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Effects of sources on time-domain finite difference models

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Recent work on excitation mechanisms in acoustic finite difference models focuses primarily on physical interpretations of observed phenomena. This paper offers an alternative view by examining the properties of models from the perspectives of linear algebra and signal processing. Interpretation of a simulation as matrix exponentiation clarifies the separate roles of sources as boundaries and signals. Boundary conditions modify the matrix and thus its modal structure, and initial conditions or source signals shape the solution, but not the modal structure. Low-frequency artifacts are shown to follow from eigenvalues and eigenvectors of the matrix, and previously reported artifacts are predicted from eigenvalue estimates. The role of source signals is also briefly discussed.

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I. INTRODUCTION

Finite difference methods have gained traction in audio and virtual room acoustics for their simplicity, efficiency, and flexibility. Two recent papers^{1,2} have presented results regarding source type, signal specification, and their collective effect on a simulated room using physical arguments and rationale. This paper discusses mathematical properties of time-stepping finite difference models to further illustrate the potential effects of excitation mechanisms.

A finite difference model is a linear system that may be excited in a number of ways. Although it is possible to apply initial conditions,³ it is more common in room acoustics to use time-varying monopole sources. Using the terminology of Schneider *et al.*,⁴ transient sources may be characterized as hard, soft, or transparent. Although monopole sources are not geometrically similar to most physical transducers, there has been a push in recent literature to make resulting solutions from idealized monopoles more physically plausible.^{1,2}

One article¹ reports empirical properties of an unintended low-frequency oscillation attributed to hard sources with a Gaussian source signal. The authors provide physical explanations of the artifact—reinforced in a more recent paper²—involving a non-zero DC (zero-frequency) velocity field introduced by the source signal. Both references also associate this effect with transparent source, but results below, and in both references, suggest that they are unrelated. To resolve the origin and nature of the artifact, the oscillation associated with a hard source will be predicted and shown to be a linear eigensolution of the numerical system.

Another article² argues for physically motivated source signals and a parameterization of implementation. Implementation of the different source types is parameterized through coefficients in the update equations, and source signals are specified by a series of filters whose properties are motivated by various physical and numerical considerations. This paper focuses primarily on the effect source

types, as opposed to implementation and signals, on possible solutions. Source signals are discussed briefly, but shown to have only the capacity to relatively weight eigensolutions of the system that are defined by geometry, scheme, and boundary conditions.

Emphasizing the perspectives of linear algebra and signal processing, this paper presents further properties of excitation mechanisms in time-domain finite difference models. First, the properties of each source type are reviewed, highlighting specific attributes which help sources to satisfy desirable constraints. The hard source artifact is explained and predicted through eigenvalue decomposition of characteristic models and, finally, the role of source signals is briefly discussed. The underlying idea, which encapsulates the effect of sources on a model, is that source type and source signals act independently on the solution.

II. FINITE DIFFERENCE SIMULATION

The classical finite difference approximation to the linear second-order wave equation uses centered second-difference approximations in space and time.⁵ Many interpolated^{6–8} and higher-order^{8,9} variants offering greater accuracy and isotropy have been developed since, but the relevant properties of the approximation are captured by the simplest scheme. The staggered two-variable formulation for pressure and velocity^{1,10} is equivalent at pressure nodes and requires twice the memory,^{11,12} so it will not be directly considered.

The linear acoustic wave equation is given by

$$\frac{1}{c^2} \frac{\partial^2 p(\mathbf{r}, t)}{\partial t^2} = \nabla^2 p(\mathbf{r}, t), \quad (1)$$

where p is acoustic pressure, and c is the speed of propagation. The pressure is a scalar field in the dimension of the problem, and $\mathbf{r}$ is a vector of spatial coordinates; for example, in three-dimensional (3D) Cartesian coordinates, $\mathbf{r} \equiv (x, y, z)$. The discrete approximation to Eq. (1) is given by the standard second-order accurate scheme

$$\delta_t^2 p_{j,k,l}^n = \lambda^2 (\delta_x^2 + \delta_y^2 + \delta_z^2) p_{j,k,l}^n, \quad (2)$$

