This is an electronic reprint of the original article.
This reprint may differ from the original in pagination and typographic detail.

Boufous, Omar; Roncoli, Claudio; Charalambous, Themistoklis
Centralized and Distributed Multi-Region Traffic Flow Control

Published in:
European Control Conference 2020, ECC 2020

Published: 01/05/2020

Document Version
Peer reviewed version

Please cite the original version:
Boufous, O., Roncoli, C., & Charalambous, T. (2020). Centralized and Distributed Multi-Region Traffic Flow Control. In *European Control Conference 2020, ECC 2020* (pp. 420-427). [9143671] IEEE.

This material is protected by copyright and other intellectual property rights, and duplication or sale of all or part of any of the repository collections is not permitted, except that material may be duplicated by you for your research use or educational purposes in electronic or print form. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone who is not an authorised user.

Centralized and Distributed Multi-Region Traffic Flow Control

Omar Boufous, Claudio Roncoli, and Themistoklis Charalambous

Abstract—In this paper, centralized and distributed multi-region perimeter flow control approaches are proposed for congestion avoidance in urban networks. First, multi-region network dynamics are modeled with Macroscopic Fundamental Diagrams (MFDs) and necessary stability conditions are derived using Lyapunov stability theory for a centralized perimeter controller. Later, an optimization problem is formulated, solved and the desired optimal states are reached by means of an algorithm based on Model Predictive Control (MPC). Finally, the paper combines the centralized controller for perimeter control as a first layer controller and a distributed controller managing the inter-transfers between regions, thus optimizing the overall state of the network. Simulations show that the distributed control scheme leads to good results maximizing the output of the traffic network, similar to the MPC controller.

Index Terms—Multi-region control, traffic control, macroscopic fundamental diagram, MPC, distributed control.

I. INTRODUCTION

Traffic congestion is a common problem faced by major cities, which is deteriorating due to urbanization and population growth, causing economic and social losses. For alleviating this problem, besides increasing the capacity of the network through new infrastructure (such as bridges, smart roundabouts and a new design of the network), the implementation of traffic management strategies represents a more efficient and sustainable alternative.

Herein, we focus on traffic control via perimeter flow control, which consists of controlling the traffic flow at the border of a protected region in order to reduce the traveling delay. The lack of a straightforward solution that properly addresses this issue, has led to advanced research in this area; see, for example, [1], [2] and references therein. Empirical studies revealed that the dynamics of homogeneous urban networks can be modeled using the concept of MFDs [3], [4], which gives the relationship between vehicle flow and density in those regions. Recent work has shed lights on traffic control strategies, including perimeter control (also called gate control), for controlling the demand or load in the region under consideration.

Urban traffic networks, however, are rarely homogeneous and as a result they are modeled as a set of connected homogeneous regions with (possibly) different characteristics (size, capacity, average trip length) [5]–[7]. In order to

maintain each region at an equilibrium point, previous work tried to develop strategies to control (i) the incoming traffic from outside of the network, and (ii) the transfer flows between regions (inter-transfers). Very recently, [8] proposed a demand management strategy restricting the amount of vehicles allowed to enter the network, combined with route guidance using MPC. In the literature of traffic control, non-linear control algorithms based on MPC are widely used (see, for example, [9]) and applied to appropriate cost functions optimizing the states of the network [10] [11]. Although they can be very efficient when applied to multi-region networks [12], [13], MPC algorithms may involve heavy computations leading to high computational complexity and require real-time state variables and prediction of demand which makes them difficult to deploy for complex large-scale urban networks. Additionally, the difficulty MPC shows when dealing with model parameters uncertainties remains problematic. On the other hand, more traditional control methods, such as PIDs, are inexpensive and able to handle uncertainties in the MFD with a robust design but are, nevertheless, used only for simple network topologies (e.g., two regions) [14]–[17].

This paper considers traffic networks subdivided into an arbitrary number of homogeneous regions, all of which communicate with outside the network and mainly focus on a strategy for perimeter control, along with a distributed scheme for monitoring the inter-transferring flows between regions inside the network, in order to distribute the traffic efficiently. The contributions of this paper are as follows:

- 1) First, we define the necessary conditions for which perimeter control is feasible.
- 2) Then, we formulate and solve an optimization problem aiming at controlling both the perimeter and the routing inside the network.
- 3) Finally, we explore different distributed policies in which each region aims, based on the information available locally, at operating at the critical network density by efficiently controlling the inflows from the outside region and by reallocating flows inside the network.

The rest of the paper is organized as follows; in Section II, we provide the notation used throughout the paper and describe the traffic network and road traffic models. Section III defines the type of network studied in this paper and the assumptions made. A centralized approach is presented in Section IV and in Section V we propose a distributed control strategy. Numerical results evaluating the performance of the proposed controllers are given in Section VI. Finally, in Section VII, we conclude this paper and draw directions for future work.

O. Boufous is with the Department of Electrical Engineering and Automation, School of Electrical Engineering, Aalto University, Espoo, Finland. E-mails: omar.boufous@aalto.fi.

T. Charalambous is with the Department of Electrical Engineering and Automation, School of Electrical Engineering, Aalto University, Espoo, Finland. E-mails: themistoklis.charalambous@aalto.fi.

