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Exact theory for a linearly elastic interior beam

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Abstract

In this paper, an elasticity solution for a two-dimensional (2D) plane beam is derived and it is shown that the solution provides a complete framework for exact one-dimensional (1D) presentations of plane beams. First, an interior solution representing a general state of any 2D linearly elastic isotropic plane beam under a uniform distributed load is obtained by employing a stress function approach. The solution excludes the end effects of the beam and is valid sufficiently far away from the beam boundaries. Then, three kinematic variables defined at the central axis of the plane beam are formed from the 2D displacement field. Using these central axis variables, the 2D interior elasticity solution is presented in a novel manner in the form of a 1D beam theory. By applying the Clapeyron's theorem, it is shown that the stresses acting as surface tractions on the lateral end surfaces of the interior beam need to be taken into account in all energy-based considerations related to the interior beam. Finally, exact 1D rod and beam finite elements are developed by the aid of the axis variables from the 2D solution.

Keywords: plane beam, elasticity solution, Airy stress function, kinematic variables, Clapeyron's theorem, finite element

1. Introduction

Elasticity solutions for plane beams are of fundamental interest in mechanical sciences. An important application of such solutions is the benchmarking of beam theories based on various kinematic assumptions. Two-dimensional (2D) interior elasticity solutions can be easily obtained, for example, for an end-loaded cantilever and a uniformly loaded simply-supported beam by employing the Airy stress function (e.g., Timoshenko and Goodier, 1970).

An interior solution excludes, by virtue of the Saint Venant's principle, the end effects that decay with distance from the ends of a beam. In the calculation of displacements, constraint conditions are applied at the beam supports to prevent it from moving as a rigid body. These constraints for the 2D elasticity solution can be chosen so that they correspond to the boundary conditions of, for example, the Timoshenko beam theory (Timoshenko, 1921). Due to the foregoing, a 2D interior plane stress solution for a plane beam acts as an ideal reference solution for narrow one-dimensional (1D) shear-deformable beam models that do not include end effects.

Many beam and plate theories are based on an assumed displacement field similar to the one first used by Vlasov (1957). These theories are commonly referred to as third-order theories

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because third-order polynomials are used in the expansion of the displacement components. For surveys on third-order kinematics and plate theories, see the works by Jemielita (1990) and Reddy (1990, 2003). Two examples of third-order beam theories are the Levinson and the Reddy–Bickford beams for which the assumed displacement field is exactly the same (Levinson, 1981; Bickford, 1982; Reddy, 1984; Heyliger and Reddy, 1988). As first shown by Bickford (1982), the Reddy–Bickford beam exhibits a boundary layer character, that is, the decaying end effects are present in the beam. The Reddy–Bickford theory is obtained through an energy-based variational formulation, which results in additional higher-order load resultants in comparison to an interior elasticity solution. If the higher-order load resultants are neglected, the Levinson theory is obtained.

In this study, a general interior elasticity solution is derived for a uniformly loaded linearly elastic homogeneous isotropic 2D plane beam. As the main novelties of the study we find that:

- The 2D solution provides the exact third-order kinematics for the beam and can be presented directly in the form of a conventional 1D beam theory.
- By applying the Clapeyron's theorem, it is shown that the stresses acting as surface tractions on the lateral end surfaces of the interior beam are an intrinsic part of all energy-based considerations.
- The 2D solution can be discretized in order to obtain 1D rod and beam finite elements, which provide exact 2D interior displacement and stress distributions.

In more detail, the paper is organized as follows. In the introductory Section 2, a polynomial Airy stress function is used to derive the interior stress field for a 2D plane beam under a uniform distributed load. The strains are calculated from the stresses according to the plane stress condition and the displacements are integrated from the strains. In Section 3, three kinematic variables defined at the central axis of the plane beam are formed from the 2D interior displacement field. Using these new central axis variables, 1D beam equations are developed. The total potential energy of the interior beam and Clapeyron's theorem are considered. Finally, exact 1D interior rod and flexural beam finite elements are developed from the 2D interior elasticity solution. Conclusions are presented in Section 4.

