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LEVELNESS OF ORDER POLYTOPES*

CHRISTIAN HAASE[†], FLORIAN KOHL[‡], AND AKIYOSHI TSUCHIYA[§]

Abstract. Since Stanley’s [*Discrete Comput. Geom.*, 1 (1986), pp. 9–23] introduction of order polytopes, their geometry has been widely used to examine (algebraic) properties of finite posets. In this paper, we follow this route to examine the levelness property of order polytopes, a property generalizing Gorensteinness. This property has been recently characterized by Miyazaki [*J. Algebra*, 480 (2017), pp. 215–236] for the case of order polytopes. We provide an alternative characterization using weighted digraphs. Using this characterization, we give a new infinite family of level posets and show that determining levelness is in co-NP. Moreover, we show how a necessary condition of levelness of [*J. Algebra*, 431 (2015), pp. 138–161] can be restated in terms of digraphs. We then turn to the more general family of alcoved polytopes. We give a characterization for levelness of alcoved polytopes using the Minkowski sum. Then we study several cases when the product of two polytopes is level. In particular, we provide an example where the product of two level polytopes is not level.

Key words. order polytopes, Bellman–Ford algorithm, posets, level algebras, alcoved polytopes

AMS subject classifications. 06A11, 52B20

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1. Introduction. Partially ordered sets—or posets for short—are ubiquitous objects in mathematics. Richard Stanley introduced in [19] two geometric objects associated with every finite poset Π : the order polytope $\mathcal{O}(\Pi)$ and the chain polytope $\mathcal{C}(\Pi)$.

In this paper, we are interested in the level property of order polytopes. We will first give a high level overview of the objects and results and refer to precise definitions and statements in the body of the paper.

Levelness was defined by Stanley [17, p. 54] as a property of graded rings, generalizing the Gorenstein property. As is customary in combinatorial commutative algebra, with a lattice polytope such as $\mathcal{O}(\Pi)$ we associate a graded semigroup algebra, its Ehrhart ring (see section 2.3 for details). The canonical module of this Ehrhart ring corresponds to a set of interior lattice points of the semigroup. We call a polytope level if the canonical module is generated by elements of one single degree (cf. Definition 2.13). Recall that Gorenstein means that the canonical module is generated by one element.

While the Gorenstein property of Ehrhart rings can be characterized purely in terms of the Ehrhart series [18], levelness is much more subtle. In Example 6.6 and Corollary 6.7 we give examples which show that no such characterization is possible. Nonetheless, levelness does pose restrictions on the Ehrhart series, which do not hold for general lattice polytopes [20, Prop. III.3.3].

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Over the past decades, the Gorenstein property has been studied extensively in combinatorial commutative algebra. Hibi [8] was the first to examine minimal elements of the canonical modules of Ehrhart rings of order polytopes, which in this context are also known as Hibi rings. In particular, he showed that a poset is Gorenstein if and only if it is graded, namely, every maximal chain has the same length. Levelness, on the other hand, has only recently come into focus [6, 11, 13]—with the exception of [9].

In an important contribution [14] Miyazaki examines and characterizes levelness of $\mathcal{O}(\Pi)$ in terms of certain alternating sequences of elements of Π with “condition N.” We provide an alternative characterization using weighted digraphs $\Gamma(\Pi')$ coming from subposets Π' of Π , possibly with an added maximum or minimum element (see Definition 4.1 and Corollary 4.4 for details).

This result enables us to use the Bellman–Ford algorithm to check levelness. As a direct consequence, we get that determining levelness belongs to the complexity class co-NP (Corollary 4.6).

We show that the necessary condition for levelness of order polytopes in [6, Thm. 4.1] is indeed equivalent to a special case of our characterization. Furthermore, we give an example that was related to us by Alex Fink showing that this condition is not sufficient; see Remark 5.4 and Figure 2(a).

In section 6, we use Corollary 4.4 to describe a new infinite family of level posets. The main ingredient is the ordinal sum $\Pi = \Pi_1 \triangleleft \Pi_2$ of two posets, making every element of Π_1 smaller than any element of Π_2 . It turns out that this poset is level if and only if both Π_1 and Π_2 are (see Theorem 6.2).

As the ordinal sum operation interacts nicely with the Ehrhart series (cf. Proposition 6.5), we can construct in Example 6.6 and Corollary 6.7 infinitely many examples of pairs of posets that have the same Ehrhart series, but where one poset is level and the other one is not.

Order polytopes belong to the more general class of alcoved polytopes. For this class, we can characterize in Proposition 7.8 levelness purely in terms of convexity without paying extra attention to lattice points. This is used in order to give a criterion in Theorem 7.11 for the Cartesian product of two alcoved polytopes to be level. It shows that the product of a level polytope with a nonlevel polytope can indeed be guaranteed to be level. This puts a corresponding result of Ene et al. [6] about the disjoint union of a poset with a long chain into a broader context, and it yields an explicit bound for what “long” means. It also shows that a poset is level if all connected components are. On the other hand, Remark 7.14 contains an explicit example of two level polytopes whose product is not level. By Theorem 7.13 this cannot happen for order or alcoved polytopes as they have the integer decomposition property.

The structure of this paper is as follows. In section 2, we recall the basics of Ehrhart theory relevant to this paper, we introduce posets, order polytopes, and chain polytopes, and we show how these relate to combinatorial commutative algebra. In section 3, we recall Miyazaki’s results on level posets. We then give an alternative characterization of level posets using weighted digraphs in section 4. In section 5, we show that this characterization generalizes the necessary condition of Ene et al. In section 6, we use this characterization to examine levelness of series-parallel posets. The last section concerns levelness of alcoved polytopes and of products of polytopes.

2. Background and notation.

2.1. Ehrhart theory and lattice polytopes. In this subsection, we give a brief introduction to (lattice) polytopes and Ehrhart theory. We refer the interested reader to the excellent books [1, 22]. A *polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r \in \mathbb{R}^d$, i.e.,

$$P = \text{conv}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\} := \left\{ \sum_{i=1}^r \lambda_i \mathbf{u}_i : \sum_{i=1}^r \lambda_i = 1, \lambda_i \geq 0 \text{ for all } i \right\}.$$