C. Roncoli is with the Department of Built Environment, School of Engineering, Aalto University, Espoo, Finland. E-mail: claudio.roncoli@aalto.fi.

II. NOTATION AND PRELIMINARIES

A. Notation

The set of real (integer) numbers is denoted by $\mathbb{R}$ ($\mathbb{Z}$) and the set of nonnegative numbers (integers) is denoted by $\mathbb{R}_+$ ($\mathbb{Z}_+$). $\mathbb{R}_+^n$ denotes the nonnegative orthant of the n -dimensional real space $\mathbb{R}^n$. Vectors are denoted by small letters whereas matrices are denoted by capital letters. The transpose of a matrix A is denoted by A^T . For $A \in \mathbb{R}^{n \times n}$, A_{ij} denotes the entry in row i and column j . The i^{th} component of a vector x is denoted by x_i , and the notation $x \geq y$ implies that $x_i \geq y_i$ for all components i . By $\mathbf{1}$ we denote the all-ones vector and by I we denote the identity matrix (of appropriate dimensions). A matrix whose elements are nonnegative, called a nonnegative matrix, is denoted by $A \geq 0$, and a matrix whose elements are positive, called positive matrix, is denoted by $A > 0$. We use $\text{diag}(x_i)$ to denote the matrix with elements $x_1, x_2, \dots$ on the leading diagonal and zeros elsewhere. The cardinality or number of elements of a finite set $\mathcal{S}$ is denoted $\text{card}(\mathcal{S})$.

B. Multi-Region Traffic Network Model

The multi-region traffic network with exchange of flows (edges) between regions (nodes) can be conveniently captured by a directed graph (digraph) $\mathcal{G}(\mathcal{N}, \mathcal{E})$ of order n ($n \geq 2$), where $\mathcal{N} = \{1, 2, \dots, n\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of edges. A directed edge from node i to node j is denoted by $\varepsilon_{ij} = (i, j) \in \mathcal{E}$ and represents a link that allows region j to receive vehicles from region i . A graph is said to be undirected if and only if $\varepsilon_{ji} \in \mathcal{E}$ implies $\varepsilon_{ij} \in \mathcal{E}$. In this paper, links are not required to be bidirectional all the time, *i.e.*, we deal with digraphs; for this reason, we use the terms “graph” and “digraph” interchangeably. All regions that can send vehicles to region i directly are said to be in-neighbors of region i and belong to the set $\mathcal{N}_i^- = \{j \in \mathcal{N} \mid \varepsilon_{ji} \in \mathcal{E}\}$. The regions that receive vehicles from region i belong to the set of out-neighbors of region i , denoted by $\mathcal{N}_i^+ = \{l \in \mathcal{N} \mid \varepsilon_{il} \in \mathcal{E}\}$.

C. Road Traffic Model

To model the traffic behavior in a region, we assume the existence of a *link* and *network* equilibrium speed and a fundamental diagram, *e.g.*, [18], [19]. Let us denote the traffic flow and the vehicle accumulation in region i by q_i and N_i , respectively. According to the model, drivers reduce their speed v_i with increasing number of vehicles in the link/network area, *i.e.*, $v_i(N_i)$, where $\partial v_i / \partial N_i \leq 0$, for $N_i \in [0, N_{\max}]$, and $v_i(0) = v_i^{\max}$, is the free speed of a vehicle on an empty network. When the number of vehicles is at its maximum, (*i.e.*, $N_i = N_{\max}$), we have $v_i(N_{\max}) = 0$, which implies that the velocity is zero and the road is fully congested. Under uniformity conditions, the product of vehicle accumulation and speed is the traffic flow, $q_i(N_i) = N_i v_i(N_i)$, which can be represented in the so-called *macroscopic (or network) fundamental diagram*. We assume the existence of a piecewise linear, triangular fundamental diagram, as depicted in Fig. 1. According to this diagram, traffic can be classified as either *free* (if $0 \leq$

$N < N^*$) or *congested* (if $N \geq N^*$), where N^* is the critical accumulation.

Fig. 1: The approximated network fundamental diagram.

III. PROBLEM FORMULATION

The problem of how vehicles traverse the regions can be abstracted, in its general case, as a directed graph, in which the (net) flow from one region (node) to the other is represented by an edge. In general, for a region i , the inflow and outflow can be represented as in the graph in Fig. 2.

Fig. 2: Graph representation of the traffic flow to / from region i .

Note that, while neighboring regions can in general be assumed to have bi-directional links, in this case the out-neighbors of a region i are the ones that accept vehicles from region i , while the in-neighbors are the ones that send vehicles to region i . It could be the case that the connection from one region to another is a one-way road, a case which is also captured by the directed graph abstraction. In principle, it is preferred that regions operate around the optimal vehicle concentration N_i^* (see Fig. 1), so that the maximum outflow is achieved. In what follows, we cast the problem as one that regions try to balance the traffic flow in such a way that *all* regions are operating at or close to their optimal vehicle concentration, if possible. We assume that all regions of the network communicate with an outer region referred to as region zero.