2. Stress function solution for a plane beam

2.1. Plane beam problem and Airy stress function

Fig. 1 presents a 2D homogeneous isotropic plane beam under a uniform pressure p. The beam has a rectangular cross-section of constant thickness t and the length and height of the beam are L and h, respectively. The load resultants N, M and Q stand for the axial force, bending moment and shear force, respectively. These cross-sectional load resultants are calculated from the equations

$$N(x) = t\int_{-h/2}^{h/2} \sigma_x(x,y)dy \ , \qquad M(x) = t\int_{-h/2}^{h/2} \sigma_x(x,y)ydy \ , \qquad Q(x) = t\int_{-h/2}^{h/2} \tau_{xy}(x,y)dy \ , \quad (1)$$

which can be used to impose the force and moment boundary conditions at $x = \pm L/2$. The boundary conditions on the upper and lower surfaces of the beam are

$$\sigma_y(x,h/2) = -p \ , \qquad \sigma_y(x,-h/2) = 0 \ , \qquad \tau_{xy}(x,\pm h/2) = 0 \ . \quad (2)$$

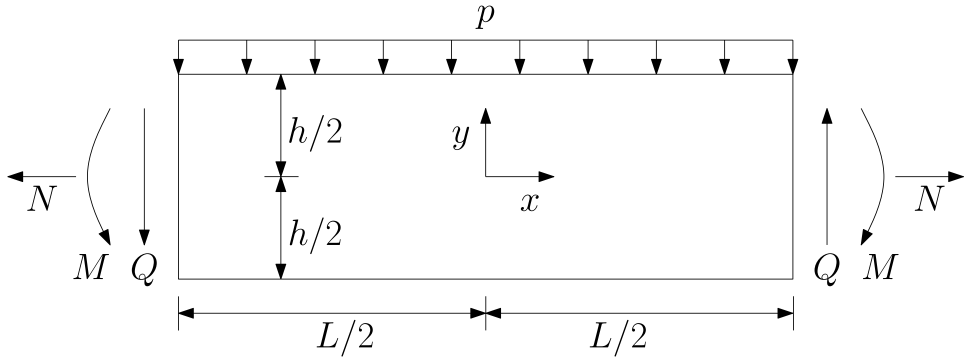

Figure 1: 2D homogeneous isotropic plane beam with a rectangular cross-section under a uniform pressure. The load resultants act at an arbitrary cross-section of the beam.

Note that the boundary conditions are satisfied in a *strong* (pointwise) sense on the upper and lower surfaces, whereas at the beam ends the tractions are not specified at each point but only through the load resultants and, thus, the boundary conditions are imposed only in a *weak* sense (Barber, 2010). In the case of Fig. 1, the replacement of the true boundary conditions at the beam ends by the statically equivalent weak boundary conditions (load resultants) implies that the exponentially decaying end effects of the plane beam are neglected by virtue of the Saint Venant's principle and only the interior solution of the beam is under consideration. The interior solution represents essentially a beam section which has been cut of from a complete beam far enough from the real lateral boundaries at which the true boundary conditions could be set. The stresses of the plane beam are obtained from the equations

$$\sigma_x = \frac{\partial^2 \Psi}{\partial y^2} \ , \qquad \sigma_y = \frac{\partial^2 \Psi}{\partial x^2} \ , \qquad \tau_{xy} = -\frac{\partial^2 \Psi}{\partial x \partial y} \ , \tag{3}$$

where $\Psi(x, y)$ is the Airy stress function. Eqs. (3) satisfy the two-dimensional equilibrium equations. To ensure compatibility, it is required that the stress function satisfies the biharmonic equation (Barber, 2010)

$$\frac{\partial^4 \Psi}{\partial x^4} + 2\frac{\partial^4 \Psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Psi}{\partial y^4} = 0 \ . \tag{4}$$

The solution to the plane beam problem is obtained by finding a solution of Eq. (4) that satisfies the stress boundary conditions (2) of the beam.

2.2. Interior stress field of a plane beam

By adapting a general solution procedure outlined by Barber (2010, chap. 5), we find that the polynomial stress function for the interior problem of any plane beam under a uniform pressure p (see Fig. 1) is

$$\Psi(x, y) = c_1 y^2 + c_2 y^3 + c_3 xy \left(1 - \frac{4y^2}{3h^2}\right) - \frac{q}{240I} \left[5h^3 x^2 + 15h^2 x^2 y + 4y^3 \left(y^2 - 5x^2\right)\right] \ , \tag{5}$$

where $q = pt$ is the uniform load, $I = th^3/12$ is the second moment of the cross-sectional area and c_1, c_2 and c_3 are to be solved by the aid of Eqs. (1). The stresses calculated from Eqs. (3) are

$$\sigma_x = 2c_1 + 6c_2 y - \frac{8c_3 xy}{h^2} + \frac{q(3x^2 y - 2y^3)}{6I} , \tag{6}$$

$$\sigma_y = -\frac{q}{24I}\left(h^3 + 3h^2 y - 4y^3\right) , \tag{7}$$

$$\tau_{xy} = c_3\left(\frac{4y^2}{h^2} - 1\right) + \frac{qx}{2I}\left(\frac{h^2}{4} - y^2\right) . \tag{8}$$