The inclusion-minimal set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s\}$ such that $P = \text{conv}\{\mathbf{v}_1, \dots, \mathbf{v}_s\}$ is called the *vertex set* of P and its elements are called the *vertices*. A polytope whose vertex set is contained in $\mathbb{Z}^d$ is called a *lattice polytope*. The dimension d of a polytope is the dimension of its affine span, and we call P a d -polytope. For a d -polytope $P \subset \mathbb{R}^d$, we define the *Ehrhart function* $\text{ehr}_P: \mathbb{Z}_{\geq 1} \rightarrow \mathbb{Z}_{\geq 0}$

$$\text{ehr}_P(k) := \#kP \cap \mathbb{Z}^d,$$

where $kP := \{k\mathbf{x} : \mathbf{x} \in P\}$ is the k th dilate of P . When P is a lattice d -polytope, ehr_P agrees with a polynomial of degree d [5] which we call the *Ehrhart polynomial* of P . The *Ehrhart series* of a lattice polytope P is the formal power series

$$\text{Ehr}_P(z) := 1 + \sum_{k \geq 1} \text{ehr}_P(k) z^k.$$

As a direct consequence of ehr_P agreeing with a polynomial, the Ehrhart series of a lattice d -polytope P can be written as a rational function

$$\text{Ehr}_P(z) = \frac{h_0^* + \dots + h_d^* z^d}{(1-z)^{d+1}}$$

and the numerator of this rational function is called the *h^* -polynomial* of P . The *degree* of P is defined as

$$\deg(P) := \max\{k : h_k^* \neq 0\},$$

and the *codegree* of P is defined as

$$\text{codeg}(P) := d + 1 - \deg(P).$$

2.2. Two poset polytopes. A *Partially Ordered Set* (or poset) $(\Pi, \leq_\Pi)$ is a set Π together with a binary relation $\leq_\Pi$ that is reflexive, antisymmetric, and transitive. The relation $\leq_\Pi$ is called a *partial order* and when there is no confusion about the poset, we simply write $\leq$. An element $j \in \Pi$ is said to *cover* an element $i \in \Pi$, denoted $j \succ i$, if $i \leq k \leq j$ implies that either $i = k$ or $j = k$. One can recover all partial orders from these cover relations. Therefore, it is convenient to illustrate the poset using these cover relations by a *Hasse diagram*; see Figure 1. A *chain* is a totally ordered subset of Π , i.e., it is a subset $\{i_1, \dots, i_k\} \subset \Pi$ where all elements are pairwise comparable. We say that a chain of the form $i_1 <_\Pi i_2 <_\Pi \dots <_\Pi i_k$ has *length* $k - 1$. If a chain is of the form $i_1 <_\Pi i_2 <_\Pi \dots <_\Pi i_k$, then the length equals the number of Hasse edges in the path from i_1 to i_2 , from i_2 to i_3 , ..., from i_{k-1} to i_k . The length of a longest chain in Π is called the *rank* of Π .

Given a poset Π , we define the poset $\overline{\Pi} = (\Pi \cup \{\infty\}, \leq_{\overline{\Pi}})$, where $i <_{\overline{\Pi}} j$ if either $j = \infty$ and $i \in \Pi$ or if $i <_\Pi j$. Similarly, we define $\underline{\Pi} = (\Pi \cup \{-\infty\}, \leq_{\underline{\Pi}})$, where $i <_{\underline{\Pi}} j$ if either $i = -\infty$ and $j \in \Pi$ or if $i <_\Pi j$. To every finite poset, Stanley associated two lattice polytopes, namely the order polytope and the chain polytope:

FIG. 1. The Hasse diagram of the poset $i < j > k$.

DEFINITION 2.1 (see [19, Def. 1.1]). The order polytope $\mathcal{O}(\Pi)$ of a finite poset Π is the subset of $\mathbb{R}^\Pi = \{f: \Pi \rightarrow \mathbb{R}\}$ defined by

$$\begin{aligned} 0 &\leq f(i) \leq 1 && \text{for all } i \in \Pi, \\ f(i) &\leq f(j) && \text{if } i \leq_\Pi j. \end{aligned}$$

DEFINITION 2.2 (see [19, Def. 2.1]). The chain polytope $\mathcal{C}(\Pi)$ of a finite poset Π is the subset of $\mathbb{R}^\Pi = \{g: \Pi \rightarrow \mathbb{R}\}$ defined by the conditions

$$\begin{aligned} 0 &\leq g(i) && \text{for all } i \in \Pi, \\ g(i_1) + g(i_2) + \dots + g(i_k) &\leq 1 && \text{for all chains } i_1 <_\Pi i_2 <_\Pi \dots <_\Pi i_k \text{ of } \Pi. \end{aligned}$$

Remark 2.3. In the following, we will use an isomorphism $\mathbb{R}^\Pi \cong \mathbb{R}^{\#\Pi}$ to make notation better.

We define an *order filter* F of a poset Π to be a subset $F \subset \Pi$ such that if $i \in F$ and $i < j$, then $j \in F$. We remark that order filters are sometimes also called upper order ideals. To every filter F , one can associate a *characteristic function* $\mathbb{1}_F$ defined as

$$\mathbb{1}_F(i) := \begin{cases} 1 & \text{if } i \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Stanley showed that vertices of $\mathcal{O}(\Pi)$ are given by the characteristic functions of order filters.

COROLLARY 2.4 (see [19, Cor. 1.3]). The vertices of $\mathcal{O}(\Pi)$ are the characteristic functions $\mathbb{1}_F$ of order filters F . In particular, the number of vertices equals the number of order filters.

Stanley also gave the vertex description of chain polytopes. We define an *antichain* A of a poset Π to be a subset $A \subset \Pi$ of pairwise incomparable elements. The characteristic function $\mathbb{1}_A$ of an antichain A is defined similarly to the characteristic function of an order filter.

THEOREM 2.5 (see [19, Thm. 2.2]). The vertices of $\mathcal{C}(\Pi)$ are given by the characteristic functions $\mathbb{1}_A$ of antichains A . In particular, the number of vertices of $\mathcal{C}(\Pi)$ equals the number of antichains of Π .

Let Π be a d -element poset and let $m \in \mathbb{Z}_{\geq 1}$. We define $\Omega(\Pi, m)$ to be the number of *order-preserving maps* $\Pi \rightarrow \{1, 2, \dots, m\}$, where we say that a map f is *order preserving* if $i \leq_\Pi j$ implies $f(i) \leq f(j)$. These order-preserving maps correspond to integer points in dilates of the order polytope as the next theorem shows.

THEOREM 2.6 (see [19, Thm. 4.1]). The Ehrhart polynomials of $\mathcal{O}(\Pi)$ and $\mathcal{C}(\Pi)$ are given by

$$\text{ehr}_{\mathcal{O}(\Pi)}(k) = \text{ehr}_{\mathcal{C}(\Pi)}(k) = \Omega(\Pi, k+1).$$

Remark 2.7. As is implicit in Stanley's proof, interior integer points are in bijection with *strictly order-preserving maps*, i.e., maps f that satisfy $i <_{\Pi} j$ implies $f(i) < f(j)$.

In a poset Π , maximal chains can have different lengths. A chain with maximum length is called a *longest chain*.

Remark 2.8. The codegree of $\mathcal{O}(\Pi)$ equals the rank of $\overline{\Pi}$, i.e., it equals the number of edges in the longest chain of $\overline{\Pi}$.