The evolution of the state N_i is given by:

$$N_i[k+1] = N_i[k] - \underbrace{\sum_{l \in \mathcal{N}_i^+} q_{il}[k]}_{\text{outflow}} + \underbrace{\sum_{j \in \mathcal{N}_i^-} q_{ji}[k]}_{\text{inflow}}, \quad (1)$$

Fig. 3: Boundary control and traffic flow to/from region i .

where the variables $q_{il}[k]$ denote the traffic flow from region i to region l between k and $k + 1$, $\sum_{l \in \mathcal{N}_i^+} q_{il}[k] = q_i[k]$ is the total outflow from region i and $\sum_{j \in \mathcal{N}_i^-} q_{ji}[k]$ is the total inflow to region i . Let $x_i[k] \triangleq N_i[k] - N_i^*$. Since $0 \leq N_i[k] \leq N_{\max}$, then $-N_i^* \leq x_i[k] \leq N_{i,\max} - N_i^*$. Therefore, Eq. (1) with respect to (w.r.t.) x_i becomes

$$x_i[k+1] = x_i[k] - \sum_{l \in \mathcal{N}_i^+} q_{il}[k] + \sum_{j \in \mathcal{N}_i^-} q_{ji}[k]. \quad (2)$$

In matrix form,

$$\mathbf{x}[k+1] = \mathbf{x}[k] - \mathbf{g}[k] + \mathbf{h}[k], \quad (3)$$

where $\mathbf{x}[k] \triangleq [x_1[k] \ \dots \ x_n[k]]^T$,

$$g_i[k] \triangleq \sum_{l \in \mathcal{N}_i^+} q_{il}[k],$$

$$h_i[k] \triangleq \sum_{j \in \mathcal{N}_i^-} q_{ji}[k].$$

The inflow $h_i[k]$ can be rewritten as

$$h_i[k] = \underbrace{\sum_{j \in \mathcal{N}_i^- \setminus \{0\}} q_{ji}[k]}_{\triangleq f_i[k] \text{ from neighboring regions}} + \underbrace{q_{0i}[k]}_{\triangleq u_i[k] \text{ from outer region}}. \quad (4)$$

Therefore, Eq. (3) can be written as

$$\mathbf{x}[k+1] = \mathbf{x}[k] - \mathbf{g}[k] + \mathbf{f}[k] + \mathbf{u}[k]. \quad (5)$$

Assumption 1: Region i cannot control the number of vehicles entering from any of the neighboring regions, apart from the outer region.

In other words, Assumption 1 says that any region i can only control $q_{0i}[k] \triangleq u_i[k]$, i.e., part of its inflow; see Fig. 3. This is a realistic assumption if one considers that there are no “buffers” in between the regions such that vehicles can reside after exiting one region and wait to get to the next.

Assumption 2: The exact values of the state variables x_i and the quantities q_{ij} can be obtained without noise.

In fact, real time estimation is a challenge that has been considered in other papers. For example, an estimation method is developed in [7], [20] to predict the aggregated traffic variables based on Kalman filtering.

The main objective of this work is to let each region i decide how much inflow it can take from the outer region, $u_i[k]$, so that the overall system serves the maximum number of vehicles while keeping the delays in the traffic network low. This is achieved when the regions operate around the critical number of vehicles, i.e., $N_i[k] \approx N_i^*$ ($x_i[k] \approx 0$). The problem definition in this work differs from other papers, such as, [8], [21], in which they also take into consideration the origin and destination of vehicles. In this paper, the total aggregated demands and accumulation of vehicles are considered, information which is lower-level and easier to obtain and to work with in the macroscopic model.

IV. CENTRALIZED APPROACH

A. Necessary conditions for perimeter control

We first consider n regions that can exchange vehicles with their neighbors and with the region outside of the controlled network, but cannot control the vehicle flows. Suppose that the system no longer evolves and is in a state of equilibrium when the state variable $\mathbf{x}$ is on average constant; this implies:

$$\exists k_0, \forall k > k_0, \mathbb{E}(\mathbf{x}[k+1] - \mathbf{x}[k]) = 0 \quad (6)$$

i.e., for each region, one can write:

$$\mathbb{E}(u_i[k] + f_i[k] - g_i[k]) = 0 \quad (7)$$

$$\mathbb{E}\left(u_i[k] + \sum_{j \in \mathcal{N}_i^- \setminus \{0\}} q_{ji}[k] - \sum_{l \in \mathcal{N}_i^+} q_{il}[k]\right) = 0. \quad (8)$$

Summing all over values of i , we obtain:

$$\mathbb{E}\left(\sum_{i=1}^n u_i[k]\right) = \mathbb{E}\left(\sum_{i=1}^n q_{i0}[k]\right). \quad (9)$$

In other words, there are on average as many *new* vehicles entering the traffic network of n regions from the outer region, as vehicles leaving it.

The purpose of this paper is to show that since it is possible to control the total inflow of the urban network $\sum_{i=1}^n u_i[k]$, condition (9) can be guaranteed, in principle, provided that a centralized perimeter controller is deployed. In what follows, we try to find controllers guaranteeing such a condition (while at the same time the outflow of the overall system is increased) for increased performance of the overall traffic network.

