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Variational formulation of the static Levinson beam theory

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Abstract

In this communication, we provide a consistent variational formulation for the static Levinson beam theory. First, the beam equations according to the vectorial formulation by Levinson are reviewed briefly. By applying the Clapeyron's theorem, it is found that the stresses on the lateral end surfaces of the beam are an integral part of the theory. The variational formulation is carried out by employing the principle of virtual displacements. As a novel contribution, the formulation includes the external virtual work done by the stresses on the end surfaces of the beam. This external virtual work contributes to the boundary conditions in such a way that artificial end effects do not appear in the theory. The obtained beam equations are the same as the vectorially derived Levinson equations. Finally, the exact Levinson beam finite element is developed.

Keywords: Levinson beam, interior beam, variational formulation, finite element

1. Introduction

The beam and plate theories by Levinson are widely known in the literature [1, 2]. Soon after their publication, Bickford gave a variational formulation for a beam theory [3], and Reddy for a plate theory [4], based on the displacement fields used by Levinson [1, 2]. Due to the fact that their variational formulations led to different equilibrium equations than the vectorial derivations by Levinson, the beam and plate theories by Levinson have since then been considered quite often as “variationally inconsistent”. In contrast to this common belief, we show in this study that the Levinson beam theory is actually variationally consistent with certain provisions. Hereafter, the scope is limited to static beam theories.

Little attention has been paid to the fact that the assumed displacement field, which is exactly the same for the Levinson and Reddy–Bickford beam theories, is exclusively an interior field. The use of interior kinematics means that the end effects that decay with distance from the ends of a beam are neglected by virtue of the Saint Venant's principle. Note that, for example, the Euler–Bernoulli and Timoshenko beam theories are interior beam theories. Furthermore, the well-known two-dimensional (2D) Airy stress function solutions for an end-loaded cantilever and a uniformly loaded simply-supported beam are interior solutions (see, e.g. [5]). The modeling of the end effects in such problems requires the use of the Papkovitch–Fadle eigenfunctions [6]. In his vectorial formulation, Levinson accounted correctly for the interior nature of his beam theory by using only the classical interior load resultants – the bending moment and the shear force [1]. Consequently,

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the theory provides the exact interior elasticity solutions, for example, for the central axis deflection and the 2D stresses of an end-loaded cantilever. As will be shown, to properly account for the interior nature in an energy-based formulation of the Levinson beam theory, one has to grasp the idea that the interior stresses of the beam act as surface tractions on the lateral interior end surfaces of the beam. If the work due to these surface tractions is not taken into account, the obtained beam theory will exhibit artificial end effects.

The current study is organized as follows. In Section 2, the static Levinson beam theory and its consistency with the Clapeyron's theorem are considered. In Section 3, a consistent variational formulation for the Levinson beam theory is carried out. An exact Levinson beam finite element is developed in Section 4 and conclusions are presented in Section 5.

2. Levinson beam theory

2.1. Stress boundary conditions and displacement field

Fig. 1 presents a beam subjected to a uniform load q , which we have chosen as a representative loading case for our developments. The beam has a narrow rectangular cross-section of constant thickness t and the length and height of the beam are L and h , respectively. The load resultants M and Q stand for the bending moment and shear force, respectively. The positive directions for the coordinates, uniform load and the load resultants are according to Levinson [1]. In his derivation of the theory, Levinson assumed that i) the transverse normal stress σ_y is zero throughout the beam and ii) the Poisson effect (lateral contraction/expansion) is not accounted for. On the basis of these assumptions, and to satisfy the homogeneous boundary conditions $\tau_{xy}(x, \pm h/2) = 0$ on the upper and lower surfaces of the beam, Levinson obtained the 2D displacement field

$$U_x(x, y) = y\phi - \frac{4y^3}{3h^2} \left(\phi + \frac{\partial u_y}{\partial x} \right), \quad (1)$$