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where the superscript n is a temporal index, subscripts indicate spatial indices, $\lambda = cT/X$ is the Courant stability factor or grid ratio, T is the temporal discretization period, and X is the spatial discretization period. Difference operators, δ , have superscripts indicating the order of the difference and subscripts indicating the variable and direction of the difference. A second difference in the spatial x -direction is given by $\delta_x^2 u_{j,k,l} = \delta_{x+} \delta_{x-} u_{j,k,l} = u_{j+1,k,l} - 2u_{j,k,l} + u_{j-1,k,l}$. Further information regarding the calculus of difference operators may be found in many texts on difference methods,^{9,13,14} as well as recent literature.^{7,8,15}

Boundaries are terminated with resistive boundary conditions, which provide absorption while keeping the spectral analysis in Sec. III B feasible. With the surface oriented such that the normal vector is $(-1, 0, 0)$, the boundary condition—using centered difference approximations and a ghost point—is given by

$$p_{j,k,l}^{n+1}(1 + \lambda\beta) = 2(1 - 3\lambda^2)p_{j,k,l}^n + (\lambda\beta - 1)p_{j,k,l}^{n-1} + \lambda^2(2p_{j-1,k,l} + p_{j,k+1,l} + p_{j,k-1,l} + p_{j,k,l+1} + p_{j,k,l-1}), \quad (3)$$

where the specific admittance is given by $\beta = \rho_0 c/Z$, and Z is the acoustic impedance of the boundary surface. Equation (3) is straightforward to derive using methods described in texts,¹⁴ but it may also be found as a special case in Ref. 16, and is equivalent to the boundary condition in Ref. 1 when it is formulated at pressure nodes.

A. Source types

Finite difference grids may be excited either by specifying initial conditions³ or transient boundary conditions. Transient monopole sources tend to be used in room acoustic simulation for computing broadband impulse responses.^{1,4,7,10,17} To maintain broadband frequency content and minimize excess computation, sources also tend to be compact in time. Alternatively, if the purpose is to auralize a signal in a model, the model could be directly excited with the signal to be auralized. However, such broadband simulations are currently not feasible in real-time even with arrays of graphics processing units (GPU),¹⁸ so we will only consider the problem of computing impulse responses or band-limited impulse responses to be convolved with other signals.

Equations (1) and (2) do not include source terms because the underlying equations depend on the type of source to be implemented. As it is typically written, the inhomogeneous wave equation is given by

$$\frac{1}{c^2} \frac{\partial^2 p(\mathbf{r}, t)}{\partial t^2} = \nabla^2 p(\mathbf{r}, t) + s(\mathbf{r}, t), \quad (4)$$

where s is a source term. Written compactly, in similar form to Refs. 2 and 4, the explicit update that follows from discretization of Eq. (4) is

$$p_{j,k,l}^{n+1} = \{\text{update}\} + c^2 T^2 \hat{s}_{j,k,l}^n \equiv \{\text{update}\} + \frac{c^2 T^2}{V} s_{j,k,l}^n, \quad (5)$$

where $\{\text{update}\}$ is all of the terms in Eq. (2) except for $p_{j,k,l}^{n+1}$, and V is the volume of a grid cell. The constant, $c^2 T^2$, comes from the time differencing on the left side of the equation, and $\hat{s}$ is a scaled version of the source signal to be discussed shortly.

This description specifies an additive source term and follows from the physics of the problem.² However, in some cases the source term is, instead, imposed on the source node independent of the update equations. It was recently pointed out that adding a coefficient which takes values zero and one to the $\{\text{update}\}$ terms allows for parameterization of both source types.² If the update equations are included as in Eq. (5), the source is known as a *soft source*, and if they are ignored or overwritten by the source signal, the source is known as a *hard source*. A *transparent source* is a special case of a soft source wherein the source signal is processed such that the radiated field mimics that of a hard source.⁴ The modified signal depends on discretization parameters, but not on the geometry of the problem and may be reused in multiple models. However, there remain only two source types—hard and soft—since the transparent source is a special case of a soft source.