B. Joint perimeter and internal flow distribution control

Instead of controlling only the incoming traffic from the outer region, another approach consists of controlling the internal flows between the regions as well. The motivation is to ensure an optimal distribution of vehicles in each region and maintain the accumulation at a desired level by (i) controlling the number of vehicles entering a region from the outside and (ii) how a region communicates with its

neighboring regions. This is similar to having a control architecture with two layers: $\mathbf{u}[k]$ is used for perimeter control as a first-layer controller, while a second-layer of distributed control optimizes locally the states of the regions. In the literature, the traffic demand for each region is a parameter that is usually not controlled but is rather part of the constraints when solving optimization problems as it is the case in [21]. A method for the management of this demand is proposed in [8], by setting a threshold; this amounts to instruct to vehicles to wait at the origin of their trip (boundary). For this purpose, we assume that the system is in a given state $\mathbf{x}[k]$ and the goal is to minimize the error $\mathbf{x}[k+1]$ in the next time step. In this case, a suitable cost function is:

$$J[k] = \mathbf{x}^T[k+1]\mathbf{x}[k+1]. \quad (10)$$

With equation (5), $J[k]$ becomes:

$$J[k] = \sum_{i=1}^n x_i[k+1]^2 = \sum_{i=1}^n (x_i[k] - g_i[k] + f_i[k] + u_i[k])^2$$

The flows $f_i[k]$ can be written for a given region i as the sum of portions w_{ji} of the outflows g_j , $j \in \mathcal{N}_i^+$ of its neighbouring regions.

$$f_i[k] = \sum_{j \in \mathcal{N}_i^+} w_{ij} g_j[k], \quad (11)$$

with w_{ij} satisfying the following constraints:

$$\begin{aligned} \forall j \sum_{i \in \mathcal{N}_j^+} w_{ij} &= 1, \\ \forall (i, j) \ i \neq j, \ w_{ij} &\geq 0. \end{aligned} \quad (12)$$

The function J is convex (proof is in the Appendix) and the equality constraint is linear. The one-step optimization problem is therefore convex and one can affirm that J has an extremum.

$$J^* = \min_{\mathbf{u}, \mathbf{w}} J(k, \mathbf{u}, \mathbf{w}). \quad (13)$$

This minimum can be found in reasonable time (when n is small) using programming solvers (e.g. *fmincon*). Although this procedure leads to optimal results, the computations and communication overheads can be heavy, especially when the number of regions, n , becomes very large.

V. DISTRIBUTED APPROACH

A. Perimeter control of a network of n regions

We consider the system of n regions depicted in Fig. 3, where each region can control the number of vehicles u_i coming from outside of the network. In order to study the stability of its corresponding system (5), we introduce a candidate Lyapunov function $V(\mathbf{x}[k]) = \mathbf{x}[k]^T P \mathbf{x}[k]$. This function is continuous in $\mathbf{x}$, equal to zero at equilibrium and positive elsewhere. The Lyapunov drift is thus given by:

$$\Delta V(\mathbf{x}[k]) = V(\mathbf{x}[k+1]) - V(\mathbf{x}[k]) < 0. \quad (14)$$

Setting $P = I$ and replacing the expression of $\mathbf{x}[k+1]$, the increment of V is then given by:

$$\begin{aligned} \Delta V(\mathbf{x}[k]) &= \mathbf{x}[k+1]^T \mathbf{x}[k+1] - \mathbf{x}[k]^T \mathbf{x}[k] \\ &= (\mathbf{f}[k] - \mathbf{g}(\mathbf{x}[k]) + \mathbf{u}[k])^T \times \\ &\quad (2\mathbf{x}[k] + \mathbf{f}[k] - \mathbf{g}(\mathbf{x}[k]) + \mathbf{u}[k]) \end{aligned}$$

Hence, a sufficient condition for stability is as follows:

$$\begin{aligned} \forall k, \forall i, \\ (f_i[k] - g_i[k] + u_i[k]) (2x_i[k] + f_i[k] - g_i[k] + u_i[k]) < 0. \end{aligned} \quad (15)$$

Note that for $x_i = 0$, $\Delta V(x_i[k])$ is zero if the outflow and inflow are the same. Otherwise, it is positive and the system deviates again from the optimal state. The energy function V is at its minimum, equal to zero. Given the constraint on the control parameter $\mathbf{u}[k] \geq 0$, it is not always possible to guarantee a negative increment. Therefore, the controller can, at best, keep the energy of the system as low as possible, with $u_i[k] = 0$ when region i is overpopulated and the inflow is greater than the outflow, i.e., $f_i - g_i > 0$ and $x_i > 0$.

One possible controller fulfilling inequality (15) requirements is:

$$u_i[k] = \begin{cases} 0, & \text{if } f_i - g_i > 0 \text{ and } x_i > 0 \\ \frac{g_i - f_i - 2x_i}{2}, & \text{if } f_i - g_i > 0, x_i < 0 \text{ and } 2x_i + f_i - g_i < 0 \\ 0, & \text{if } f_i - g_i > 0, x_i < 0 \text{ and } 2x_i + f_i - g_i > 0 \\ g_i - f_i - x_i, & \text{if } f_i - g_i < 0. \end{cases} \quad (16)$$

Despite the total flow coming from the outside is regulated, the spontaneous routing of vehicles may determine a suboptimal redistribution of traffic among regions. In order to deal with this issue, we propose in the following part a strategy to efficiently redistribute the outflow of each region to neighboring regions by accounting for the traffic information in its surrounding; see Fig. 4.

Fig. 4: Traffic flow to / from region i .