$$U_y(x, y) = u_y, \quad (2)$$

where $u_y(x)$ is the transverse deflection of the central axis of the beam and $\phi(x)$ is the clockwise positive rotation of the lateral cross-section at the central axis. The homogeneous boundary conditions are satisfied in a *strong* (pointwise) sense on the upper and lower surfaces of the beam. Levinson did not discuss in detail the stress boundary conditions on the lateral end surfaces of the beam, but it is crucial to note that in his theory the tractions at the beam ends are not specified at each point but only through the load resultants and, thus, the boundary conditions are imposed only in a *weak* sense [6]. The replacement of the true stress boundary conditions at the beam ends by the statically equivalent weak boundary conditions (load resultants) means that the exponentially decaying end effects of the beam are neglected by virtue of the Saint Venant's principle and only the interior solution of the beam is under consideration. The cross-sectional load resultants are calculated from the equations

$$M(x) = t \int_{-h/2}^{h/2} \sigma_x y dy, \quad Q(x) = t \int_{-h/2}^{h/2} \tau_{xy} dy \quad (3)$$

and can be used to impose natural interior boundary conditions at $x = \pm L/2$. The interior solution represents essentially a beam section which has been cut off from a complete beam far enough from the real lateral boundaries at which the true boundary conditions could be set.

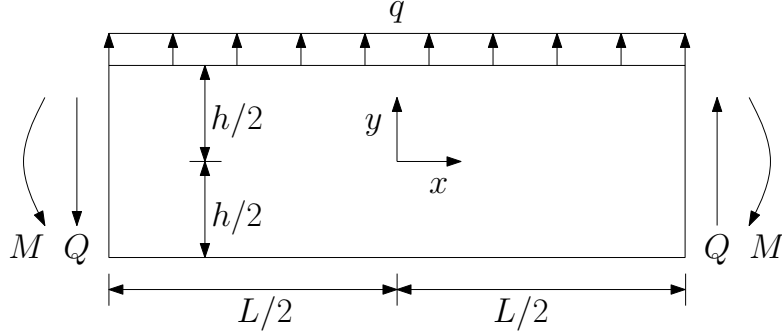


Figure 1: Homogeneous isotropic beam with a narrow rectangular cross-section. The positive directions are according to [1]. The load resultants act at an arbitrary cross-section of the beam.

2.2. General static solution

The static equilibrium equations for the Levinson beam theory are [1]

$$\frac{2}{3}GA \left(\phi + \frac{\partial u_y}{\partial x} \right) + \frac{EI}{5} \left(\frac{\partial^3 u_y}{\partial x^3} - 4 \frac{\partial^2 \phi}{\partial x^2} \right) = 0 , \quad (4)$$

$$\frac{2}{3}GA \left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 u_y}{\partial x^2} \right) = -q . \quad (5)$$

where E and G are the Young's modulus and shear modulus, respectively. In addition, $A = ht$ and $I = th^3/12$ are the area of the cross-section and the second moment of the cross-sectional area, respectively. The kinematic and constitutive relations for the Levinson beam are

$$\epsilon_x = \frac{\partial U_x}{\partial x} , \quad \gamma_{xy} = \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} , \quad (6)$$

$$\sigma_x = E\epsilon_x , \quad \tau_{xy} = G\gamma_{xy} , \quad (7)$$

respectively. The general polynomial solution to Eqs. (4) and (5) can be written as

$$u_y = c_1 + c_2 x + c_3 x^2 - c_4 \frac{2x^3}{3h^2(1+\nu)} + \frac{qx^2}{120EI} [5x^2 - 12h^2(1+\nu)] , \quad (8)$$

$$\phi = -c_2 - 2c_3 x + c_4 \left[1 + \frac{2x^2}{h^2(1+\nu)} \right] - \frac{qx}{60EI} [10x^2 + 3h^2(1+\nu)] . \quad (9)$$

Note that the constant c_1 corresponds to rigid body translation in the y -direction and c_2 to a small counterclockwise rigid body rotation. We calculate the load resultants (3) using the stresses (7) and the general solution (8) and (9). Then, we can express the constants c_3 and c_4 in terms of the load resultants and substitute them back into the stresses (7) to obtain

$$\sigma_x = \frac{My}{I} + \frac{qy}{60I}(1+\nu)(20y^2 - 3h^2) , \quad (10)$$

$$\tau_{xy} = \frac{Q}{8I}(h^2 - 4y^2) . \quad (11)$$

Static bending solutions for the Levinson beam can be found in the papers by Levinson [1], Reddy et al. [7] and Reddy [8].