Note that the above interior stress distributions are universal, that is, they are valid for any plane beam under a uniform load since they are not associated with any particular constraint conditions at the beam ends. Using Eqs. (6) and (8), the load resultants calculated from Eqs. (1) are

$$N = 2Ac_1 , \qquad M = 6Ic_2 - \frac{2}{3}Ac_3 x + \frac{q}{2}\left(x^2 - \frac{h^2}{10}\right) , \qquad Q = qx - \frac{2}{3}Ac_3 , \tag{9}$$

where $A = ht$ is the area of the cross-section. As a first step towards presenting the solution in the form of a 1D beam theory, it can be easily verified that the following global equilibrium equations, which can also be obtained by integrating the 2D stress equilibrium equations, hold for the load resultants (9)

$$\frac{\partial N}{\partial x} = 0 , \qquad \frac{\partial M}{\partial x} = Q , \qquad \frac{\partial Q}{\partial x} = q . \tag{10}$$

We note that Schneider and Kienzler (2015) arrived at the same equilibrium equations (10) through their recent exact 3D representation of linear elasticity. When c_1, c_2 and c_3 are solved from Eqs. (9) and substituted into Eqs. (6) and (8), we obtain

$$\sigma_x = \frac{N}{A} + \frac{My}{I} + \frac{3qy}{5A} - \frac{qy^3}{3I} , \tag{11}$$

$$\tau_{xy} = \frac{Q}{8I}(h^2 - 4y^2) , \tag{12}$$

The stress distribution of Eq. (11) has been called by Rehfield and Murthy (1982) the refined (nonclassical) axial stress distribution in the context of their beam theory. More complicated distributed loads lead to different additional load terms in the stresses. By setting $q = 0$, Eqs. (11) and (12) give the stress distribution of the classical Euler-Bernoulli beam.

2.3. Example – Simply-supported beam

As an example, let us consider a simply-supported beam under a constant uniform load q. In a setting according to Fig. 1, the axial force, bending moment and shear force along the beam are given by

$$N(x) = 0 , \quad M(x) = q(x^2/2 - L^2/8) , \quad Q(x) = qx , \tag{13}$$

respectively. We find that the interior stress state in the beam calculated from Eqs. (7) and (11)–(13) is the same as the one found in any elasticity textbook for the particular problem at hand (e.g., Barber, 2010). The benefit of using Eqs. (7), (11) and (12) is that they provide the interior stress field also for different support conditions, granted that the load resultants along the beam are known. In other words, instead of seeking a particular stress function suitable for the case at hand, one can take use of the systematic approach provided by Eqs. (7), (11) and (12). The procedure of creating a general interior stress field for a uniform distributed load presented above may be extended to more complicated loads by updating the load dependent part of the stress function (5). As a rule of thumb in isotropic cases, the interior solution is typically considered to be a good approximation when the axial distance to an end of a beam is at least equal to the height of the beam.

2.4. Interior strains and displacements of a plane beam

Under plane stress, the stress-strain relations read

$$\epsilon_x = \frac{\sigma_x - \nu\sigma_y}{E} \ , \qquad \epsilon_y = \frac{\sigma_y - \nu\sigma_x}{E} \ , \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy} \ , \tag{14}$$

where ϵ_x, ϵ_y and γ_{xy} are the axial normal strain, transverse normal strain and transverse shear strain, respectively, and E, G and ν are the Young's modulus, shear modulus and the Poisson ratio, respectively. The strain-displacement relations are

$$\epsilon_x = \frac{\partial U_x}{\partial x} \ , \qquad \epsilon_y = \frac{\partial U_y}{\partial y} \ , \qquad \gamma_{xy} = \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \ , \tag{15}$$

where $U_x(x,y)$ and $U_y(x,y)$ are the displacements in the directions of x and y, respectively. The displacements are integrated from the first two of Eqs. (15), and the resulting arbitrary functions $f(y)$ and $g(x)$ are resolved by substituting the calculated displacements into the last of Eqs. (15). As the final result, we get for the interior displacements the expressions

$$\begin{aligned} U_x(x,y) &= \frac{1}{E}\left\{2c_1x + 6c_2xy - \frac{2c_3y}{3h^2}\left[3h^2(1+\nu) + 6x^2 - 2y^2(2+\nu)\right] - Cy + D_1\right\} \\ &\quad + \frac{qx}{24EI}\left[\nu h^3 + 3\nu h^2 y + 4x^2y - 4y^3(2+\nu)\right] \ , \end{aligned} \tag{16}$$

$$\begin{aligned} U_y(x,y) &= \frac{1}{E}\left[-2c_1\nu y - 3c_2(x^2+\nu y^2) + \frac{4c_3(x^3+3\nu xy^2)}{3h^2} + Cx + D_2\right] \\ &\quad + \frac{q}{48EI}\left\{-2h^3y + 3h^2\left[x^2(2+\nu) - y^2\right] + 2\left[y^4(1+2\nu) - x^4 - 6\nu x^2y^2\right]\right\} \ , \end{aligned} \tag{17}$$

where the constants C, D_1 and D_2 define the degrees of freedom of the plane beam as a rigid body. The constant C relates to anticlockwise rotation about the origin and D_1 and D_2 correspond to translations in the directions of x and y, respectively. Note that C, D_1 and D_2 are determined by the boundary region displacements of the beam, whereas the constant coefficients c_1, c_2 and c_3 can be calculated from the axial force, bending moment and shear force according to Eqs. (9).