2.3. Level affine semigroups. This subsection is based on [3, Chap. 6]. Let $P \subset \mathbb{R}^d$ be a lattice d -polytope with vertex set $V(P)$ and let $\mathbb{k}$ be an algebraically closed field of characteristic zero. We define the *cone over P* as

$$\text{cone}(P) := \text{span}_{\mathbb{R}_{\geq 0}} \{(\mathbf{v}, 1) : \mathbf{v} \in V(P)\} \subset \mathbb{R}^d \times \mathbb{R}.$$

The set

$$\mathcal{S}(P) := \left\{ \mathbf{x} : \mathbf{x} = \sum_{i \in I} \lambda_i (\mathbf{v}_i, 1), \text{ for } \mathbf{v}_i \in P \cap \mathbb{Z}^d \text{ and } \lambda_i \in \mathbb{Z}_{\geq 0}, \text{ for all } i \in I \right\}$$

forms an additive *semigroup*, i.e., a set that is closed under addition, which contains a neutral element, and where addition is associative. Moreover, we define the semigroup $C_{\mathbb{Z}}(P) := \text{cone}(P) \cap \mathbb{Z}^{d+1}$. We say that P has the *integer decomposition property* if $\mathcal{S}(P) = C_{\mathbb{Z}}(P)$. This semigroup gives rise to the *Ehrhart ring of P*

$$\mathbb{k}[P] := \mathbb{k}[C_{\mathbb{Z}}(P)] = \mathbb{k}[\mathbf{x}^{\mathbf{p}} \cdot y^m : (\mathbf{p}, m) \in C_{\mathbb{Z}}(P)] \subset \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y]$$

with $\deg(\mathbf{x}^{\mathbf{p}} \cdot y^m) = m$.

Remark 2.9. The Hilbert series (resp., h -polynomial) of $\mathbb{k}[P]$ coincides with the Ehrhart series (resp., h^* -polynomial) of P .

Furthermore, the semigroup $C_{\mathbb{Z}}(P)$ is normal, i.e., the graded ring $\mathbb{k}[P]$ is normal. By a seminal result of Melvin Hochster [12], $\mathbb{k}[P]$ is Cohen–Macaulay.

Remark 2.10. Since every order polytope has a unimodular triangulation [19, sect. 5], order polytopes have the integer decomposition property. Hence, we have

$$\mathbb{k}[\mathcal{O}(\Pi)] = \mathbb{k}[C_{\mathbb{Z}}(\mathcal{O}(\Pi))] = \mathbb{k}[\mathcal{S}(\mathcal{O}(\Pi))].$$

DEFINITION 2.11 (Danilov–Stanley [3, Thm. 6.31]). *Let the notation be as above and let $\mathbb{k}$ be a field. We define the canonical module of $\mathbb{k}[P]$, denoted $\omega_{\mathbb{k}[P]}$, to be the ideal of $\mathbb{k}[P]$ generated by the monomials corresponding to the interior integer points $\text{int}(C_{\mathbb{Z}}(P))$.*

Remark 2.12. The codegree of a lattice d -polytope P is the smallest positive integer c such that cP contains an interior integer point. This is a consequence of a reciprocity theorem relating the Hilbert series of $\mathbb{k}[P]$ to the Hilbert series of the canonical module $\omega_{\mathbb{k}[P]}$; see, for instance, [3, Thm. 6.41].

We can now state our main definition.

DEFINITION 2.13. *A lattice polytope P is level if and only if for all $\mathbf{x} \in \text{int}(\text{cone}(P)) \cap \mathbb{Z}^{d+1}$ there exists a point $\mathbf{y} \in \text{int}(\text{cone}(P)) \cap \mathbb{Z}^{d+1}$ at height $\text{codeg}(P)$ such that $\mathbf{x} - \mathbf{y} \in \text{cone}(P)$. A lattice polytope is Gorenstein if it is level and there is a unique integer point $\mathbf{y} \in \text{int}(\text{cone}(P)) \cap \mathbb{Z}^{d+1}$ at height $\text{codeg}(P)$.*

Remark 2.14. This is equivalent to saying that a lattice d -polytope P is level if and only if the $\mathbb{k}[P]$ -module $\omega_{\mathbb{k}[P]}$ is generated by elements of degree $\text{codeg}(P)$ (as a $\mathbb{k}[P]$ -module). Furthermore, a lattice polytope P is *Gorenstein* if $\omega_{\mathbb{k}[P]}$ is generated by a single element, which necessarily is of degree $\text{codeg}(P)$. We say that a finite poset Π is *level* (Gorenstein) if $\mathcal{O}(\Pi)$ is level (Gorenstein).

DEFINITION 2.15. We say that $\mathbf{x} \in \text{int}(\text{cone}(P)) \cap \mathbb{Z}^{d+1}$ is minimal if the corresponding monomial is a minimal generator of $\omega_{\mathbb{k}[P]}$. Geometrically speaking, this means that $\mathbf{x} \in \text{int}(\text{cone}(P)) \cap \mathbb{Z}^{d+1}$ is minimal if and only if there is no $\mathbf{y} \in \text{int}(\text{cone}(P)) \cap \mathbb{Z}^{d+1}$ such that $\mathbf{x} - \mathbf{y} \in \text{cone}(\mathcal{O}(\Pi))$.

3. Miyazaki's characterization. In this section, we recall the characterization for levelness of order polytopes which was introduced by Miyazaki; see [14]. In order to give a characterization of the level property, Miyazaki defined sequences with condition N .

DEFINITION 3.1 (see [14, Def. 3.1]). Let $i_1, j_1, i_2, j_2, \dots, i_t, j_t$ be a possibly empty sequence of elements in a finite poset Π . We say the sequence satisfies condition N if

1. $i_1 < j_1 > i_2 < j_2 > \dots > i_t < j_t$ and
2. for any m, n with $1 \leq m < n \leq t$, $i_m \not\leq j_n$.

Remark 3.2. Our definition of condition N is slightly different from Miyazaki's original definition. In [14], for a poset Π , he investigated properties of the ring $\mathbb{k}[\mathcal{O}(\Pi^{op})]$, where Π^{op} is the dual poset of Π , i.e., the poset on the same ground set with all orders reversed. Hence conditions 1 and 2 in Definition 3.1 are the same as conditions (2) and (3) in [14, Def. 3.1]. Moreover, in our definition, we do not need condition (1) of [14, Def. 3.1], since we take elements in Π , whereas in [14, Def. 3.1], elements from $\bar{\Pi}$ are taken.

Before we state the next definition, let us recall the definition of $\text{rank}[i, j]$,

$$\text{rank}[i, j] := \#\text{Hasse edges in a longest chain from } i \text{ to } j,$$

where $i \leq j \in \Pi$ and $[i, j] := \{k \in \Pi : i \leq_{\Pi} k \leq_{\Pi} j\}$.