1. Scaling

The reason for scaling the source signal may be motivated by either physics² or a finite volume interpretation of the scheme.¹⁵ All single-node source implementations amount to modifying a volume of fluid the size of one grid cell—for the standard scheme in Cartesian coordinates, cells are cubic and the volume is X^3 . Since a soft source does not impose a boundary condition, the effective source may be reduced to a simple monopole at the center of the source cell. However, the source signal that is added at the source node corresponds to a cell average, so to achieve consistent amplitudes—or to introduce the same amount of the scalar field—with varying discretization, the intended source signal must be scaled by $1/V = 1/X^3$. This leads to the definition $\hat{s} \equiv s/V$ in Eq. (5). Note that the volume of a grid cell may change with scheme,^{15,19} and similar explanations may be found in Refs. 2 and 20.

If the source is implemented as in Eq. (5), the numerical solution will converge at a rate proportional to X^2 or T^2 to an exact solution given by

$$p(\mathbf{r}, t) = \frac{1}{4\pi r} s(t - r/c). \quad (6)$$

This solution is a simplified form of that in Ref. 2 corresponding to a simplified form of Eq. (5).

III. EFFECT OF HARD SOURCES

The most notable effect of hard sources is an oscillatory artifact reported in Refs. 1 and 2, or mode shift reported in Ref. 21. A similar, but unrelated, DC solution component is linear DC drift in models with rigid boundaries. Any linear function of time and space is a solution of the wave equation. Homogeneous Neumann boundary conditions constrain this to functions that are linear in time and constant in space, and

Dirichlet conditions constrain this to constant solutions in both time and space. With absorption at boundaries, there is still DC drift, but it is not linear. This behavior has been conflated with the oscillatory hard source artifact, but DC drift is actually prevented by hard sources when using the standard scheme. Section III A will describe the oscillatory artifact with a physical analogy, and Sec. III B will predict the behavior of the system through a matrix analysis.

A. Mechanical analogy

The wave equation in its semi-discrete form, $d^2p(\mathbf{r}, t)/dt^2 = c^2/X^2(\delta_x^2 + \delta_y^2 + \delta_z^2)p(\mathbf{r}, t)$, governs a regular lattice of masses connected by linear springs or elastic strings. Note that similar analogies appear, for example, in Ref. 22 as lumped mass-spring networks. Figure 1 illustrates the system as it would appear in one-dimensional (1D) and two-dimensional (2D) finite difference wave propagation problems. The components of the system are masses, dashpots, and springs, which produce linear forces proportional to acceleration, velocity, and displacement, respectively. All displacements are assumed to be small to allow the linear, small-angle approximation.

Figure 1(a) shows a portion of the analogous 2D lattice of oscillators implied by the 2D update equation; extension to a 3D lattice is straightforward, but more difficult to visualize. The 2D case is included to facilitate extrapolation to 3D, but the essence of the DC effects is captured in the 1D cartoons. Figures 1(b)–1(d) illustrate the scenarios of interest in one dimension. The relative displacement of masses is analogous to acoustic pressure, so there is no implied gravity or other external forces on the system.

Figure 1(b) represents the intended condition: masses (pressure nodes) are elastically connected to each other, oscillation occurs about equilibrium, and oscillation is damped at the boundaries. Figure 1(c) illustrates the response of the system to a zero-frequency excitation, and Fig. 1(d) shows the case where one (source) node is fixed with resistance boundary conditions and DC excitation. Once the DC force is released, the system in Fig. 1(d) oscillates at its lowest natural frequency. Displacing all nodes, except a single fixed node, provides an initial condition that strongly excites the

FIG. 1. Physical interpretation of the semi-discrete approximation to the wave equation with resistance boundaries. (a) 2D lattice of oscillators. (b) 1D lattice with high-frequency excitation. (c) 1D lattice with only DC excitation. (d) 1D lattice with DC excitation and a fixed interior node.

lowest natural mode of the system; in fact, Fig. 1(d) quite closely resembles the eigenvector associated with this mode. With no fixed nodes, it is the zero-frequency mode in Fig. 1(c) that is excited, so no oscillation is observed.