Finally, as the key item of this section, let us consider the strain energy of the beam and the external work done by the surface tractions. The strain energy and the external work due to the uniform load are

$$U = \frac{1}{2} \int_V (\sigma_x \epsilon_x + \tau_{xy} \gamma_{xy}) dV, \quad W_q = \int_{-L/2}^{L/2} q u_y dx, \quad (12)$$

respectively. As a novel contribution, we consider the work due to the interior stresses on the lateral end surfaces of the interior beam and obtain

$$\begin{aligned} W_s = & t \int_{-h/2}^{h/2} \sigma_x(L/2, y) U_x(L/2, y) dy - t \int_{-h/2}^{h/2} \sigma_x(-L/2, y) U_x(-L/2, y) dy \\ & + t \int_{-h/2}^{h/2} \tau_{xy}(L/2, y) U_y(L/2, y) dy - t \int_{-h/2}^{h/2} \tau_{xy}(-L/2, y) U_y(-L/2, y) dy. \end{aligned} \quad (13)$$

By substituting the polynomial expressions for σ_x , ϵ_x , τ_{xy} , γ_{xy} , U_x and U_y according to the general solution (8) and (9) into Eqs. (12) and (13), we find that

$$2U - W_q - W_s = 0. \quad (14)$$

The above calculation shows that in static equilibrium the strain energy of the beam is equal to one-half of the work done by the surface tractions if they were to move (slowly) through their respective displacements. This is exactly in line with the *Clapeyron's theorem* (e.g. [9, 10]). We conclude that the work done by the stresses on the lateral interior end surfaces of the beam is an integral part of the Levinson beam theory.

3. Variational formulation of the Levinson beam theory

The variational formulation is carried out according to the coordinate system of Fig. 1. We re-write the kinematics of the Levinson beam in the form

$$U_x(x, y) = y\phi - \frac{4y^3}{3h^2} (\phi + u'_y), \quad (15)$$

$$U_y(x, y) = u_y, \quad (16)$$

where the comma denotes differentiation with respect to coordinate x . To carry out the variational formulation, we assume that the kinematic central axis variables $u_y(x)$ and $\phi(x)$ are sufficiently smooth but otherwise arbitrary functions. The axial normal and the transverse shear stresses calculated using the displacements (15) and (16) are

$$\sigma_x = E\epsilon_x = E \left[y\phi' - \frac{4y^3}{3h^2} (\phi' + u''_y) \right], \quad (17)$$

$$\tau_{xy} = G\gamma_{xy} = G \left(1 - \frac{4y^2}{h^2} \right) (\phi + u'_y), \quad (18)$$

respectively. The internal virtual work (virtual strain energy) of the beam is

$$\begin{aligned} \delta U &= \int_{-L/2}^{L/2} \int_A (\sigma_x \delta \epsilon_x + \tau_{xy} \delta \gamma_{xy}) dA dx \\ &= \int_{-L/2}^{L/2} \left[M \delta \phi' - \frac{4P}{3h^2} (\delta \phi' + \delta u''_y) + \left(Q - \frac{4R}{h^2} \right) (\delta \phi + \delta u'_y) \right] dx, \end{aligned} \quad (19)$$

where

$$M = \int_A \sigma_{xy} dA = \frac{EI}{5} (4\phi' - u_y'') , \quad (20)$$

$$Q = \int_A \tau_{xy} dA = \frac{2}{3} GA (\phi + u_y') \quad (21)$$

are the classical interior load resultants and

$$R = \int_A \tau_{xy} y^2 dA = \frac{GAh^2}{30} (\phi + u_y') , \quad (22)$$

$$P = \int_A \sigma_{xy} y^3 dA = \frac{EIh^2}{140} (16\phi' - 5u_y'') \quad (23)$$

are higher-order load resultants and their presence in the beam theory leads to exponential end effects. Exponential effects are known to appear in beam theories based on gradient elasticity [11]. However, in the present context of classical elasticity, it is very difficult to assign a physical meaning to the higher-order load resultants and the related exponential effects. In fact, in the end it turns out that the higher-order load resultants are not present in the final equilibrium equations of the Levinson beam theory. The external virtual work due to the uniform load is given by

$$\delta W_q = \int_{-L/2}^{L/2} q \delta u_y dx . \quad (24)$$

The external virtual work due to the interior stresses on the end surfaces is calculated using Eqs. (13) and (15)–(18) and yields

$$\delta W_s = \left[\left(M - \frac{4P}{3h^2} \right) \delta \phi - \frac{4P}{3h^2} \delta u_y' + Q \delta u_y \right]_{-L/2}^{L/2} . \quad (25)$$