B. Basic routing with fixed weights

A simplified distributed approach is proposed in which each region shares its traffic outflow to some of its neighboring regions. A basic strategy consists in distributing the

outgoing traffic from each region i *equally* to all its neighbors regardless of their state, *i.e.*,

$$x_i[k+1] = x_i[k] + \sum_{j \in \mathcal{N}_i^-} \frac{q_j[k]}{1 + \text{card}(\mathcal{N}_j^+)} - q_i[k] + u_i[k].$$

In matrix form, this can be written as

$$\mathbf{x}[k+1] = \mathbf{x}[k] + W[k]\mathbf{q}[k] + \mathbf{u}[k], \quad (17)$$

where $\mathbf{q}[k] \triangleq [q_1[k] \dots q_n[k]]^T$ and

$$W_{ij}[k] = \begin{cases} \frac{1}{1 + \text{card}(\mathcal{N}_j^+)} & \text{if } j \in \mathcal{N}_i^-, \\ -1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

Since $q_i[k]$ is a piecewise linear function with respect to N_i^* (see Fig. 1), we can express it as follows:

$$q_i[k] = \begin{cases} \alpha_i N_i[k], & \text{if } N_i[k] < N_i^*, \\ \gamma_i - \beta_i N_i[k], & \text{if } N_i[k] \geq N_i^*, \end{cases} \quad (19)$$

and, hence, we can write $\mathbf{q}[k]$ as follows:

$$\mathbf{q}[k] = D[k]N[k] + \gamma[k], \quad (20)$$

where $N[k] \triangleq [N_1[k] \dots N_n[k]]^T$, $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix with

$$D_{ii}[k] = \begin{cases} \alpha_i, & \text{if } N_i[k] < N_i^*, \\ -\beta_i, & \text{if } N_i[k] \geq N_i^*, \end{cases} \quad (21)$$

and $\gamma \in \mathbb{R}^{n \times 1}$ is a vector with

$$\gamma_i[k] = \begin{cases} 0, & \text{if } N_i[k] < N_i^*, \\ \gamma_i, & \text{if } N_i[k] \geq N_i^*. \end{cases} \quad (22)$$

Substituting (20) into (17), and $\mathbf{x}[k]$ with $N[k] - N_i^* \mathbf{1}$,

$$N[k+1] = (I + W[k]D[k])N[k] + W[k]\gamma[k] + \mathbf{u}[k]. \quad (23)$$

C. Routing for time-dependent weights

An alternative policy is to first examine the state of the neighbors of each region i and direct the traffic only towards the regions that are less populated than i . Let

$$\mathbb{I}_{ij}[k] = \begin{cases} 1, & \text{if } x_i[k] > x_j[k], \\ 0, & \text{otherwise.} \end{cases} \quad (24)$$

In the case of a uniform redistribution (*i.e.*, all out-neighbors receive the same proportion of the outflow of the sending regions), the evolution of the state x_i is given by:

$$x_i[k+1] = x_i[k] + \sum_{j \in \mathcal{N}_i^-} \mathbb{I}_{ji}[k] \frac{q_j[k]}{1 + \sum_{l \in \mathcal{N}_j} \mathbb{I}_{jl}[k]} - q_i[k] + u_i[k] \quad (25)$$

Equation (23) still holds with the following new definition

of the matrix W :

$$W_{ij}[k] = \begin{cases} \frac{1}{1 + \sum_{l \in \mathcal{N}_j} \mathbb{I}_{jl}[k]} & \text{if } j \in \mathcal{N}_i^- \text{ and } x_j[k] > x_i[k], \\ -1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

D. Totally distributed policy

We assume that each region knows the state of all its neighboring regions and can decide what amount of vehicles is allowed to enter from the outer region and how to redistribute the outflow, through the weights, thus optimizing its state as well as the overall state of the network. In the case of a weighted distribution of the traffic outflow, *i.e.*, the more a region is underpopulated, the more vehicles it receives, the evolution of state x_i is given by:

$$x_i[k+1] = x_i[k] - q_i[k] + u_i[k] + q_j \sum_{j \in \mathcal{N}_i^-} \frac{\mathbb{I}_{ji}[k](x_j[k] - x_i[k])}{\sum_{l \in \mathcal{N}_j^+ \cup \{0\}} \mathbb{I}_{jl}[k](x_j[k] - x_l[k])} \quad (27)$$

Hence,

$$W_{ij}[k] = \begin{cases} \frac{x_j[k] - x_i[k]}{\sum_{l \in \mathcal{N}_j^+ \cup \{0\}} \mathbb{I}_{jl}[k](x_j[k] - x_l[k])} & \text{if } j \in \mathcal{N}_i^- \text{ and } x_j[k] > x_i[k], \\ -1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \quad (28)$$

Equation (27) can be written as:

$$x_i[k+1] = x_i[k] + \sum_{j \in \mathcal{N}_i^-} w_{ij} \times q_j[k] - q_i[k] + u_i[k] \quad (29)$$

where,

$$w_{ij}[k] = \mathbb{I}_{ji}[k] \frac{x_j[k] - x_i[k]}{\sum_{l \in \mathcal{N}_j^+ \cup \{0\}} \mathbb{I}_{jl}[k](x_j[k] - x_l[k])} \quad (30)$$