Simulations are not generally excited with a strictly DC source, but a Gaussian pulse, for example, has most of its energy concentrated toward zero frequency. The response at each node is then dominated by the DC solution component. To extrapolate the analogy to 3D finite difference simulation, we compare the responses of hard, soft, and transparent sources with strong low-frequency content on a cubic $30 \times 30 \times 30$ grid. The measured acoustic response is analogous to tracking the displacement of one of the masses in Fig. 1. One may also reference Fig. 12 of Ref. 1 and Fig. 6 of Ref. 2 for a similar, but truncated, comparison. The source is placed at node (15, 15, 15) and the receiver at node (10, 8, 6), with resistance boundaries corresponding to normal incidence pressure reflection coefficient 0.95. Recorded responses are shown in Fig. 2.

The soft and transparent source responses both exhibit non-oscillatory drift and differ primarily in magnitude for this smooth, band-limited excitation signal. These are, again, analogous to the system in Fig. 1(c), where a DC component in the solution causes the mean of the solution to shift from equilibrium. With a hard source, the fixed source node prevents DC drift and introduces a new, low-frequency mode. With the low-frequency mode available, large-scale oscillation in Fig. 2 is emphasized by the low-frequency content of the source signal.

It should also be noted that this type of oscillation is not caused exclusively by hard sources. If nodes are fixed for any other reason, similar oscillation may occur. For instance, if edge or corner nodes are neglected, they are implicitly fixed, resulting in an unintended Dirichlet boundary. The mechanical analogy provides an intuitive description, but the precise explanation is that a fixed node in the mesh shifts the numerical modes of the system. Section III B analyzes the finite difference system directly in order to predict frequencies of oscillation.

B. Matrix analysis

The oscillation can be predicted through a spectral analysis of the scheme and boundary conditions, which define a

FIG. 2. Response on a $30 \times 30 \times 30$ grid to soft, hard, and transparent sources with strong low-frequency content. Resistance boundary conditions, corresponding to normal incidence pressure reflection coefficient 0.95, are applied on the geometrical boundaries.

global linear operator. Equations (2) and (5) are linear algebraic equations defining how each node is locally updated using nearest neighbor field values, so all updates may be collected into a linear system or matrix operator.^{13,22–24} Consider a vector, $\mathbf{p}^n$, of field values everywhere on the grid at time step n . The update in Eq. (2) requires pressure at two time levels, so let $(\mathbf{q}^n)^T = [(\mathbf{p}^n)^T (\mathbf{p}^{n-1})^T]$, where T indicates transposition. Then, with a state vector, $\mathbf{q}$, the set of local updates for arbitrary simulation parameters define a matrix operator, A , such that

$$\mathbf{q}^{n+1} = A\mathbf{q}^n = A^{n+1}\mathbf{q}^0, \quad (7)$$

for initial conditions $\mathbf{q}^0$. The solution is entirely defined by powers of the matrix and initial conditions, and the long-term behavior of the solution can be understood by examining its eigenvalues and eigenvectors. Further information on matrix analyses, spectra of operators, and pseudospectra may be found in the literature.^{13,22–24}

The eigenvalues for two example operators are shown in Fig. 3. The example rooms are $U \times V \times W = 10 \times 10 \times 10$ nodes, and scaling to physical dimensions can be chosen arbitrarily. Resistance boundary conditions are defined to have normal pressure reflection coefficient 0.99, as in Ref. 1, but the time step is set at the maximum allowed by the Courant limit for presentation. The normalized analytical

modal frequencies for a rectangular room with rigid boundaries are indicated by radial lines in the figure and given by

$$\omega T = \pi \lambda \sqrt{\left(\frac{u}{U-1}\right)^2 + \left(\frac{v}{V-1}\right)^2 + \left(\frac{w}{W-1}\right)^2}, \quad (8)$$

for non-negative integer indices u, v, w . The modal frequencies depend on λ , but are independent of other discretization parameters. The eigenvalues in the complex plane may be written in polar form and assigned a frequency corresponding with their angle. This is related to the z -domain frequency variable, $z = \exp(i\omega T) = \exp(i2\pi fT)$, where f is frequency in Hz.

