2\pi\omega_0^2} . \quad (12)$$

The solutions for c_n and d_n can be found by introducing an auxiliary complex modal coefficient $u_n = d_n + jc_n$. Thus, multiplying Eq. (8) by $j = \sqrt{-1}$ and then taking the sum of the resulting equation with Eq. (9) leads to

$$\ddot{u}_n + \omega_n^2(1 + j\eta)u_n = b_n e^{jn\Omega t} , \quad (13)$$

where

$$b_n = \frac{P_0}{\pi} \cos\left(\frac{n\Delta}{2}\right) e^{-jn\Delta/2} . \quad (14)$$

The solution to Eq. (13) is

$$u_n(t) = U_n(jn\Omega) e^{jn\Omega t} . \quad (15)$$

Furthermore, let us define the frequency response function (FRF) as

$$H_n(jn\Omega) = \frac{U_n(jn\Omega)}{b_n} = \frac{1}{\omega_n^2 - (n\Omega)^2 + j\omega_n^2\eta} . \quad (16)$$

The FRF can be expressed in the form

$$H_n(jn\Omega) = |H_n(jn\Omega)| e^{-j\psi_n} , \quad (17)$$

where the magnitude of $H_n(jn\Omega)$ is given by

$$|H_n(jn\Omega)| = \sqrt{H_n(jn\Omega)\overline{H}_n(jn\Omega)} , \quad (18)$$

in which $\overline{H}_n(jn\Omega)$ is the complex conjugate of $H_n(jn\Omega)$. The modal phase shift is calculated from

$$\psi_n = \tan^{-1} \left[\frac{-\text{Im}[H_n(jn\Omega)]}{\text{Re}[H_n(jn\Omega)]} \right] , \quad 0 \leq \psi_n < \pi . \quad (19)$$

Now, Eq. (15) can be written as

$$u_n(t) = \frac{P_0}{\pi} \cos\left(\frac{n\Delta}{2}\right) |H_n(jn\Omega)| e^{j[n\Omega t - \psi_n - n\Delta/2]} . \quad (20)$$

Finally, substituting $d_n(t) = \text{Re}[u_n(t)]$, $c_n(t) = \text{Im}[u_n(t)]$ and $d_0 = P_0/2\pi\omega_0^2$ into Eq. (7) yields for the twin point load the steady-state solution

$$u(\theta, t) = \sum_{n=1}^{\infty} B'_n \cos[n(\Omega t - \theta) - \psi'_n] + \frac{P_0}{2\pi\omega_0^2} , \quad (21)$$

where the modal amplitude is

$$B'_n = B_n \cos\left(\frac{n\Delta}{2}\right) = \frac{P_0}{\pi} |H_n(jn\Omega)| \cos\left(\frac{n\Delta}{2}\right) = \frac{P_0}{\pi \sqrt{[\omega_n^2 - (n\Omega)^2]^2 + (\omega_n^2\eta)^2}} \cos\left(\frac{n\Delta}{2}\right) . \quad (22)$$

The modal phase shift is calculated from

$$\psi'_n = \psi_n + \frac{n\Delta}{2} , \quad (23)$$

where

$$\tan \psi_n = \frac{\omega_n^2\eta}{\omega_n^2 - (n\Omega)^2} , \quad 0 \leq \psi_n < \pi \quad (n = 1, 2, \dots) . \quad (24)$$

Note that by setting $\Delta = 0$ in Eqs. (22) and (23), the case for a single moving constant radial point load with magnitude P_0 is obtained. Other loading cases can be developed on the basis of the steady-state response given by Eq. (21). These are presented in Appendix A.

2.3. Viscoelastic behavior, power dissipation and resonant speeds

The viscoelastic cover material is a rubber-like amorphous polymer. The mechanical behavior of the polymer is modeled by the generalized Maxwell model presented in Fig. 2. In the model, the spring associated with the long-term modulus E_∞ describes the restoring effect of the cross-links of the molecular polymer chains. The Maxwell elements in parallel represent the elastic deformations of the polymer chains and the damping (dissipation) due to friction between the chains. Every Maxwell element has a relaxation time $\tau_i = \eta_i/E_i$ ($i = 1, 2, \dots, m$). Together the elements determine the relaxation spectrum of the polymer.

Figure 2: Generalized Maxwell model for viscoelastic materials.