DEFINITION 3.3. Let $i_1, j_1, i_2, j_2, \dots, i_t, j_t$ be a sequence of elements in a finite poset Π with condition N , and set $j_0 = \infty$ and $i_{t+1} = -\infty$. We set

$$r(i_1, j_1, \dots, i_t, j_t) := \sum_{s=1}^t (\text{rank}[i_s, j_{s-1}] - \text{rank}[i_s, j_s]) + \text{rank}[i_{t+1}, j_t].$$

Moreover, set

$$r_{\max} := \max\{r(i_1, j_1, \dots, i_t, j_t) : i_1, j_1, \dots, i_t, j_t \text{ is a sequence with condition } N\}.$$

For every sequence with condition N , Miyazaki defines a special element of $\mathbb{C}_{\mathbb{Z}}(\mathcal{O}(\Pi))$.

DEFINITION 3.4 (see [14, Def. 3.6]). Let $i_1, j_1, i_2, j_2, \dots, i_t, j_t$ be a sequence of elements in a finite poset Π with condition N , and set $j_0 = \infty$ and $i_{t+1} = -\infty$. We define

$$x(i_1, j_1, \dots, i_t, j_t)_{i_m} := \sum_{s=m}^t (\text{rank}[i_{s+1}, j_s] - \text{rank}[i_s, j_s])$$

for $1 \leq m \leq t$, and $x(i_1, j_1, \dots, i_t, j_t)_{i_{t+1}} = 0$, and we define a lattice point $\mathbf{y}(i_1, j_1, \dots, i_t, j_t)$ in $\mathbb{R}^{\bar{\Pi}}$ whose k th coordinate is

$$y(i_1, j_1, \dots, i_t, j_t)_k := \max\{\text{rank}[i_s, k] + x(i_1, j_1, \dots, i_t, j_t)_{i_s} : k \geq i_s\}$$

for $k \in \bar{\Pi}$.

These elements give rise to an important class of minimal elements, as the next lemma shows.

LEMMA 3.5 (see [14, Lem. 3.8]). *Let $i_1, j_1, i_2, j_2, \dots, i_t, j_t$ be a sequence of elements in a finite poset Π with condition N , and set $j_0 = \infty$ and $i_{t+1} = -\infty$. If $r(i_1, j_1, \dots, i_t, j_t) = r_{\max}$, then the element $\mathbf{y}(i_1, j_1, \dots, i_t, j_t)$ is minimal in the sense of Definition 2.15. In particular, it is an interior lattice point in the cone. Furthermore,*

$$y(i_1, j_1, \dots, i_t, j_t)_{i_m} = x(i_1, j_1, \dots, i_t, j_t)_{i_m},$$

$$y(i_1, j_1, \dots, i_t, j_t)_{j_{m-1}} = \text{rank}[i_m, j_{m-1}] + x(i_1, j_1, \dots, i_t, j_t)_{i_m}$$

for $1 \leq m \leq t+1$. In particular, $y(i_1, j_1, \dots, i_t, j_t)_{j_0} = r_{\max}$.

We give an example to illustrate Definitions 3.1, 3.3, and 3.4 and Lemma 3.5.

Example 3.6. Let Π be the poset from Figure 2(a) and $(i_1, j_1, i_2, j_2) = (p_9, p_7, p_5, p_3)$. Then the sequence i_1, j_1, i_2, j_2 satisfies condition N . Moreover, one has $r(i_1, j_1, i_2, j_2) = 6$, in particular, $r_{\max} = r(i_1, j_1, i_2, j_2)$. On the other hand, we obtain $x(i_1, j_1, i_2, j_2)_{i_1} = 3, x(i_1, j_1, i_2, j_2)_{i_2} = 2$ and $x(i_1, j_1, i_2, j_2)_{i_3} = 0$, where $i_3 = -\infty$. Now let us compute each $y_k := y(i_1, j_1, i_2, j_2)_k$. For instance, we consider y_7 . Then $i_1, i_2, i_3 \leq p_7$ in $\bar{\Pi}$. Hence one has

$$y_7 = \max\{\text{rank}[i_1, 7] + 3, \text{rank}[i_2, 7] + 2, \text{rank}[i_3, 7] + 0\} = 4.$$

In fact, Figure 2(b) shows the value y_k for each $k \in \Pi$. Moreover, $y(i_1, j_1, i_2, j_2)_{\infty} = 6$. In particular, this element $\mathbf{y}(i_1, j_1, i_2, j_2)$ is minimal in the sense of Definition 2.15. This example plays a key role in Remark 5.4.

Now, we can introduce Miyazaki's characterization.

LEMMA 3.7 (see [14, Thm. 3.9]). *Let Π be a poset and $r = \text{codeg}(\mathcal{O}(\Pi))$. Then Π is level if and only if $r_{\max} = r$.*

FIG. 2. Fink's poset and the minimal element $\mathbf{y}$ illustrated.

In this paper, we characterize levelness of order polytopes in terms of weighted digraphs. A directed, weighted edge (i, j, w) is an edge from vertex i to another vertex j with weight w . Given a poset Π , we define the *Hasse graph* $H(\Pi)$ of Π to be the digraph with nodes coming from $\overline{\Pi}$ and with directed, weighted edges $(i, j, -1)$ and $(j, i, 1)$, where $i < j$.

In our language, a sequence $i_1, j_1, \dots, i_t, j_t$ with condition N can be reinterpreted as a path $\mathcal{P}$ in $H(\Pi)$ from ∞ to $-\infty$ with up-edges (or down-edges) coming from longest chains in $[i_m, j_m]$ (or $[i_{m+1}, j_m]$), where $j_0 = \infty$ and $i_{t+1} = -\infty$, and we say that such a path satisfies condition N . Moreover, we set $r(\mathcal{P}) := r(i_1, j_1, \dots, i_t, j_t)$ and $\mathbf{y}(\mathcal{P}) := \mathbf{y}(i_1, j_1, \dots, i_t, j_t)$. We chose the weights of the edges, so that $-r(\mathcal{P})$ equals the weighted length.

Remark 3.8. This special minimal element $\mathbf{y}(\mathcal{P})$ has the property that for every up (or down) interval $[i, j]$ in the path we have $y(\mathcal{P})_i - y(\mathcal{P})_j = \text{rank}[i, j]$. This is a direct consequence of Lemma 3.5 and a brief computation.

Now, we can characterize the level property as the following.