The numerator is always positive, $x_j[k] > x_i[k]$ because when $x_j[k] < x_i[k]$, $w_{ij}[k] = 0$. Similarly, the denominator is a sum of positive terms, provided that the *new* variables x_0^j satisfy:

$$\forall k \forall j, \quad x_0^j[k] < x_j[k]. \quad (31)$$

Inequality (31) ensures that all regions are able to direct vehicles (and hence part of their traffic outflow) to region 0. The parameters $\{x_0^1, x_0^2, \dots, x_0^n\}$ are part of the design of the distributed scheme. They determine for each region the routing towards the external region. x_0^j sets the amount of vehicles to be sent outside of the network, from region j to region 0. Their choice depends on the configuration of the network and the traffic situation. This corresponds to traffic routing which “forces” vehicles to abandon the protected multi-region traffic network.

VI. NUMERICAL EVALUATION

In this section, we study the performance of the controllers applied to a hypothetical urban network symbolizing a city center that can be modeled as a multi-region traffic network consisting of 3 homogeneous regions (see Fig. 5).

Fig. 5: Test case network and partitioning into three regions.

The MFDs used have similar shapes (see Fig. 6), which means that the regions of the network have comparable behaviors and properties. Their parameters, *i.e.*, critical accumulation, capacity, and maximum number of vehicles are as follows:

	$i = 1$	$i = 2$	$i = 3$
$N_i^*(veh/m)$	571	500	411
$N_{i,max}(veh/m)$	1333	1125	1400
$q_{i,max}(veh/h)$	228	250	247

TABLE I: Characteristics of the MFDs.

Fig. 6: MFDs of the three regions of the network.

We first run simulations with the perimeter controller (16) derived in V-A. For this purpose, vehicle inter-transfers inside the network between the regions are assumed to be random. In the simulation figures, gridlock levels are represented by dotted lines while the optimal accumulations are drawn with dashed lines.

Fig. 7: Perimeter control of three regions.

Fig. 7 shows that the three regions never reach the gridlock level indicated by the dashed lines. The system is operating near the optimal point, thus maximizing the throughput of each region. It should be noted that, this controller has limitations, that are set by the characteristics of the system (MFDs and states). This controller cannot handle some cases where the demand on the network is high. In fact, when a region i is congested ($x_i > 0$) and the inflow (from the neighboring regions) is greater than the outflow, the number of vehicles will inevitably reach the congestion level. Therefore, this centralized controller cannot be relied on alone to manage a network.

The controller derived in Section IV-B, based on an MPC formulation yields good results as shown in Fig. 8. The states converge towards their optimal value. The convergence speed is relatively high, since the steady state is reached only after 3 iterations.

Fig. 8: One-step optimization controller performing boundary and internal flows control of three regions.

With regards to the distributed approach, simulations show that the proportional time-dependant weights controller (26) used alone is not able to bring the system to the optimal state. Without controlling the inflow u_i for all regions, convergence towards the optimal state is not guaranteed. In the case where

u_i is important for a region, the congestion level is reached. At best, we obtain the results illustrated in Fig. 9 for small well chosen values of the demand u_i .

Fig. 9: Distributed control with proportional weights of three regions.

However, combining the perimeter controller (16) with the weight distribution strategy (28) in Section V-D, we obtain the following results that are similar to the ones for the one-step optimization algorithm (see Fig. 8) with rapid convergence, only after ≈ 3 iterations. One advantage of using this distributed approach is that the temporal and algorithmic complexity is much less than that of the one-step optimization algorithm.

Fig. 10: Centralized perimeter control and distributed internal flow control with proportional weights.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

A. Conclusions

This paper presents the problem of traffic congestion in an n -region network with the aim of maximizing the outflow of the regions with regards to their respective MFDs. The dynamics of the system are studied by analyzing the necessary conditions of stability for perimeter control. Later, the distribution of vehicles between the regions is taken into

account and a convex optimization problem is formulated and solved, leading to a convergence of the system towards the optimal states. Finally, the paper investigates several strategies to reach the same goal in a distributed approach. The proposed algorithm combines boundary control and distributed control for the other vehicle exchanges between regions. Simulation results show that the controller yields encouraging results, keeping the regions close the optimal point, in a similar way to the MPC controller.

B. Future Directions

A possible future direction could consist in including Quality of Service constraints [22], [23]. Additionally, we can take into consideration the uncertainties on the values of the measured parameters in the model, in order to investigate the impact of the presence of noise on the performance of the controllers and their robustness [24]–[26]. Furthermore, we aim at designing estimation schemes that utilize real-time measurements of circulating flow and accumulation of vehicles in the regions, in order to produce accurate estimates of the currently prevailing critical accumulation [7], [20].

ACKNOWLEDGMENT

This project has received funding from the European Union's Horizon 2020 research and innovation program under grant agreement No 856602.