It can be seen from Eq. (21) that the steady-state cover response is polyharmonic due to the traveling point loads. The frequency-dependent viscoelasticity is accounted for in the model by utilizing the frequency-dependent moduli. The tension-compression storage and loss moduli are calculated for each mode by (Ferry, 1980)

$$E_s(n\Omega) = E_\infty + \sum_{i=1}^m E_i \frac{(n\Omega\tau_i)^2}{1 + (n\Omega\tau_i)^2}, \quad (25)$$

$$E_l(n\Omega) = \sum_{i=1}^m E_i \frac{n\Omega\tau_i}{1 + (n\Omega\tau_i)^2}, \quad (26)$$

respectively, and the shear storage and loss moduli and the loss factor η are given by

$$G_s(n\Omega) = \frac{E_s(n\Omega)}{3}, \quad (27)$$

$$G_l(n\Omega) = \frac{E_l(n\Omega)}{3}, \quad (28)$$

$$\eta(n\Omega) = \frac{E_l(n\Omega)}{E_s(n\Omega)}, \quad (29)$$

respectively. The viscoelastic material parameters used in this work are given in Table 1 of Section 3.1.

With the frequency-dependency included, the total dissipation power of the cover under a single moving point load in steady-state can be decomposed into compression and shear parts as

$$P_{\text{tot}} = P_c + P_s = \frac{\pi R_e l}{h} \Omega \sum_{n=1}^{\infty} n E_l(n\Omega) B_n^2 + \frac{\pi \kappa h l}{R_e} \Omega \sum_{n=1}^{\infty} n^3 G_l(n\Omega) B_n^2, \quad (30)$$

where l is the length of the cylinder in axial direction. The modal amplitude term B_n corresponding to the single point load can be replaced by B'_n , B''_n , B'''_n or B''''_n to obtain the total dissipation powers for the other loading cases presented in Appendix A.

As noted earlier, in the calculation of the natural angular frequencies ω_n of the free vibrations from Eq. (11), the frequency-dependent behavior of the viscoelastic polymer has to be taken into account. It can be seen from Eq. (22) that without damping ($E_l = 0, \eta = 0$), the system is in resonance when $\omega_n = n\Omega$. Thus, a resonant load angular velocity for each mode in this case can be calculated from

$$\Omega_n = \frac{\omega_n}{n} , \quad (31)$$

where $\omega_n = 2\pi f_n$ is obtained from Eq. (11) by matching the angular frequency used to determine the storage moduli with the computed natural angular frequency through successive approximations for each mode.

3. Computational results and discussion

3.1. Viscoelastic parameters of the cover material

The viscoelastic material parameters used in the calculations are given in Table 1. The storage modulus E_s and the loss factor η of the viscoelastic material under harmonic excitation are plotted as a function of the angular excitation frequency for the range $1 - 10^6$ rad/s in Fig. 3. The frequencies of main interest, in terms of the cylinder cover problem, reside in the rubbery range well-below the glass transition, which starts to take place at very high frequencies (not shown).

Table 1: Viscoelastic material parameters E_i and τ_i of the polymeric cylinder cover of an industrial machine. The parameters were obtained by standard DMTA tests (Dynamic Mechanical Thermal Analysis) and curve fitting procedures. The long-term tension-compression modulus of the polymer is $E_\infty = 9.114$ MPa. The rest of the system parameters are as in (Karttunen and von Hertzen, 2014): the Poisson ratio is $\nu = 0.5$ and the density of the polymer is $\rho = 2000$ kg/m³. The cover thickness, and cylinder radius and length are $h = 0.01$ m, $R_c = 0.204$ m and $l = 0.2$ m, respectively. The shape factor κ is equal to $6/7$.

i	1	2	3	4	5	6	7	8	9	10
E_i [MPa]	18.94	10.68	4.720	2.630	1.644	1.241	0.854	0.767	0.726	0.610
τ_i [s]	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}	10^0	10^1	10^2

Figure 3: (a) Storage modulus E_s and (b) loss factor η of the viscoelastic material under harmonic excitation.

3.2. Dynamic response of the cover under a single moving point load

The case of a single moving constant point load (i.e., $\Delta = 0$) is studied first. Fig. 4 shows the absolute values of the modal amplitudes B_n calculated from Eq. (22) as a function of the rotational frequency $f_{\text{rot}} = \Omega/2\pi$ of the moving load and the mode number n . Fig. 5 is a similar plot for the modal phase shifts ψ_n ($n = 1, 2, \dots$) calculated from Eq. (24). The resonant rotational frequencies $f_{\text{rot},n} = f_n/n$ and the locations of the maximums of the modal amplitudes $|B_n|_{\text{max}}$ calculated from Eqs. (31) and (22), respectively, as a function of n have been added as continuous curves into Fig. 5. It can be seen from Fig. 4 that for increasing rotational frequencies, the lower modes attain clearly higher amplitude values than the higher modes. It can be read from the contour that the modal amplitude maximums for any mode do not occur below the rotational frequency $f_{\text{rot}} = 39$ Hz. Fig. 5 shows in more detail that there exists a small difference between the curves for $f_{\text{rot},n}$ and $|B_n|_{\text{max}}$; due to the loss factor $\eta(n\Omega)$, the amplitude maximums occur at slightly lower rotational frequencies.