PROPOSITION 3.9. *Let Π be a finite poset and $r = \text{codeg}(\mathcal{O}(\Pi))$. Then Π is level if and only if there is no path $\mathcal{P}$ in $H(\Pi)$ with condition N such that $r(\mathcal{P}) > r$.*

Proof. This directly follows from Lemmas 3.7 and 3.5. $\square$

4. A new characterization of levelness. In this section, we introduce an algorithm for checking levelness of order polytopes. First, we need to associate a weighted digraph to a poset Π together with a subposet $\Pi' \subset \overline{\Pi}$, where we require that $i <_{\Pi'} j$ implies $i <_{\Pi} j$.

DEFINITION 4.1. *Let Π be a finite poset and let Π' be a subposet of $\overline{\Pi}$ such that $i <_{\Pi'} j$ implies $i <_{\Pi} j$. Let $\Gamma(\Pi, \Pi') = (\overline{\Pi}, E)$ be the weighted digraph with weighted, directed edges:*

1. $(i, j, -1) \in E$ if and only if $j >_{\overline{\Pi}} i$;
2. $(j, i, 1) \in E$ if and only if $j >_{\Pi'} i$.

Clearly, $\Gamma(\Pi, \overline{\Pi}) = H(\Pi)$. If Π is clear from the context, we will write $\Gamma(\Pi')$.

A *negative cycle* is a directed cycle whose sum of weights is negative. A *wedge of cycles* is a closed directed path (where repetition of vertices is allowed) whose sum of weights is negative.

In the following, we will be especially interested in integer points in the interior of $\text{cone}(\mathcal{O}(\Pi))$. These points lie in a polyhedron defined by the inequalities $x_i \geq 1$, $x_j \geq x_i + 1$ for every $j > i$, and $x_{\infty} \geq x_k + 1$ for all $\infty > k$. We call an integer point $\mathbf{x} \in \text{int}(\text{cone}(\mathcal{O}(\Pi))) \cap \mathbb{Z}^{d+1}$ *sharp* along a covering pair $j > i$ if $x_j = x_i + 1$.

Next, we associate a weighted digraph to every integer point in $\text{int}(\text{cone}(\mathcal{O}(\Pi)))$. The following lemma shows that the associated digraph does not have any negative cycles.

LEMMA 4.2. *Let $\mathbf{b} \in \text{int}(\text{cone}(\mathcal{O}(\Pi))) \cap \mathbb{Z}^{d+1}$ be given. Then the weighted digraph $\Gamma_{\mathbf{b}}$ whose nodes are given by the elements of $\overline{\Pi}$ and whose weighted, directed edges are*

- $\{(i, j, -1) : i < j\} \cup \{(j, i, 1) : i < j, b_j - b_i = 1\}$ for $1 \leq i, j \leq d$,
- $\{(-\infty, i, -1) : i > -\infty\} \cup \{(i, -\infty, 1) : i > -\infty, b_i = 1\}$,
- and $\{(i, \infty, -1) : \infty > i\} \cup \{(\infty, i, 1) : \infty > i, b_i = \max_j b_j\}$.

does not have any negative cycles. In particular, every subgraph contains no negative cycles.

Proof. For $i, j \in \overline{\Pi}$, let $u(i, j)$ denote a directed path from i to j where $i < j$ and let $d(l, k)$ denote a directed path from l to k where $l > k$. Let $u(i_1, i_2)$, $d(i_2, i_3)$, $u(i_3, i_4)$, $\dots$, $d(i_s, i_1)$ be a directed cycle in $\Gamma_{\mathbf{b}}$ with $i_1 < i_2$, $i_3 < i_2$, $\dots$, $i_1 < i_s$. We first remark that the weight of any down-path $d(i, i+1)$ with vertices $b_i = b^{(0)} > b^{(1)} > \dots > b^{(r)} > b^{(r+1)} = b_{i+1}$ are given by

$$\overbrace{b_i - b^{(1)}}^{=1} + \overbrace{b^{(1)} - b^{(2)}}^{=1} + \dots + b^{(r)} + \overbrace{b^{(r)} - b_{i+1}}^{=1} = b_i - b_{i+1},$$

where we set $b_{\infty} = \max_k b_k + 1$ and $b_{-\infty} = 0$. Therefore, the sum of the weights in the cycle is equal to

$$\begin{aligned} & (b_{i_2} - b_{i_3}) + \dots + (b_{i_s} - b_{i_1}) - \text{length}(i_1, i_2) - \dots - \text{length}(i_{s-1}, i_s) \\ &= (b_{i_2} - b_{i_1}) + \dots + (b_{i_s} - b_{i_{s-1}}) - \text{length}(i_1, i_2) - \dots - \text{length}(i_{s-1}, i_s) \geq 0, \end{aligned}$$

since $\mathbf{b} \in \text{int}(\text{cone}(\mathcal{O}(\Pi)))$ implies that $b_j - b_i \geq \text{length}(i, j)$ for all $i < j$, where $\text{length}(i, j)$ is the number of edges of the path from i to j . $\square$

The following theorem uses the *Bellman–Ford algorithm*, which was introduced by Bellman and Ford; see, for instance, [2]. We are using this algorithm as a black box. Instead of explicitly describing it, we will merely state some basic facts about it:

- The Bellman–Ford algorithm finds the shortest path from a sink to any other node in a weighted digraph. In contrast to other algorithms, it can also deal with negative weights assuming that the digraph does not contain any negative cycles (that can be reached from the starting node); see [16, Thm. 8.5].
- If there is such a negative cycle, the Bellman–Ford algorithm can detect the negative cycle [16, Thm. 8.6].
- The Bellman–Ford algorithm runs in $O(\#V \cdot \#E)$, where V is the vertex set and E is the edge set of the underlying graph; see [16, Thm. 8.5].

Given a path $\mathcal{P}$ in $H(\Pi)$ with condition N , let $\Pi'(\mathcal{P})$ be the subposet of $\overline{\Pi}$ whose covering pairs are given by the up-paths of $\mathcal{P}$. Then $\Gamma(\Pi'(\mathcal{P}))$ has no negative cycles. Now, we give a new characterization of level order polytopes.

THEOREM 4.3. *Let Π be a finite poset on d elements. Then Π is level if and only if for any path $\mathcal{P}$ in $H(\Pi)$ with condition N , $\Gamma(\Pi'(\mathcal{P}) \cup \{\text{longest chains in } \overline{\Pi}\})$ has no negative cycles.*

Proof. Let $r := \text{codeg}(\mathcal{O}(\Pi))$ be the codegree of $\mathcal{O}(\Pi)$. Let us assume Π is level and let $\mathcal{P}$ be a path in $H(\Pi)$ with condition N . We prove that $\Gamma(\Pi'(\mathcal{P}) \cup \{\text{longest chains in } \overline{\Pi}\})$ has no negative cycles by verifying the following two claims:

1. There is an integer point $\mathbf{b} \in \text{int}(\text{cone}(\mathcal{O}(\Pi))) \cap \mathbb{Z}^{d+1}$ associated to $\Gamma(\Pi'(\mathcal{P}))$ such that $\Gamma(\Pi'(\mathcal{P})) \subset \Gamma_{\mathbf{b}}$.
2. Since $\mathcal{O}(\Pi)$ is level, there is a point $\tilde{\mathbf{b}} \in \text{int}(\text{cone}(\mathcal{O}(\Pi))) \cap \mathbb{Z}^{d+1}$ of height r such that $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ is a subgraph of $\Gamma_{\tilde{\mathbf{b}}}$.