APPENDIX

PROOF OF THE CONVEXITY OF THE COST FUNCTION J

The quadratic cost function is given by

$$J[k] = \sum_{i=1}^n J_i[k] = \sum_{i=1}^n x_i[k+1]^2, \quad (32)$$

where

$$\begin{aligned} J_i[k] &= (x_i[k] - g_i[k] + f_i[k] + u_i[k])^2 \\ &= (x_i[k] - g_i[k])^2 + 2u_i[k] \left(\sum_{j \in \mathcal{N}_i^+} w_{ij} g_j[k] \right) \\ &\quad + 2(x_i[k] - g_i[k]) \left(\sum_{j \in \mathcal{N}_i^+} w_{ij} g_j[k] \right) + 2u_i[k] (x_i[k] - g_i[k]) \\ &\quad + \left(\sum_{j \in \mathcal{N}_i^+} w_{ij} g_j[k] \right)^2 + u_i[k]^2. \end{aligned} \quad (33)$$

The Hessian matrix of J_i (index k dropped for simplicity) is given by:

$$H_i = \begin{bmatrix} \frac{\partial^2 J_i}{\partial w_{i1}^2} & \frac{\partial^2 J_i}{\partial w_{i1} \partial w_{i2}} & \cdots & \frac{\partial^2 J_i}{\partial w_{i1} \partial w_{in}} & \frac{\partial^2 J_i}{\partial w_{i1} \partial u_i} \\ \frac{\partial^2 J_i}{\partial w_{i2} \partial w_{i1}} & \frac{\partial^2 J_i}{\partial w_{i2}^2} & \cdots & \frac{\partial^2 J_i}{\partial w_{i2} \partial w_{in}} & \frac{\partial^2 J_i}{\partial w_{i2} \partial u_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 J_i}{\partial w_{in} \partial w_{i1}} & \frac{\partial^2 J_i}{\partial w_{in} \partial w_{i2}} & \cdots & \frac{\partial^2 J_i}{\partial w_{in}^2} & \frac{\partial^2 J_i}{\partial w_{in} \partial u_i} \\ \frac{\partial^2 J_i}{\partial u_i \partial w_{i1}} & \frac{\partial^2 J_i}{\partial u_i \partial w_{i2}} & \cdots & \frac{\partial^2 J_i}{\partial u_i \partial w_{in}} & \frac{\partial^2 J_i}{\partial u_i^2} \end{bmatrix}.$$

Thus, the Hessian of J_i is

$$H_i = 2 \times \begin{bmatrix} g_1 & g_1 g_2 & \cdots & g_1 g_n & g_1 \\ g_2 g_1 & g_2 & \cdots & g_2 g_n & g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_n g_1 & g_n g_2 & \cdots & g_n & g_n \\ g_1 & g_2 & \cdots & g_n & 1 \end{bmatrix}.$$

This matrix is symmetric and can therefore be diagonalized according to the spectral theorem. To find the eigenvalues, we compute the characteristic polynomial:

$$\det(H_i - \lambda I) = \begin{vmatrix} 2g_1 - \lambda & 2g_1 g_2 & \cdots & 2g_1 g_n & 2g_1 \\ 2g_2 g_1 & 2g_2 - \lambda & \cdots & 2g_2 g_n & 2g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2g_n g_1 & 2g_n g_2 & \cdots & 2g_n - \lambda & 2g_n \\ 2g_1 & 2g_2 & \cdots & 2g_n & 2 - \lambda \end{vmatrix}.$$

In order to calculate this determinant, we transform the columns with the following linear combinations: $\forall i \in [1, n] \ C_i \leftarrow C_i - g_i \times C_{n+1}$

$$\begin{aligned} & \det(H_i - \lambda I) \\ &= \begin{vmatrix} 2g_1 - \lambda - 2g_1^2 & 0 & \cdots & 0 & 2g_1 \\ 0 & 2g_2 - \lambda - 2g_2^2 & \cdots & 0 & 2g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2g_n - \lambda - 2g_n^2 & 2g_n \\ g_1 \lambda & g_2 \lambda & \cdots & g_n \lambda & 2 - \lambda \end{vmatrix} \\ &= (2 - \lambda) \prod_{i=1}^n (2g_i - \lambda - 2g_i^2). \end{aligned}$$

The eigenvalues are:

$$\Lambda = \{2, 2g_1(g_1 - 1), 2g_2(g_2 - 1), \dots, 2g_n(g_n - 1)\}.$$

These are all positive, given that $g_i > 1$, $i \in \{1, 2, \dots, n\}$. Hence, the Hessian matrix H is positive semidefinite. We can conclude that the function J_i is convex. and by summation of all functions J_i , J is convex. ■