Figure 4: Absolute values of the steady-state modal amplitudes B_n for a single moving point load.

In Fig. 5, the 90-degree-curve for the modal phase shifts ψ_n is actually marked by the same line as $f_{\text{rot},n}$. When the modal phase shifts approach the resonance value of 90 degrees, the displacements localized in the vicinity of the load become increasingly asymmetric with respect to the load. The lowest rotational frequency values for the curves $f_{\text{rot},n}$ and $|B_n|_{\text{max}}$ are attained by the mode $n = 182$. In terms of the resonance condition given by Eq. (31), the critical resonant load speed is then $f_{\text{rot,cr}} = f_{\text{rot},182} = 39.82$ Hz. The rotational frequency corresponding to $|B_{182}|_{\text{max}}$ is $f_{\text{rot}} = 39.7$ Hz. The curves rise at higher modes due to the stiffening of the viscoelastic material caused by increasing vibration frequencies.

The steady-state response of the cylinder cover for the rotational frequency range 1–100 Hz of the moving point load is shown in Fig. 6. Above $f_{\text{rot}} = 37$ Hz, the displacements start to accumulate behind the load and form into a traveling wave, which gets stronger as the rotational frequency increases. The steady-state cover response shows that the determination of an exact critical speed

Figure 5: Steady-state modal phase shifts ψ_n , the resonant rotational frequency curve $f_{\text{rot},n} = f_n/n$ and the locations of the maximums of the modal amplitudes $|B_n|_{\text{max}}$. The critical resonant load speed is $f_{\text{rot},\text{cr}} = 39.82$ Hz. Below (above) the dashed line the modal phase shifts are smaller (larger) than 90°

in a real (damped) case is hardly possible. However, the resonance interpretation is still a proper backbone for the traveling wave phenomenon and gives a good estimate for the location of the critical speed range of the system within which the cover response shifts from a local quasi-static deformation to a non-local traveling wave. It is especially noteworthy that the stiffening of the viscoelastic cover for the higher modes makes the determination of the critical speed range easier in comparison to the damped elastic case studied in (Karttunen and von Hertzen, 2014) because the lowest rotational frequency corresponding to the amplitude maximum of a mode is clearly-defined, and close to the resonant load speed, unlike in the damped elastic case, where the curve for the locations of the maximums of the modal amplitudes falls monotonously as the mode number n increases.

In Fig. 7, as a final numerical experiment for the moving point load, the loss factor has been taken as an exceptionally small constant, $\eta = 10^{-4}$, to study and illustrate the behavior of the system near the undamped ($\eta = 0$) critical speed $f_{\text{rot},\text{cr}} = 39.82$ Hz. It can be seen in Fig. 7(a) that at the critical speed, approximately ten modes resonate nearly simultaneously. Correspondingly, in Fig. 7(b), presenting the steady-state cover response, the superposed modal amplitude peaks are seen as a strong shock response when the critical speed is crossed. Above the critical speed in Fig. 7(a), the cover response clearly divides into two separate branches, one for higher, and the other for lower modes. The modal amplitudes decrease gradually in the high-mode branch and, thus, the lower modes become dominant in the cover response at supercritical speeds. The same conclusion holds, when the frequency-dependent loss factor $\eta(n\Omega)$ according to the parameters of Table 1 is used.

Figure 6: Steady-state response of the cylinder cover for the rotational frequency range 1–100 Hz of the moving point load.

Figure 7: (a) Absolute values of the modal amplitudes B_n for a single moving point load for $\eta = 10^{-4}$. (b) Corresponding steady-state response of the cylinder cover. Several modes are active nearly simultaneously at $f_{\text{rot,cr}} = 39.82$ Hz and the cover exhibits a strong shock response.