3. 2D solution in the form of a 1D beam theory

3.1. Displacement and strain fields by central axis variables

The general interior solution of Section 2 can be presented in terms of three kinematic variables derived from the displacements (16) and (17) at the central axis so that the variables depend only on x. We obtain for the axial displacement and the transverse deflection of the central axis, and for the clockwise positive rotation of the cross-section at the central axis the expressions

$$u_x(x) = U_x(x,0) = \frac{1}{E}(2c_1 x + D_1) + \frac{q\nu x}{2Et} , \tag{18}$$

$$u_y(x) = U_y(x,0) = \frac{1}{E}\left(-3c_2x^2 + \frac{4c_3x^3}{3h^2} + Cx + D_2\right) + \frac{qx^2}{48EI}\left[3h^2(2+\nu) - 2x^2\right] , \tag{19}$$

$$\phi(x) = \frac{\partial U_x}{\partial y}(x,0) = \frac{1}{E}\left\{6c_2x - C - c_3\left[\frac{4x^2}{h^2} + 2(1+\nu)\right]\right\} + \frac{qx}{24EI}\left[4x^2 + 3\nu h^2\right] , \tag{20}$$

respectively. Using these kinematic *central axis variables*, we can present the displacements (16) and (17) in the form

$$U_x(x,y) = u_x + y\phi - \frac{4y^3}{3h^2}\left(\phi + \frac{\partial u_y}{\partial x}\right) + \frac{\nu y^3}{6}\frac{\partial^2 \phi}{\partial x^2} , \tag{21}$$

$$U_y(x,y) = u_y - \nu y\frac{\partial u_x}{\partial x} - \frac{\nu y^2}{2}\frac{\partial \phi}{\partial x} + \frac{qy}{48EI}\left[(2h+3y)(\nu^2-1)h^2 + 2y^3(1+2\nu)\right] . \tag{22}$$

These displacements represent the exact third-order interior kinematics of a linear homogeneous isotropic beam having a narrow rectangular cross-section. If we neglect the load term and the Poisson effect in Eqs. (21) and (22), the 2D displacement field is exactly of the same form as in the Levinson and Reddy–Bickford beam theories. Note that instead of the Airy stress function, one may also consider the Marguerre function for the case of plane stress and the Love strain function for plane strain in order to arrive at the displacements (21) and (22) (e.g., Soutas-Little, 1973). In terms of the central axis variables, the strains (15) may be readily written in the form

$$\epsilon_x = y\frac{\partial \phi}{\partial x} + \frac{\partial u_x}{\partial x} - \frac{qy^3}{6EI}(2+\nu) , \tag{23}$$

$$\epsilon_y = -\nu\left(y\frac{\partial \phi}{\partial x} + \frac{\partial u_x}{\partial x}\right) + \frac{q}{24EI}\left[(3y+h)(\nu^2-1)h^2 + 4y^3(1+2\nu)\right] , \tag{24}$$

$$\gamma_{xy} = \left(1 - \frac{4y^2}{h^2}\right)\left(\phi + \frac{\partial u_y}{\partial x}\right) . \tag{25}$$

Note that if the strains are calculated directly from the expressions (21) and (22) using the definitions (15), the results simplify to Eqs. (23)–(25) due to the specific low-order polynomial form of the central axis variables.

3.2. Load resultants and beam equilibrium equations

Using the strains (23)–(25) and applying the plane stress constitutive relations (14), the load resultants (1) in terms of the central axis variables can be written as

$$N = EA\frac{\partial u_x}{\partial x} - \frac{qh\nu}{2} , \tag{26}$$

$$M = EI\frac{\partial \phi}{\partial x} - \frac{qh^2}{40}(2+5\nu) , \tag{27}$$

$$Q = \frac{2}{3}GA\left(\phi + \frac{\partial u_y}{\partial x}\right) . \tag{28}$$

Substitution of the load resultants into the equilibrium equations (10) leads to

$$EA\frac{\partial^2 u_x}{\partial x^2} = 0 \ , \tag{29}$$

$$EI\frac{\partial^2 \phi}{\partial x^2} - \frac{2}{3}GA\left(\phi + \frac{\partial u_y}{\partial x}\right) = 0 \ , \tag{30}$$

$$\frac{2}{3}GA\left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 u_y}{\partial x^2}\right) = q \ . \tag{31}$$