Figure 3(a) shows the spectrum with no fixed interior nodes, corresponding to a soft source or transparent source, and most eigenvalues have algebraic multiplicity 3 due to symmetry of the room geometry. Figure 3(b) shows the spectrum with a single fixed node in the center of the room, corresponding to a hard source type, and the newly introduced eigenvalues are emphasized by arrows.

With a fixed interior node, the DC eigenvalues are forced to the origin. The constant eigenfunctions associated with the DC eigenvalues cannot have non-zero mean if one node is fixed at zero. However, there is an additional mode introduced above DC and below any of the natural room modes, which is precisely the artifact reported in Ref. 1. This analysis shows that the artifact is an eigensolution when an isolated *boundary condition* (hard source) is introduced on the interior. Soft and transparent sources do not introduce boundaries and only affect the initial conditions, $\mathbf{q}^0$, so they cannot *cause* the oscillatory artifact reported in Refs. 1 and 2. With soft and transparent sources, the solution may drift, initially resembling the hard source solution (see Fig. 2, Fig. 12 in Ref. 1, or Fig. 6 in Ref. 2), but it will not oscillate because the hard source mode is not available for excitation.

C. Prediction of previously reported artifacts

The entire eigenvalue spectrum is too expensive to compute for problems the size of those in Ref. 1. Matrices for the middle-sized problems already occupy 2×10^6 dimensions for the scalar formulation in Eq. (2), and 4×10^6 dimensions for the staggered, two-variable, vector formulation used in Ref. 1. Matrix sizes for the larger problems increase by another order of magnitude. Nevertheless, the eigenvalue of interest occurs in a predictable location, allowing an iterative solver to converge quickly once the matrix is constructed.

Using an iterative solver to compute the single eigenvalue of interest, we compare to the smallest examples in Figs. 5 and 7 of Ref. 1. The domain is $50 \times 50 \times 50$ nodes, and the spatial discretization periods are $X = 0.02$ m and $X = 0.04$ m, respectively. The Courant factor for each is set to $\lambda = 0.5941/\sqrt{3}$ and $\lambda = 0.2970/\sqrt{3}$, respectively, as in the original article.¹ Using these settings, the lowest eigenfrequencies are estimated to be 15.9536 Hz and 7.9768 Hz. Visual inspection of the figures,¹ and the reported estimate of 16 Hz for the first case, appear to coincide to the degree that comparison is possible.

FIG. 3. Eigenvalues of finite difference operators with resistance boundaries. Radial lines indicate normalized analytical modal frequencies, and circles are computed eigenvalues. (a) Operator with resistance boundary conditions for a $10 \times 10 \times 10$ node cubic room. (b) Same operator as (a), but center node is fixed. The dashed radial line is the frequency of the oscillatory mode introduced by the fixed node.

Changing the source position changes the natural frequencies of the system, but the reported variation is relatively small.¹ The previous experiment with varying source position¹ is currently too large for spectral analysis, so the experiment is repeated on a $30 \times 30 \times 30$ domain instead of a $100 \times 100 \times 100$ domain, with source and receiver positions scaled accordingly. The mean frequency of oscillation across the scaled source positions is 33.6189 Hz with standard deviation 2.0825, or 6.2% of the mean. The resolution of the Fourier magnitude spectrum in Fig. 4 of Ref. 1 is insufficient for detailed comparison, but 6.2% of 5 Hz and 6 Hz are 0.31 Hz and 0.37 Hz, respectively. Since the peak appears to be split between frequency points at 5 Hz and 6 Hz, this amount of variation, at least, does not contradict the original results.¹

It is also worth noting that a hard source and its associated oscillation can have some benefits. With a soft source, the eigenvalues at ± 1 [see Fig. 3(a)] can, in some cases, lead to late-time instability, particularly when using single-precision arithmetic. The fixed source node forces the DC and Nyquist eigenvalues to the origin, leaving only non-defective, nearly DC and nearly Nyquist eigenvalues, which are better conditioned. However, this behavior does not, in general, hold for interpolated schemes or schemes on non-rectangular grids.