3.3. Dynamic response under a wave-attenuating twin load

It was shown experimentally by Chatterjee et al. (1999) that a traveling wave on a balloon tire can be suppressed actively by applying an additional external force on the tire. In the case of the cylinder cover problem, the wave suppression can be studied by the aid of a twin point load. It can be found that an optimal way to attenuate a traveling wave at a supercritical speed is to set the angle Δ between the moving point loads to a value which equals the half of the wavelength of the dominant resonating mode. In the range within which the traveling wave starts to appear there are approximately ten modes which resonate practically simultaneously. Under such circumstances, the wave suppression cannot be accomplished effectively by any choice of the angle Δ . The resonances become more distinct at higher rotational frequencies and only one mode dominates at a time. Fig. 8 shows the cover response at the frequency $f_{\text{rot}} = 80.75$ Hz, which has been chosen according to the resonance condition of the mode $n = 20$. It can be seen that when the angle Δ for a twin load is adjusted according to the resonant mode, the wave is attenuated effectively. Note that the resultant of the twin load is twice that of the single load and still the emerging traveling wave due to the twin load is much smaller. At supercritical speeds, the point loads “climb up the hill” from the displacement minimums, because for the supercritically excited modes, the cover response lags the excitation due to the resonance phase shift.

Figure 8: Suppression of the wave of a moving twin load by the aid of an optimal separation angle Δ . The point loads move to clockwise direction, v is the tangential load velocity on the cover surface. The resultant of the twin load is twice that of the single load. The rotational frequency is $f_{\text{rot}} = 80.75$ Hz, which corresponds to the resonance condition of the mode $n = 20$.

Fig. 9 shows the effect of the optimally adjusted twin point loads on the total dissipation power of the cover at supercritical speeds in comparison to a single point load with magnitude P_0 . For the twin loads, the dissipation power has been calculated at the resonant rotational frequencies. The two point loads, each with a magnitude P_0 , cause an increase in the dissipation power below $f_{\text{rot}} \approx 55$ Hz. At higher speeds, the mode resonances are more distinct and a single mode is dominant, causing the dissipation power to be reduced more effectively. In the balloon tire experiment conducted by Chatterjee et al. (1999), a single mode is clearly dominant in the response and the suppression method is effective. A radical decrease in the dissipation power can be seen, if the single point load of magnitude P_0 is divided into two equal point loads.

Figure 9: Total dissipation power for a single point load and optimally adjusted twin point loads at supercritical rotational frequencies.

3.4. Dynamic response under a distributed load

The cover response under a distributed load can be studied by the aid of Eqs. (A.6) and (A.7). Fig. 10(a) displays the maximum radial displacement within the cylinder circumference in the rotational frequency range 1 – 100 Hz for a single moving point load and three different distributed loads. The distributed load having a central angle of 2.5 degrees corresponds to a realistic nip width in a rolling contact machine. For car tires, for example, the contact area is typically a lot wider than in rolling contact applications such as paper calenders. It can be seen that for $\gamma = 2.5^\circ$, the distributed load causes a similar type of response as the point load. For $\gamma = 5^\circ$, the cover response is still qualitatively similar, although the peak value of $u_{\max}$ appears at a higher rotational frequency than for smaller angles. For the even larger central angle of $\gamma = 10^\circ$, the response starts to show peculiar characteristics, which can be explained by Fig. 10(b), showing the cover response for a distributed load having a large central angle of $\gamma = 60^\circ$. It can be seen that the traveling wave is generated under the load, beginning from the leading edge. The wave attenuates under the load, but outside of it, the wave gets stronger again and appears almost as a mirror image of the part under the load. When there is an integer number of full waves plus a half wave under the load, the wave motion attains its minimum at the trailing edge of the load area and reverses its direction. When this is combined with the release of the compressive load, the reinforcement is strong, as in Fig. 10(b) and in Fig. 10(a) at $f_{\text{rot}} = 48$ Hz for $\gamma = 10^\circ$. The phenomenon is similar to the reinforced resonance (Karttunen and von Hertzen, 2014), which takes place when the traveling wave extends over the cylinder's whole circumference and the head and tail of the wave interact. When there is a full number of waves on the cylinder circumference, the head and tail join smoothly, or in other words, a mode is resonating, and the reinforcement is maximal. Analogously, the reinforcement of the wave at the lift-off point can be seen to be caused by the “smoothness of the fit” at the interface of the loaded and unloaded areas.

Figure 10: (a) Maximum radial displacement $u_{\max}$ in the rotational frequency range 1 – 100 Hz for a single moving point load and three different distributed loads. All loads have the same resultant magnitude P_0 . (b) The cover deformation for $\gamma = 60^\circ$. The traveling wave forms under the compressed part and attenuates under it, but is reinforced due to a favourable phasing of the wave at the loaded-unloaded-interface, that is to say, at the lift-off point.