By uncoupling Eqs. (30) and (31), the axis variables u_y and ϕ can be solved from

$$EI\frac{\partial^3 \phi}{\partial x^3} = q \ , \tag{32}$$

$$EI\frac{\partial^4 u_y}{\partial x^4} = -q \ . \tag{33}$$

In summary, the general solution to Eqs. (29)–(31) is given by Eqs. (18)–(20), and the solution is expanded into the whole interior region of the 2D plane beam through Eqs. (21) and (22). These equations constitute an alternative representation of the elasticity solution presented in Section 2. We can see, for example, that the obtained beam equations (32) and (33) are exactly the same as those in the Timoshenko beam theory with a constant uniform distributed load (e.g., Dym and Shames, 2013). Eqs. (28) and (31) are the same as in the static Levinson beam theory. The provided exact 1D interior presentation enables a more thorough study of the pros and cons of approximate interior beam theories, such as those of Levinson and Timoshenko, on the level of governing differential equations instead of simplistic (numerical) comparisons between specific 2D elasticity and 1D beam solutions. Following the methodology presented above for the uniform load, 1D beam presentations may be obtained for more complicated loads by updating the load-dependent part of the stress function (5).

3.3. Example – Simply-supported beam revisited

We are now able to solve a 2D interior plane beam problem in terms of the 1D beam presentation. Using the load resultants given by Eqs. (13) and the interior beam conditions $u_y(\pm L/2) = 0$ and $u_x(0) = 0$, integration of Eqs. (26)–(28) yields for the central axis variables the expressions

$$\begin{aligned} u_x &= \frac{q\nu h^3 x}{24EI} \ , \\ u_y &= -\frac{q}{1920EI}(L^2 - 4x^2)\left[5(5L^2 - 4x^2) + 6h^2(8+5\nu)\right] \ , \\ \phi &= -\frac{qx}{120EI}\left[15L^2 - 20x^2 - 3h^2(2+5\nu)\right] \ . \end{aligned}$$

Substitution of the central axis variables into the 2D displacements (21) and (22) results in the exact 2D interior displacement field. Alternatively, to obtain the 2D displacements, we can solve u_y and ϕ from Eqs. (32) and (33) with the interior boundary conditions

$$u_y(\pm L/2) = 0 \ , \qquad M(\pm L/2) = 0 \quad \rightarrow \quad \frac{\partial \phi}{\partial x}(\pm L/2) = \frac{qh^2}{40EI}(2+5\nu) \ .$$

Detailed end effects are out of the scope of this study, but one should keep in mind that the moment-related boundary conditions at the beam ends might take a different form in a solution which includes the boundary layer behavior. Note also that, unlike in the Timoshenko beam theory, the distributed load q has an effect on the moment-related boundary condition.

3.4. Applying Clapeyron's theorem

Let us briefly study the energetic aspects of the interior beam. The strain energy of the 2D beam and the external work due to the uniform load are given by

$$U = \frac{1}{2}\int_V (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy}\gamma_{xy})dV \ , \quad W_q = -\int_{-L/2}^{L/2} qU_y(x, h/2)dx \ , \tag{34}$$

respectively. While the contributions (34) to the total potential energy of the beam are apparent, the following consideration may not be that obvious at first. We recall from the problem definition in Section 2.1 that the interior solution part represents essentially a beam section with fully-developed interior stresses which has been cut off from a complete beam so that the boundary layer is not modeled. Thus, we obtain for the work due to the stresses on the lateral end surfaces of the interior beam

$$\begin{aligned} W_s = t\int_{-h/2}^{h/2} \sigma_x(L/2, y)U_x(L/2, y)dy - t\int_{-h/2}^{h/2} \sigma_x(-L/2, y)U_x(-L/2, y)dy \\ + t\int_{-h/2}^{h/2} \tau_{xy}(L/2, y)U_y(L/2, y)dy - t\int_{-h/2}^{h/2} \tau_{xy}(-L/2, y)U_y(-L/2, y)dy \ . \end{aligned} \tag{35}$$

By substituting the polynomial expressions according to the general solution presented in Section 2 for σ_x, ϵ_x, σ_y, ϵ_y, τ_{xy}, γ_{xy}, U_x and U_y into (34) and (35) we find that

$$2U - W_q - W_s = 0 \ . \tag{36}$$

The above calculation shows that in static equilibrium the strain energy of the beam is equal to one-half of the work done by the surface tractions if they were to move (slowly) through their respective displacements. This is exactly according to the *Clapeyron's theorem* (e.g., Sadd, 2014). In a recent paper (Karttunen and von Hertzen, 2015), in contrast to the long-standing belief that the interior beam theory by Levinson (1981) is "variationally inconsistent", we provided a consistent variational formulation for the Levinson beam by accounting for the external (virtual) work (35). The resulting beam equations were exactly the same as those in the case of the vectorial derivation by Levinson (1981). The work (35) is an integral part of all energy-based considerations related to interior beams.