By applying the principle of virtual displacements, $\delta U = \delta W_q + \delta W_s$, we obtain through integration by parts the equilibrium equations

$$Q - M' - \frac{4R}{h^2} + \frac{4P'}{3h^2} = 0 , \quad (26)$$

$$Q' - \frac{4R'}{h^2} + \frac{4P''}{3h^2} = -q . \quad (27)$$

By combining the boundary terms of Eq. (25) with those produced by integration by parts of Eq. (19) we obtain

$$\left[\left(\frac{4P'}{3h^2} - \frac{4R}{h^2} \right) \delta u_y \right]_{-L/2}^{L/2} = 0 . \quad (28)$$

Consequently, we require that the virtual displacement δu_y or the expression multiplying it must vanish at $x = \pm L/2$. It follows from the interior problem definition in Section 2.1 that the virtual displacement δu_y is free in the whole interior beam region, including the lateral interior end surfaces (cf. Eq. (13)). Therefore, the natural interior boundary conditions become

$$\left(\frac{4P'}{3h^2} - \frac{4R}{h^2} \right) (\pm L/2) = 0 , \quad (29)$$

By applying Eq. (26) to the boundary conditions (29), we can write an alternative form of the boundary conditions as $(\phi'' + u_y''')(\pm L/2) = 0$. By using Eqs. (22), (23), (26) and (27), we can write

$$f \equiv \frac{4P'}{3h^2} - \frac{4R}{h^2} = -\frac{EI}{105}(\phi'' + u_y''') , \quad (30)$$

$$f'' = -\frac{2}{3}GA(\phi'' + u_y''') . \quad (31)$$

It follows from Eqs. (30) and (31) that

$$f'' = \beta^2 f \quad \rightarrow \quad f = \alpha_1 e^{\beta x} + \alpha_2 e^{-\beta x} , \quad (32)$$

where $\beta^2 = 70GA/EI$. The boundary conditions (29), $f(\pm L/2) = 0$, give us $\alpha_1 = \alpha_2 = 0$. Thus, the equilibrium equations (26) and (27) simplify to

$$M' = Q \quad \text{and} \quad Q' = -q , \quad (33)$$

which are the same as the vectorially derived Levinson beam equations. Furthermore, we note that the general solution of the equilibrium equations (26) and (27) is

$$u_y = B_1 + B_2 x + B_3 x^2 + B_4 x^3 + B_5 e^{\beta x} + B_6 e^{-\beta x} + \frac{qx^2}{24EI} \left(x^2 - \frac{72}{5} \frac{EI}{GA} \right) , \quad (34)$$

$$\phi = -B_2 - 2B_3 x - 3B_4 x^2 - 9 \frac{EI}{GA} B_4 + \frac{1}{4} \beta \left(B_5 e^{\beta x} - B_6 e^{-\beta x} \right) - \frac{qx}{6EI} \left(x^2 + \frac{9}{5} \frac{EI}{GA} \right) . \quad (35)$$

It is easy to verify that the interior beam conditions (29) imply that $B_5 = B_6 = 0$ and the solution (34) and (35) becomes the same as the one given by Eqs. (8) and (9). Setting essential boundary conditions, that is, by specifying u_y on the basis of Eq. (28) at $x = \pm L/2$ instead of the natural boundary conditions (29), would create artificial end effects. For example, by choosing $u_y(\pm L/2) = 0$ for a uniformly-loaded simply-supported beam, with the other conditions chosen according to Reddy [8], leads to exponential end effects although the used kinematic description is valid only for interior behavior.