To prove claim 1, let us run the Bellman–Ford algorithm on $\Gamma(\Pi'(\mathcal{P}))$ with starting node $-\infty$. The Bellman–Ford algorithm minimizes the distance c_i from $-\infty$ to $i \in \overline{\Pi}$. Since all up-weights are negative and the distance is minimal, all c_i 's are negative. Moreover, for any covering pair $j > i$ we have $c_j \leq c_i - 1$, since there is a path from $-\infty$ to i and from i to j using the edge $(i, j, -1)$. This inequality is in general not sharp, as there might be a path from $-\infty$ to j of smaller total weight. However, if there is also an edge $(j, i, 1)$, we get the inequality $c_i \leq c_j + 1$, since there is a path

of total weight $c_j + 1$ from $-\infty$ to i via j using the edge $(j, i, 1)$. Combining these inequalities yields $c_j = c_i - 1$ whenever there is an edge $(j, i, 1)$. After multiplying all entries of $\mathbf{c}$ by -1 , we get a point $\mathbf{b} := (-c_i)_{i \in \overline{\Pi}}$ such that for all covering pairs $j \succ i$ in $\overline{\Pi}$, we have $b_j \geq b_i + 1$. This together with the fact that $b_i \geq 1$ for all $i \in \overline{\Pi}$ implies $\mathbf{b} \in \text{int}(\text{cone}(\mathcal{O}(\Pi))) \cap \mathbb{Z}^{d+1}$. Furthermore, from our reasoning above we see that $b_j = b_i + 1$ for any weighted edge $(j, i, 1)$ in $\Gamma(\Pi'(\mathcal{P}))$ and hence $\Gamma(\Pi'(\mathcal{P})) \subset \Gamma_{\mathbf{b}}$. This has the following geometric interpretation: The interior lattice points of $\text{cone}(\mathcal{O}(\Pi))$ are in the polyhedron defined by the inequalities $x_i \geq 1$, $x_j \geq x_i + 1$ for all $j \succ i$, and $x_\infty - x_k \geq 1$ for all $\infty \succ k$. Thus, whenever there is a down-edge $(j, i, 1)$ the point $\mathbf{b}$ is on the face defined by $x_j = x_i + 1$.

It remains to prove claim 2. Since Π is level, there exists a point $\tilde{\mathbf{b}} \in \text{int}(\text{cone}(\mathcal{O}(\Pi))) \cap \mathbb{Z}^{d+1}$ on height r such that $\mathbf{b} - \tilde{\mathbf{b}} \in \text{cone}(\mathcal{O}(\Pi)) \cap \mathbb{Z}^{d+1}$. This implies that for every covering pair $j \succ i$ in $\overline{\Pi}$, we have

$$b_j - b_i \geq \tilde{b}_j - \tilde{b}_i.$$

Hence for any weighted edge $(j, i, 1)$ in $\Gamma(\Pi'(\mathcal{P}))$, we have $\tilde{b}_j - \tilde{b}_i = 1$. Since $\tilde{\mathbf{b}}$ is on height r , we also know that $\tilde{\mathbf{b}}$ is sharp along the longest chains in $\overline{\Pi}$. Since $\mathcal{G} := \Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ is a subgraph of $\Gamma_{\tilde{\mathbf{b}}}$, using Lemma 4.2, we get that $\mathcal{G}$ does not contain a negative cycle.

We prove the other direction by contraposition. Let us assume that Π is not level. Then there exists a path $\mathcal{P}$ with condition N such that $r(\mathcal{P}) = r_{\max} > r$. Moreover, $\Gamma(\Pi'(\mathcal{P}))$ is a subgraph of $\Gamma_{\mathbf{y}(\mathcal{P})}$. Hence by using Lemma 4.2, it follows that $\Gamma(\Pi'(\mathcal{P}))$ has no negative cycle. On the other hand, since $\text{rank}(\overline{\Pi}) = r$, it follows that $\Gamma(\Pi'(\mathcal{P}) \cup \{\text{longest chains in } \overline{\Pi}\})$ has a negative cycle. $\square$

This directly implies the following.

COROLLARY 4.4. *Let Π be a finite poset. Then Π is level if and only if for all $\Gamma(\Pi')$ that do not have a negative cycle, the digraph $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ does not have a negative cycle.*

Proof. One direction directly follows from Theorem 4.3, so we only need to show that if Π is level and if $\Gamma(\Pi')$ does not contain a negative cycle, then $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ does not have a negative cycle. However, this follows from the proof of Theorem 4.3. $\square$

Remark 4.5. For practical purposes this corollary is more convenient than the previous characterization. This is due to the fact that it is hard to determine (all) paths with condition N . We will use this corollary to give an infinite family of level posets; see Theorem 6.2.

Moreover, we get that—given the input Π —determining the levelness of order polytopes is in co-NP, i.e., the complexity class having a short certificate for rejection. For more about complexity classes, we refer to [15, sect. 2.5].

COROLLARY 4.6. *Levelness of order polytopes is in co-NP.*

Proof. If $\mathcal{O}(\Pi)$ is not level, then there exists a short certificate $\Pi'(\mathcal{P})$ such that $\Gamma(\Pi'(\mathcal{P}))$ does not have a negative cycle but $\Gamma(\Pi'(\mathcal{P}) \cup \{\text{longest chains in } \overline{\Pi}\})$ has a negative cycle. This can be tested by the Bellman–Ford algorithm in polynomial time, since we need to run the Bellman–Ford algorithm twice, once for $\Gamma(\Pi'(\mathcal{P}))$ and once for $\Gamma(\Pi'(\mathcal{P}) \cup \{\text{longest chains in } \overline{\Pi}\})$. Therefore, we can verify nonlevelness in polynomial time. $\square$

We now explicitly describe the algorithm underlying Corollary 4.4.

Algorithm 4.7

For $\Gamma(\Pi') \subset H(\Pi)$:
 Run Bellman–Ford for $\Gamma(\Pi')$
 If negative cycle:
 Do nothing
 Else:
 Run Bellman–Ford for $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$
 If negative cycle:
 Return NOT LEVEL
 Else:
 Do nothing
 Return LEVEL

THEOREM 4.8. *A poset Π is level if and only if Algorithm 4.7 returns level.*

Proof. This directly follows from Corollary 4.4. $\square$

5. A necessary condition of Ene, Herzog, Hibi, and Saeedi Madani. We now want to show that [6, Thm. 4.1] is a special case of Corollary 4.4. We first need to define the depth and the height of an element, where we follow again [6]. The *height of an element* $i \in \Pi$, denoted $\text{height}(i)$, is the maximum length of a chain in $\overline{\Pi}$ descending from i . Similarly, we define the *depth of an element* i , denoted $\text{depth}(i)$, to be the maximum length of a chain in $\overline{\Pi}$ ascending from i .