3.5. Exact rod and beam finite elements

The polynomial 2D interior elasticity solution presented in Section 2 can be used to derive exact 1D interior rod and beam finite elements. Fig. 2 presents the setting according to which the elements are developed. Both nodes in Fig. 2 have three degrees of freedom. For nodes $i = 1, 2$, we have axial displacements $u_{x,i}$, transverse displacements $u_{y,i}$ and rotations of the cross-section ϕ_i. By the aid of the central axis variables (18)–(20), we obtain for nodes 1 and 2 six equations

$$\begin{aligned} u_{x,1} &= u_x(-L/2) \ , & u_{x,2} &= u_x(L/2) \ , \\ u_{y,1} &= u_y(-L/2) \ , & u_{y,2} &= u_y(L/2) \ , \\ \phi_1 &= -\phi(-L/2) \ , & \phi_2 &= -\phi(L/2) \ . \end{aligned} \tag{37}$$

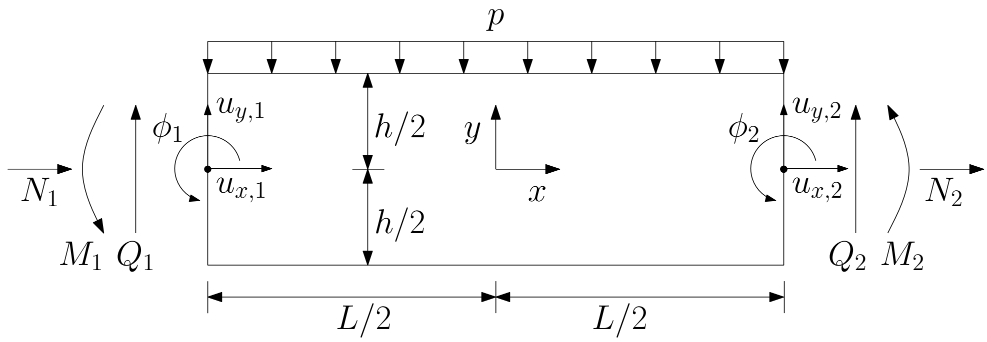

Figure 2: Set-up according to which exact rod and flexural beam finite elements are developed.

We can solve the six unknowns c_1, c_2, c_3, C, D_1 and D_2 from Eqs. (37). Explicit expressions for these are given in Appendix A. To obtain the finite element equations, we substitute c_1, c_2 and c_3 into Eqs. (9) to calculate the load resultants at nodes $i = 1, 2$, with the notion that the positive directions are taken to be according to Fig. 1 so that

$$\begin{aligned} N_1 &= -N(-L/2)\ , \qquad & N_2 &= N(L/2)\ , \\ Q_1 &= -Q(-L/2)\ , \qquad & Q_2 &= Q(L/2)\ , \\ M_1 &= M(-L/2)\ , \qquad & M_2 &= -M(L/2)\ . \end{aligned} \tag{38}$$

The conventional presentation for the 1D rod and beam elements is obtained by writing Eqs. (38) in matrix form. Before doing so, we also derive the FE equations from the total potential energy

$$\Pi = U - W_q - W_s\ . \tag{39}$$

The stresses on the end surfaces in Eq. (35) are written as given by Eqs. (11) and (12), where the load resultants are expressed as nodal forces according to Eqs. (38). Then, by calculating (34) and (35) and by applying the principle of minimum total potential energy

$$\frac{\partial \Pi}{\partial u_{x,i}} = 0\ , \quad \frac{\partial \Pi}{\partial u_{y,i}} = 0\ , \quad \frac{\partial \Pi}{\partial \phi_i} = 0 \qquad (i = 1, 2) \tag{40}$$

we obtain the finite element equilibrium equations. The force-based method and the total potential energy approach result in the same equations, which can be written in the form