IV. EFFECT OF SOURCE SIGNALS

The role of source signals is to define how prominently the modes of the system appear in the solution. Design of signals could be motivated by a number of criteria including physics,² an undesirable mode,¹ or, more commonly, by the dispersive properties of the scheme. This section briefly describes effects of source signals not already addressed in recent literature.^{1,2,21}

The natural excitation signal for a discrete-time linear system is the unit impulse, since the response to all other distributed signals may be constructed through convolution. However, the measured response on the grid to an impulse is, in general, quite dispersed, leading many to introduce band-limited pulses, such as a Gaussian,¹ its derivatives,^{4,10,17} or a raised cosine.¹⁵ While a smooth source signal causes the solution to *appear* more like linear wave propagation—a desirable property for visualization—any practical signal other than an impulse can only remove information from the response. The impulse may be recovered by deconvolving the input signal, but some information will be lost to round-off error. For this reason, to extract the most possible information from a model, the delta source should be used. However, not all of this information is useful and, as with any source signal, it is difficult to argue that one signal should be generally preferred to others.

The discussion in Refs. 1 and 2 regarding excitation of DC and low-frequency components may also be clarified. Source signals do not *cause* drift or oscillation; drift of the DC component and the hard source oscillation are intrinsic to the system independent of source signals. Any field initialized to a state where the mean of the solution at adjacent time levels is not equal will exhibit drift. Drift is not an

eigensolution (scaling), but translation, excited when DC components at adjacent time levels are not equal. Setting means of each half of the state vector, $\mathbf{q}$ in Eq. (7), to be equal eliminates drift, as does introducing a zero-mean source signal. The differentiation constraint in Ref. 2 is another way of stating that the source signal should have zero mean, but the existence of drift follows from the wave equation and the matrix update independent of specific initial conditions or forcing terms.

The interpretation of sources as continually active because of their pulse shape^{1,2} should not apply also to excitation of the hard source oscillation. Consider a continually active soft source—a signal with non-zero mean—in a model with rigid boundaries where some other interior node is fixed. By the matrix analysis, this introduces an oscillatory (hard source) mode, but retains the continually active source. If the source were continually exciting this mode, presumably, this component would continue to grow. However, the solution does not grow, which means that either the interpretation of the source as continually active does not apply or the continually active source does not excite the oscillatory mode as was previously supposed.^{1,2} This example supports the argument that source signals are not the cause of the reported artifacts.

Source signals can affect the appearance of the underlying solution, but the linear nature of the system makes the importance of a particular signal secondary to the model's structure. In other words, when specifying sources, the source type has far greater impact on the solution than source signal. This may be the reason for the variety of signals used in the literature.

V. CONCLUDING REMARKS

Multiple recent papers have addressed issues surrounding source implementation in time-domain finite difference models.^{1,2,21} This paper seeks to further clarify results regarding the effects of the hard source type. Section III B shows that oscillation due to a hard source is a linear eigensolution resulting from the fixed source node. Section III A provides a physical interpretation for why the mode is introduced, and Sec. III predicts the oscillation reported for smaller models in Ref. 1. The transparent source is shown to be unassociated with the hard source oscillation, and Sec. IV makes the argument that soft sources, which have been interpreted as continually active, cannot excite the oscillatory hard source mode. These results address only cuboid rooms with resistance boundaries, but the effect of fixing source nodes and source signals should qualitatively transfer to more general schemes and geometries as well.

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