The authors of [6] show that the following is a necessary condition for levelness.

THEOREM 5.1 (see [6, Thm. 4.1]). *Suppose Π is level. Then*

$$(5.1) \quad \text{height}(j) + \text{depth}(i) \leq \text{rank}(\overline{\Pi}) + 1$$

for all $j \succ i \in \Pi$.

Our next result shows that this is weaker than Corollary 4.4. In fact, it is equivalent to Corollary 4.4, where Π' is a single edge.

THEOREM 5.2. *Let Π be a finite poset on d elements and let $r = \text{codeg}(\mathcal{O}(\Pi))$. The following are equivalent:*

1. *inequality (5.1) is satisfied by all covering pairs,*
2. *for all $\Gamma(\Pi')$ with only a single down-edge, the digraph $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ does not have a negative cycle.*

Remark 5.3. We remark that claim 2 is equivalent to the statement that for all Hasse edges $j \succ i \in \Pi$ there is an integer point $\mathbf{x} \in r \text{int}(\mathcal{O}(\Pi))$ such that $x_j = x_i + 1$, as we will now prove. For the first direction, let $\Gamma(\Pi')$ have only a single down-edge $(j, i, 1)$ and assume that the digraph $\mathcal{G} := \Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ does not have a negative cycle. Similar to the proof of Theorem 4.3, running the Bellman–Ford algorithm on $\mathcal{G}$ gives rise to a point $\mathbf{b} \in \text{int}(\text{cone}(\mathcal{O}(\Pi))) \cap \mathbb{Z}^{d+1}$ such that $b_k = b_l + 1$ for every $(k, l, 1) \in \mathcal{G}$. In particular, $b_j = b_i + 1$. Since the length of the longest chain equals r and since $b_k = 1$ for every $k \succ -\infty$, this implies that $b_\infty = r$. In other words, $\mathbf{b}$ is an integer point in $\text{int}(\text{cone}(\mathcal{O}(\Pi)))$ on height r , which is equivalent to saying that $(b_i)_{i \in \Pi} \in r \text{int}(\mathcal{O}(\Pi)) \cap \mathbb{Z}^d$.

To show the other direction, let $\Gamma(\Pi)$ have a single down edge $(j, i, 1)$ and let $\mathbf{x} \in r \operatorname{int}(\mathcal{O}(\Pi)) \cap \mathbb{Z}^d$ satisfying $x_j = x_i + 1$. The integer point $\mathbf{x}$ is in bijection with an integer point $\tilde{\mathbf{x}} \in \operatorname{int}(\operatorname{cone}(\mathcal{O}(\Pi)))$ where

$$\tilde{x}_i = \begin{cases} r & \text{if } i = \infty, \text{ and} \\ x_i & \text{otherwise.} \end{cases}$$

Then the graph $\Gamma_{\tilde{\mathbf{x}}}$ has no negative cycles by Lemma 4.2, and $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ is a subgraph thereof. Therefore, it cannot have a negative cycle.

Proof. Let us assume that all covering pairs satisfy (5.1) and fix a covering pair $j \succ i$. As mentioned in Remark 2.8, $\operatorname{rank}(\overline{\Pi})$ equals the codegree r of the order polytope $\mathcal{O}(\Pi)$. Following Remark 5.3, we need to show that there exists an integer point $\mathbf{x} \in r \operatorname{int}(\mathcal{O}(\Pi))$ such that $x_j = x_i + 1$. To create such an $\mathbf{x}$, we can label the elements in Π using labels from $\{1, 2, \dots, r-1\}$. We first label $x_j = \operatorname{height}(j)$ and hence $x_i = \operatorname{height}(j) - 1$. For $k \in \Pi \setminus \{i, j\}$ we label x_k by

$$(5.2) \quad x_k = \begin{cases} \max\{\operatorname{height}(k), x_i + \operatorname{rank}[i, k]\} & \text{if } k \succ i, \\ \operatorname{height}(k) & \text{otherwise.} \end{cases}$$

To show that this indeed gives an interior integer point in $r \operatorname{int} \mathcal{O}(\Pi)$, we need to show that $r > x_k \geq \operatorname{height}(k)$ for all k , since this ensures that $x_k \geq x_l + 1$ for all $k \succ l$ and that $x_k < r$ which are exactly the inequalities describing $r \operatorname{int} \mathcal{O}(\Pi)$. We say a label x_k is well-defined if it satisfies this condition. There are two cases:

1. $k \succ i$, then (5.1) ensures that (5.2) only yields well-defined labels;
2. $k \not\succ i$, then the recursive definition gives us $\operatorname{height}(k)$, which by definition is well-defined.

This proves the first direction.

Now let assume that for all Hasse edges $(j \succ i)$ in Π there exists an integer point $\mathbf{x} \in r \operatorname{int}(\mathcal{O}(\Pi))$ such that $x_j = x_i + 1$. Let us fix a covering pair $j \succ i$. Then we have an integer point $\mathbf{x} \in r \operatorname{int}(\mathcal{O}(\Pi))$ with $x_j = x_i + 1$ and it follows that $\operatorname{height}(j) \leq x_j$. Since we have an integer point in the interior of $r\mathcal{O}(\Pi)$, we also get that

$$\operatorname{depth}(i) \leq \operatorname{rank}(\overline{\Pi}) - x_i.$$

Putting everything together, we obtain

$$\operatorname{height}(j) + \operatorname{depth}(i) \leq \operatorname{rank}(\overline{\Pi}) + 1,$$

as desired. $\square$

However, this result is not sufficient, as the following example, which is due to Alex Fink, shows.

Remark 5.4. Let Π be the poset from Figure 2(a). Then $\operatorname{codeg}(\mathcal{O}(\Pi)) = 5$, since the length of a longest chain in $\overline{\Pi}$ equals 5. Moreover, for any covering pair $i \prec j$ in Π , there is a minimal element $\mathbf{x}$ on height 5 with $x_i + 1 = x_j$, which can quickly be checked by hand. Thus, by Theorem 5.2 the condition of [6, Thm. 4.1] is satisfied. However, the element $\mathbf{y}(9, 7, 5, 3)$ is on height 6 and minimal by Lemma 3.5; see Figure 2(b). Therefore, Π cannot be level by Lemma 3.7.

FIG. 3. The ordinal sum of a chain of length 3 and an antichain of length 2.