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x,1} \\ u_{x,2} \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} - \frac{q\nu h}{2}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}\ , \tag{41}$$

$$\frac{EI}{(1+\Phi)L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4+\Phi)L^2 & -6L & (2-\Phi)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2-\Phi)L^2 & -6L & (4+\Phi)L^2 \end{bmatrix}\begin{Bmatrix} u_{y,1} \\ \phi_1 \\ u_{y,2} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{Bmatrix} - \frac{q}{2}\begin{Bmatrix} L \\ \frac{10L^2 - 3h^2(2+5\nu)}{60} \\ L \\ -\frac{10L^2 - 3h^2(2+5\nu)}{60} \end{Bmatrix} \tag{42}$$

where $\Phi = 3h^2(1+\nu)/L^2$. Eqs. (41) and (42) can be written concisely as

$$\mathbf{K}_r \mathbf{u}_r = \mathbf{f}_r - \mathbf{q}_r\ , \tag{43}$$

$$\mathbf{K}_b\mathbf{u}_b = \mathbf{f}_b - \mathbf{q}_b \ , \tag{44}$$

where $\mathbf{K}_r$ and $\mathbf{K}_b$ are the rod and beam element stiffness matrices, respectively, and $\mathbf{u}_r$ and $\mathbf{u}_b$ are the rod and beam nodal displacement vectors, respectively. The force vectors are $\mathbf{f}_r$ and $\mathbf{q}_r$ for the rod element, and $\mathbf{f}_b$ and $\mathbf{q}_b$ for the beam element, respectively. Note that the stiffness matrix $\mathbf{K}_b$ is the same as for the Levinson beam (Karttunen and von Hertzen, 2015) and also equal to the stiffness matrix of a Timoshenko beam element when the Timoshenko shear coefficient has the value of 2/3 (e.g., cf. Kosmatka, 1995). Finally, by substituting c_1, c_2, c_3, C, D_1 and D_2 (Appendix A) into the displacements (16) and (17) we can write the 2D displacements in the form

$$U_x(x,y) = \mathbf{N}_x\mathbf{u} + qL_1 \ , \tag{45}$$

$$U_y(x,y) = \mathbf{N}_y\mathbf{u} + qL_2 \ , \tag{46}$$

where the nodal displacement vector is

$$\mathbf{u} = \{u_{x,1} \quad u_{y,1} \quad \phi_1 \quad u_{x,2} \quad u_{y,2} \quad \phi_2\}^{\mathrm{T}} \ . \tag{47}$$

In addition we have

$$L_1 = -\frac{xy}{24EI}\left\{L^2 - 4\left[x^2 - y^2(2+\nu)\right]\right\} \ , \tag{48}$$

$$\begin{aligned} L_2 = \frac{1}{384EI}&\left\{16h^3y(\nu^2-1) - 4\Phi L^2\left[L^2 - 4x^2 - 2y^2(\nu-1)\right]\right. \\ &\left.+8\nu y^2\left[L^2 + 4(y^2 - 3x^2)\right] - (L^2 - 4x^2)^2 + 16y^4\right\} \ . \end{aligned} \tag{49}$$

The shape functions $\mathbf{N}_x$ and $\mathbf{N}_y$ are given in Appendix A. Once the nodal displacements have been solved from Eqs. (41) and (42), the exact interior 2D displacement field can be calculated by substituting the nodal displacements into Eqs. (45) and (46), after which the calculation of 2D interior strains and stresses is straightforward.

4. Conclusions

In this paper, a general interior elasticity solution for a 2D plane beam under a uniform load was derived. It was shown that the solution can be presented in the form of a conventional 1D beam theory by the aid of kinematic variables defined at the central axis of the 2D beam. In addition, the solution can be presented in the form of rod and beam finite elements. The presentation of the 2D interior elasticity solution as a 1D beam theory offers a new point of view on 2D plane beam solutions and reveals the underlying structure of an exact 1D interior beam. It was also shown that in all energy-based considerations related to the interior beam one has to take into account the fact that the interior stresses do work on the lateral end surfaces of the interior beam. Many higher-order beam theories can be found in the literature which are founded exclusively on interior kinematics (e.g. Bickford, 1982). However, these higher-order constructions are incomplete as interior beam theories because they lack the aforementioned work due to the stresses at the beam ends.

Appendix A. Equations for finite element developments

We obtain from Eqs. (37) the following relations

$$
\begin{aligned}
c_1 &= \frac{E}{2L}(u_{x,2} - u_{x,1}) - \frac{q\nu}{4t} \ , \\
c_2 &= \frac{E}{6L}(\phi_1 - \phi_2) - \frac{q}{144I}(L^2 + 3\nu h^2) \ , \\
c_3 &= \frac{3Eh^2}{4L^3(1+\Phi)}\left[2(u_{y,1} - u_{y,2}) + L(\phi_1 + \phi_2)\right] \ ,
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
C &= -\frac{E}{4L^3(1+\Phi)}\left\{4\Phi L^2(u_{y,1} - u_{y,2}) + L^2\left[6(u_{y,1} - u_{y,2}) + L(\phi_1 + \phi_2)\right]\right\} \ , \\
D_1 &= \frac{E}{2}(u_{x,1} + u_{x,2}) \ , \\
D_2 &= \frac{E}{8}\left[4(u_{y,1} + u_{y,2}) + L(\phi_1 - \phi_2)\right] - \frac{qL^4(1+4\Phi)}{384I} \ ,
\end{aligned} \tag{A.2}
$$