4. Conclusions

A comparison of the viscoelastic cover model subjected to a single moving point load with the corresponding elastic case with Kelvin-Voigt damping shows that in the viscoelastic case, a critical load speed, or a very narrow speed range, within which traveling waves start to appear in the cover, can be easily found in terms of a resonance condition unlike in the damped elastic case. This is due to the fact that the viscoelastic cover material stiffens for increasing excitation frequencies bringing about a distinct minimum in the rotational resonance curve and causing a specific natural mode and the related modal solution to give a clear indication of the critical speed. The stiffening also causes the cover response spectrum to bifurcate into two separate mode branches above the critical load speed. The response is dominated by the low-mode branch at supercritical speeds.

For the viscoelastic case, it was shown that the traveling waves in the cover can be suppressed at supercritical speeds using a moving twin load. An optimal angle between the two moving point loads was determined to be such that it corresponds to the half of the wavelength of the dominant resonating mode. The method is effective when a single mode is clearly dominant in the cover response. It was found that a wide-enough moving distributed load may cause notable qualitative differences in the cover response in comparison to a point load. A traveling wave is generated and attenuated under the distributed load, but at the lift-off point at the trailing edge of the load, the wave may be reinforced significantly depending on its wavelength.

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Appendix A. Other loading cases

For N moving twin loads, with each point load having the amplitude $P_0/(2N)$ and separated by the angle Δ , the steady-state response Eq. (21) becomes

$$u(\theta, t) = \sum_{n=1}^{\infty} \frac{B'_n S_n}{N} + \frac{P_0}{2\pi\omega_0^2}, \quad (\text{A.1})$$

where, by using the notation $\Theta = n(\Omega t - \theta) - \psi'_n$, the sum term S_n can be written as (Jeffrey and Zwillinger, 2007)

$$S_n = \sum_{p=0}^{N-1} \cos(\Theta - 2pn\Delta) = \cos[\Theta - (N-1)n\Delta] \frac{\sin(Nn\Delta)}{\sin(n\Delta)}. \quad (\text{A.2})$$

Now the steady-state response is obtained when the substitutions $B'_n \rightarrow B''_n$ and $\psi'_n \rightarrow \psi''_n$ are made in Eq. (21), where

$$B''_n = \frac{B_n}{2N} \frac{\sin(Nn\Delta)}{\sin(n\Delta/2)}, \quad (\text{A.3})$$

$$\psi''_n = \psi_n + n\Delta(N-1/2). \quad (\text{A.4})$$

Let us consider next, by means of twin point loads, a moving constant distributed load with a resultant load of P_0 acting along an arc on the cylinder circumference for which the corresponding central angle is

$$\gamma = (2N-1)\Delta \quad \text{or} \quad N\Delta = \frac{\gamma + \Delta}{2}. \quad (\text{A.5})$$

In the limit $N \rightarrow \infty$ and $\Delta \rightarrow 0$, we get for the distributed load, by the aid of Eqs. (A.3)–(A.5),

$$B'''_n = B_n \frac{\sin(n\gamma/2)}{n\gamma/2}, \quad (\text{A.6})$$

$$\psi'''_n = \psi_n + \frac{n\gamma}{2}. \quad (\text{A.7})$$

The steady-state response for a moving constant distributed load with a central angle γ is then obtained by the substitutions $B'_n \rightarrow B'''_n$ and $\psi'_n \rightarrow \psi'''_n$ in Eq. (21). Note that in the limit $\gamma \rightarrow 0$, the case for the point load is obtained. As a final loading case, let us consider M moving distributed loads of the aforementioned kind, each with a resultant load P_0/M and separated from each other by an angle σ . By virtue of Eqs. (A.3), (A.4), (A.6) and (A.7), we get

$$B''''_n = \frac{B_n}{M} \frac{\sin(n\gamma/2)}{n\gamma/2} \frac{\sin(Mn\sigma/2)}{\sin(n\sigma/2)}, \quad (\text{A.8})$$

$$\psi''''_n = \psi_n + \frac{n}{2} [\gamma + \sigma(M-1)]. \quad (\text{A.9})$$

The steady-state response for M moving constant distributed loads is now obtained by the substitutions $B'_n \rightarrow B''''_n$ and $\psi'_n \rightarrow \psi''''_n$ in Eq. (21).

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