REFERENCES

- [1] H. S. Mahmassani, M. Saberi, and A. Zockaie, "Urban network gridlock: Theory, characteristics, and dynamics," *Transportation Research Part C: Emerging Technologies*, vol. 36, pp. 480–497, 2013.
- [2] I. I. Sirmatel and N. Geroliminis, "Economic model predictive control of large-scale urban road networks via perimeter control and regional route guidance," *IEEE Transactions on Intelligent Transportation Systems*, vol. 19, no. 4, pp. 1112–1121, April 2018.
- [3] C. F. Daganzo, "Urban gridlock: Macroscopic modeling and mitigation approaches," *Transportation Research Part B: Methodological*, vol. 41, no. 1, pp. 49–62, 2007.
- [4] N. Geroliminis and C. F. Daganzo, "Existence of urban-scale macroscopic fundamental diagrams: Some experimental findings," *Transportation Research Part B: Methodological*, vol. 42, no. 9, pp. 759–770, 2008.
- [5] M. Saberi, H. S. Mahmassani, T. Hou, and A. Zockaie, "Estimating Network Fundamental Diagram Using Three-Dimensional Vehicle Trajectories: Extending Edie's Definitions of Traffic Flow Variables to Networks," *Transportation Research Record: Journal of the Transportation Research Board*, vol. 2422, pp. 12–20, 2014.
- [6] M. Yildirimoglu and N. Geroliminis, "Approximating dynamic equilibrium conditions with macroscopic fundamental diagrams," *Transportation Research Part B: Methodological*, vol. 70, pp. 186–200, 2014.
- [7] A. Kouvelas, M. Saeedmanesh, and N. Geroliminis, "Real-time estimation of aggregated traffic states of multi-region urban networks," in *IEEE International Conference on Intelligent Transportation Systems (ITSC)*, Oct. 2017, pp. 1–6.
- [8] C. Menelaou, S. Timotheou, P. Kolios, and C. G. Panayiotou, "Joint route guidance and demand management for multi-region traffic networks," in *European Control Conference (ECC)*, June 2019, pp. 2183–2188.
- [9] R. Findeisen and F. Allgöwer, "An introduction to nonlinear model predictive control," in *21st Benelux meeting on systems and control*, vol. 11. Technische Universiteit Eindhoven Veldhoven Eindhoven, The Netherlands, 2002, pp. 119–141.
- [10] N. Geroliminis, J. Haddad, and M. Ramezani, "Optimal Perimeter Control for Two Urban Regions With Macroscopic Fundamental Diagrams: A Model Predictive Approach," *IEEE Transactions on Intelligent Transportation Systems*, vol. 14, no. 1, pp. 348–359, March 2013.
- [11] K. Yang, N. Zheng, and M. Menendez, "Multi-scale perimeter control approach in a connected-vehicle environment," *Transportation Research Part C: Emerging Technologies*, vol. 94, pp. 32–49, 2018.
- [12] M. Ramezani, J. Haddad, and N. Geroliminis, "Dynamics of heterogeneity in urban networks: Aggregated traffic modeling and hierarchical control," *Transportation Research Part B: Methodological*, vol. 74, April 2015.
- [13] M. Ramezani and M. Nourinejad, "Dynamic modeling and control of taxi services in large-scale urban networks: A macroscopic approach," *Transportation Research Part C Emerging Technologies*, vol. 94, August 2017.
- [14] J. Haddad and N. Geroliminis, "On the stability of traffic perimeter control in two-region urban cities," *Transportation Research Part B: Methodological*, vol. 46, pp. 1159–1176, November 2012.
- [15] J. Haddad and A. Shraiber, "Robust perimeter control design for an urban region," *Transportation Research Part B: Methodological*, vol. 68, pp. 315 – 332, 2014.
- [16] R. Mohajerpoor, M. Saberi, H. Vu, T. Garoni, and M. Ramezani, "H infinity robust perimeter flow control in urban networks with partial information feedback," *Transportation Research Part B: Methodological*, March 2019.
- [17] M. Keyvan-Ekbatani, A. Kouvelas, I. Papamichail, and M. Papageorgiou, "Exploiting the fundamental diagram of urban networks for feedback-based gating," *Transportation Research Part B: Methodological*, vol. 46, no. 10, pp. 1393 – 1403, 2012.
- [18] C. F. Daganzo, "The cell transmission model, part ii: Network traffic," *Transportation Research Part B: Methodological*, vol. 29, no. 2, pp. 79–93, 1995.
- [19] C. F. Daganzo and N. Geroliminis, "An analytical approximation for the macroscopic fundamental diagram of urban traffic," *Transportation Research Part B: Methodological*, vol. 42, no. 9, pp. 771–781, 2008.
- [20] K. Ampountolas and A. Kouvelas, "Real-time estimation of critical vehicle accumulation for maximum network throughput," in *American Control Conference (ACC)*, 2015, pp. 2057–2062.
- [21] A. Kouvelas, M. Saeedmanesh, and N. Geroliminis, "Linear-parameter-varying model predictive control for multi-region traffic systems," in *Transportation Research Board 98th Annual Meeting*, 2019.
- [22] A. Csikós, H. Farhadi, B. Kulcsár, T. Charalambous, and H. Wymeersch, "Traffic flow optimization with qos constrained network admission control," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 5275–5280, 2017, 20th IFAC World Congress.
- [23] A. Csikós, T. Charalambous, H. Farhadi, B. Kulcsár, and H. Wymeersch, "Network traffic flow optimization under performance constraints," *Transportation Research Part C: Emerging Technologies*, vol. 83, pp. 120–133, 2017.
- [24] B. Kulcsar, K. Ampountolas, and A. Dabiri, "Single-region robust perimeter traffic flow control," in *European Control Conference (ECC)*, July 2015, pp. 2628–2633.
- [25] K. Ampountolas, N. Zheng, and N. Geroliminis, "Macroscopic modelling and robust control of bi-modal multi-region urban road networks," *Transportation Research Part B: Methodological*, vol. 104, pp. 616–637, 2017.
- [26] A. Csikós, T. Charalambous, and B. Kulcsár, "Towards a robust traffic admission control in homogeneous urban vehicular networks under QoS constraints," in *European Control Conference (ECC)*, June 2018, pp. 1307–1313.