where $\Phi = 3h^2(1+\nu)/L^2$. The shape functions in Eqs. (45) and (46) are

$$
\mathbf{N}_x^{\mathrm{T}} = \left\{
\begin{array}{c}
\frac{1}{2} - \frac{x}{L} \\
\frac{y\left[3L^2 - 12x^2 + 4y^2(2+\nu)\right]}{2L^3(1+\Phi)} \\
\frac{y\left\{-2\Phi L^2(L-2x) + L\left[(L-2x)(L+6x) + 4y^2(2+\nu)\right]\right\}}{4L^3(1+\Phi)} \\
\frac{1}{2} + \frac{x}{L} \\
-\frac{y\left[3L^2 - 12x^2 + 4y^2(2+\nu)\right]}{2L^3(1+\Phi)} \\
\frac{y\left\{-2\Phi L^2(L+2x) + L\left[(L+2x)(L-6x) + 4y^2(2+\nu)\right]\right\}}{4L^3(1+\Phi)}
\end{array}
\right\} \ , \tag{A.3}
$$

$$
\mathbf{N}_y^{\mathrm{T}} = \left\{
\begin{array}{c}
\frac{y\nu}{L} \\
\frac{\Phi L^2(L-2x) + 12\nu x y^2 + (L+x)(L-2x)^2}{2L^3(1+\Phi)} \\
\frac{\Phi L^2\left[L^2 - 4(x^2+\nu y^2)\right] - 4\nu L(L-6x)y^2 + L(L+2x)(L-2x)^2}{8L^3(1+\Phi)} \\
-\frac{y\nu}{L} \\
\frac{\Phi L^2(L+2x) - 12\nu x y^2 + (L-x)(L+2x)^2}{2L^3(1+\Phi)} \\
\frac{\Phi L^2\left[-L^2 + 4(x^2+\nu y^2)\right] + 4\nu L(L+6x)y^2 - L(L-2x)(L+2x)^2}{8L^3(1+\Phi)}
\end{array}
\right\} \ . \tag{A.4}
$$

References

Barber, J. R., 2010. Elasticity, 3rd Edition. Springer, New York.

Bickford, W. B., 1982. A consistent higher order beam theory. Dev. Theor. Appl. Mech. 11, 137–150.

Dym, C. L., Shames, I. H., 2013. Solid Mechanics – A Variational Approach (Augmented Edition). Springer, New York.

Heyliger, P. R., Reddy, J. N., 1988. A higher order finite element for bending and vibration problems. J. Sound Vib. 126 (2), 309–326.

Jemielita, G., 1990. On kinematical assumptions of refined theories of plates – a survey. J. Appl. Mech. 57 (4), 1088–1091.

Karttunen, A. T., von Hertzen, R., 2015. Variational formulation of the static Levinson beam theory. Mech. Res. Commun. 66, 15–19.

Kosmatka, J. B., 1995. An improved two–node finite element for stability and natural frequencies of axial–loaded Timoshenko beams. Comput. Struct. 57 (1), 141–149.

Levinson, M., 1981. A new rectangular beam theory. J. Sound Vib. 74 (1), 81–87.

Reddy, J. N., 1984. A simple higher–order theory for laminated composite plates. J. Appl. Mech. 51 (4), 745–752.

Reddy, J. N., 1990. A general nonlinear 3rd–order theory of plates with moderate thickness. Int. J. Nonlin. Mech. 25 (6), 677–686.

Reddy, J. N., 2003. Mechanics of Laminated Composite Plates and Shells: Theory and Analysis, 2nd Edition. CRC Press, Boca Raton, Florida.

Rehfield, L. W., Murthy, P. L. N., 1982. Toward a new engineering theory of bending – fundamentals. AIAA J. 20 (5), 693–699.

Sadd, M. H., 2014. Elasticity – Theory, Applications and Numerics, 3rd Edition. Academic Press, Oxford.

Schneider, P., Kienzler, R., 2015. On exact rod/beam/shaft-theories and the coupling among them due to arbitrary material anisotropies. Int. J. Solids Struct. 56, 265–279.

Soutas-Little, R. W., 1973. Elasticity. Dover, New York.

Timoshenko, S. P., 1921. On the correction for shear of the differential equation for transverse vibration of prismatic bars. Philos. Mag. 41, 744–746.

Timoshenko, S. P., Goodier, J. N., 1970. Theory of Elasticity, 3rd Edition. McGraw-Hill, Singapore.

Vlasov, B. F., 1957. On equations of bending of plates (in Russian). Dokla Ak. Nauk Azerbeijanskoi SSR 3, 955–959.