4. Exact Levinson beam finite element

The general polynomial solution (8) and (9) can be used as the basis for the derivation of an exact 1D Levinson beam finite element. Fig. 2 presents the setting according to which the element is developed. Both nodes in Fig. 2 have two degrees of freedom. For nodes $i = 1, 2$, we have transverse displacements $u_{y,i}$ and rotations of the cross-section ϕ_i . Using Eqs. (8) and (9), we obtain for nodes 1 and 2 the following four equations

$$\begin{aligned} u_{y,1} &= u_y(-L/2) , & u_{y,2} &= u_y(L/2) , \\ \phi_1 &= -\phi(-L/2) , & \phi_2 &= -\phi(L/2) . \end{aligned} \quad (36)$$

By solving the four integration constants c_1, c_2, c_3 and c_4 from Eqs. (36) we obtain

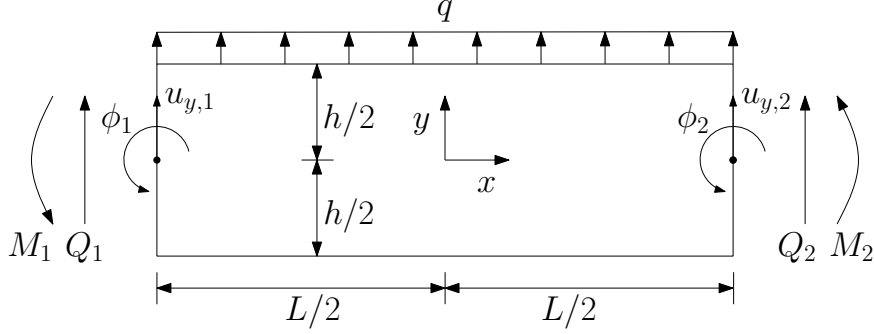


Figure 2: Set-up according to which the exact Levinson beam finite element is developed.

$$c_1 = \frac{1}{8} [4(u_{y,1} + u_{y,2}) + L(\phi_1 - \phi_2)] + \frac{qL^4}{384EI} (1 + 4\Phi) , \quad (37)$$

$$c_2 = \frac{\Phi}{L(1 + \Phi)} (u_{y,2} - u_{y,1}) - \frac{1}{4L(1 + \Phi)} [6(u_{y,1} - u_{y,2}) + L(\phi_1 + \phi_2)] ,$$

$$c_3 = \frac{\phi_2 - \phi_1}{2L} - \frac{qL^2}{240EI} (5 + 2\Phi) , \quad (38)$$

$$c_4 = -\frac{\Phi}{2L(1 + \Phi)} [2(u_{y,1} - u_{y,2}) + L(\phi_1 + \phi_2)] ,$$

where $\Phi = 3h^2(1 + \nu)/L^2$ is exactly the same as the similar factor present in a Timoshenko beam element with the shear coefficient $\kappa = 2/3$ (cf. [12]). By substituting Eqs. (37) and (38) into Eqs. (8) and (9) to calculate the load resultants at nodes $i = 1, 2$ using Eqs. (3), with the notion that the positive directions are taken to be according to Fig. 1 so that

$$Q_1 = -Q(-L/2) , \quad Q_2 = Q(L/2) , \quad M_1 = M(-L/2) , \quad M_2 = -M(L/2) , \quad (39)$$

we get

$$\mathbf{K}\mathbf{u} = \mathbf{f} + \mathbf{q} , \quad (40)$$

where

$$\mathbf{K} = \frac{EI}{(1 + \Phi)L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4 + \Phi)L^2 & -6L & (2 - \Phi)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2 - \Phi)L^2 & -6L & (4 + \Phi)L^2 \end{bmatrix} \quad (41)$$

$$\mathbf{u} = \{u_{y,1} \quad \phi_1 \quad u_{y,2} \quad \phi_2\}^T \quad (42)$$

$$\mathbf{f} = \{Q_1 \quad M_1 \quad Q_2 \quad M_2\}^T \quad (43)$$

$$\mathbf{q} = \frac{q}{2} \left\{ L \quad -\frac{L^2}{30}(\Phi - 5) \quad L \quad \frac{L^2}{30}(\Phi - 5) \right\}^T . \quad (44)$$

The stiffness matrix (41) is the same as for a Timoshenko beam element ($\kappa = 2/3$) but the load vector (44) for the uniform load is different (cf. [12]). By substituting Eqs. (37) and (38) into Eqs. (8) and (9) we can write the displacements (1) and (2) in the form

$$U_x(x, y) = \mathbf{N}_x \mathbf{u} + qL_1 , \quad (45)$$

$$U_y(x, y) = \mathbf{N}_y \mathbf{u} + qL_2 , \quad (46)$$

where the shape functions are

$$\mathbf{N}_x^T = \left\{ \begin{array}{l} \frac{y[3L^2 - 12x^2 + 8y^2(1+\nu)]}{2L^3(1+\Phi)} \\ \frac{y[-2\Phi L(L-2x) + (L-2x)(L+6x) + 8y^2(1+\nu)]}{4L^2(1+\Phi)} \\ -\frac{y[3L^2 - 12x^2 + 8y^2(1+\nu)]}{2L^3(1+\Phi)} \\ \frac{y[-2\Phi L(L+2x) + (L+2x)(L-6x) + 8y^2(1+\nu)]}{4L^2(1+\Phi)} \end{array} \right\}, \quad (47)$$

$$\mathbf{N}_y^T = \left\{ \begin{array}{l} \frac{(L-2x)[(L-2x)(L+x) + \Phi L^2]}{2L^3(1+\Phi)} \\ \frac{(L^2 - 4x^2)[L(L-2x) + \Phi L^2]}{8L^3(1+\Phi)} \\ \frac{(L+2x)[(L+2x)(L-x) + \Phi L^2]}{2L^3(1+\Phi)} \\ -\frac{(L^2 - 4x^2)[L(L+2x) + \Phi L^2]}{8L^3(1+\Phi)} \end{array} \right\}. \quad (48)$$

In addition, we have

$$L_1 = \frac{xy}{24EI} [L^2 - 4x^2 + 8y^2(1+\nu)], \quad (49)$$

$$L_2 = \frac{(L^2 - 4x^2)}{384EI} [L^2(1+4\Phi) - 4x^2]. \quad (50)$$

Therefore, once the nodal displacements have been solved from Eq. (40), the 2D displacement field can be calculated by substituting them into Eqs. (45) and (46), after which the calculation of 2D strains and stresses is straightforward.

Finally, we note that the exact beam element can also be derived from the total potential energy

$$\Pi = U - W_q - W_s. \quad (51)$$

The stresses on the end surfaces in Eq. (13) are written as given by Eqs. (10) and (11), where the load resultants are expressed as nodal forces according Eqs. (39). Then, by using the general solution (8) and (9) in terms of Eqs. (37) and (38) to calculate (12) and (13) and by applying the principle of minimum total potential energy

$$\frac{\partial \Pi}{\partial u_{y,i}} = 0, \quad \frac{\partial \Pi}{\partial \phi_i} = 0 \quad (i = 1, 2) \quad (52)$$

we obtain the finite element equations (40).

5. Conclusions

The variational formulation for the static Levinson beam theory was given. It was shown that exponential solutions which decay with distance from the beam ends cannot exist in the theory. The exponential solutions vanish because the formulation includes the external virtual work done by the interior stresses on the lateral interior end surfaces of the beam. The validity of this scheme was confirmed by the Clapeyron's theorem. The novel virtual work contribution in the variational formulation is a consequence of the fact that the utilized assumed displacement field is based

exclusively on interior kinematics. The boundary layer is not modeled and, thus, the beam theory would be severely inconsistent if it produced any exponentially decaying boundary layer effects.

We also point out that the variationally derived equilibrium equations (26) and (27) are exactly the same as those of the Reddy–Bickford theory (see, e.g. [3]). However, the Reddy–Bickford theory does not include the external virtual work contribution due to the stresses on the lateral end surfaces of the beam. The problem with the Reddy–Bickford theory is essentially that the interior solution is extended to the actual physical boundary of a complete beam. For example, at a clamped end of a complete beam the work done by the stresses would vanish due to zero displacements. However, in the interior context this sort of clamping leads to artificial boundary layer effects. Although the sixth–order Reddy–Bickford and other similar theories may ultimately provide good results in engineering calculations regardless of their unphysical features, they are unnecessarily complicated in comparison to the fourth–order theory by Levinson, which has been validated here.

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