6. Series-parallel posets. The goal of this section is to describe a new family of level posets. The main character of this section is the *ordinal sum*; see [21, sect. 3.2]. Let Π_1 and Π_2 be two posets. Then their ordinal sum $\Pi_1 \triangleleft \Pi_2$ is the poset with elements from the union $\Pi_1 \cup \Pi_2$ and with relations $s \leq t$ if

- $s, t \in \Pi_1$ with $s \leq_{\Pi_1} t$, or
- $s, t \in \Pi_2$ with $s \leq_{\Pi_2} t$, or
- $s \in \Pi_1$ and $t \in \Pi_2$.

Posets that can be built up as ordinal sums of posets are called *series-parallel posets*.

Remark 6.1. For a poset Π , let $\mathbb{k}[\Pi] := \mathbb{k}[\mathcal{O}(\Pi)]$. The tensor product $\mathbb{k}[\Pi_1] \otimes_{\mathbb{k}} \mathbb{k}[\Pi_2]$ can be expressed in terms of series-parallel posets. Namely, one has $\mathbb{k}[\Pi_1] \otimes_{\mathbb{k}} \mathbb{k}[\Pi_2] = \mathbb{k}[\Pi_1 \triangleleft (\{1\}, \leq) \triangleleft \Pi_2]$, where $(\{1\}, \leq)$ is a one-element poset. It is a well-known fact that the tensor product $\mathbb{k}[\Pi_1] \otimes_{\mathbb{k}} \mathbb{k}[\Pi_2]$ is level if and only if every component is level. However, this tensor product is *not* the same as the Ehrhart ring $\mathbb{k}[\Pi_1 \triangleleft \Pi_2]$.

The following result characterizes levelness of series-parallel posets.

THEOREM 6.2. *The ordinal sum $\Pi = \Pi_1 \triangleleft \Pi_2$ of two posets Π_1, Π_2 is level if and only if both Π_1 and Π_2 are level.*

Proof. We prove the first direction by contraposition. So let us assume that Π_1 is not level. By Corollary 4.4, there exists a weighted digraph $\Gamma(\Pi'_{\Pi_1})$ with nodes coming from $\overline{\Pi_1}$ which does not contain a negative cycle, but the weighted directed graph $\Gamma(\Pi'_{\Pi_1} \cup \{\text{longest chains in } \overline{\Pi_1}\})$ has a negative cycle. However, we also get that $\Gamma(\Pi')$ has a weighted digraph with up-edges of weight 1 only coming from up-edges of $\Gamma(\Pi'_{\Pi_1})$ which does not contain a negative cycle, but $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ contains one, proving that $\Pi_1 \triangleleft \Pi_2$ is not level. The case where Π_2 is not level follows analogously.

We prove the other direction again by contraposition. So let us assume that $\Pi_1 \triangleleft \Pi_2$ is not level. By Corollary 4.4, there exists a weighted digraph $\Gamma(\Pi')$ with nodes coming from $\overline{\Pi}$ such that $\Gamma(\Pi')$ does not have a negative cycle and $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ has a negative cycle. In order to show that either Π_1 or Π_2 is not level, we will construct graphs $\Gamma(\Pi'_{\Pi_1})$ and $\Gamma(\Pi'_{\Pi_2})$ without negative cycles such that $\Gamma(\Pi'_{\Pi_1} \cup \{\text{longest chains in } \overline{\Pi_1}\})$ or $\Gamma(\Pi'_{\Pi_2} \cup \{\text{longest chains in } \overline{\Pi_2}\})$ have negative cycles. The following two quotient maps will be essential for this:

$$\begin{aligned} \overline{\Pi_1 \triangleleft \Pi_2} &\xrightarrow{q_1} \overline{\Pi_1 \triangleleft \Pi_2} / (p_2 \sim p'_2 \sim \infty) \cong \overline{\Pi_1}, \\ \overline{\Pi_1 \triangleleft \Pi_2} &\xrightarrow{q_2} \overline{\Pi_1 \triangleleft \Pi_2} / (p_1 \sim p'_1 \sim -\infty) \cong \overline{\Pi_2}, \end{aligned}$$

where $p_1, p'_1 \in \Pi_1$ and $p_2, p'_2 \in \Pi_2$.

Note that these quotient maps also induce weighted directed graphs $\Gamma(\Pi'_{\Pi_1})$ and $\Gamma(\Pi'_{\Pi_2})$ on the underlying posets $\overline{\Pi_1}$ and $\overline{\Pi_2}$, respectively. We will show the following:

1. Both $\Gamma(\Pi'_{\Pi_1})$ and $\Gamma(\Pi'_{\Pi_2})$ do not have a negative cycle.
2. Either $\Gamma(\Pi'_{\Pi_1} \cup \{\text{longest chains in } \overline{\Pi_1}\})$ or $\Gamma(\Pi'_{\Pi_2} \cup \{\text{longest chains in } \overline{\Pi_2}\})$ or both have a negative cycle.

This implies that either Π_1 or Π_2 or both cannot be level, proving the claim. The first claim follows by contraposition. If either $\Gamma(\Pi'_{\Pi_1})$ or $\Gamma(\Pi'_{\Pi_2})$ had a negative cycle, then one can *lift* this cycle to obtain a negative cycle in $\Gamma(\Pi')$. This is due to the fact that every maximal element in Π_1 is comparable to every minimal element in Π_2 , together with the fact that every up-edge has the same weight, namely -1 .

Now let us prove the second claim. We remark that maximal chains $-\infty < p_1 < \dots < p_{r_1} < q_1 < \dots < q_{r_2} < \infty$ in $\overline{\Pi_1 \triangleleft \Pi_2}$ can be obtained from a longest chain $-\infty < p_1 < \dots < p_{r_1} < \infty$ in $\overline{\Pi_1}$ and a longest chain $-\infty < q_1 < \dots < q_{r_2} < \infty$ in $\overline{\Pi_2}$ and vice versa. This means that

$$\text{im}_{q_1}(\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi_1 \triangleleft \Pi_2}\})) = \text{im}_{q_1}(\Gamma(\Pi')) \cup \{\text{longest chains in } \overline{\Pi_1}\}$$

and

$$\text{im}_{q_2}(\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi_1 \triangleleft \Pi_2}\})) = \text{im}_{q_2}(\Gamma(\Pi')) \cup \{\text{longest chains in } \overline{\Pi_2}\},$$

where im_{q_1} (or im_{q_2}) denotes the image of the quotient map onto $\overline{\Pi_1}$ (or $\overline{\Pi_2}$) with all loop edges removed. Moreover, if a negative cycle of $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi_1 \triangleleft \Pi_2}\})$ is entirely contained in either $q_1^{-1}(\overline{\Pi_1}) \cup \{\text{minimal elements of } \Pi_2\}$ or $q_2^{-1}(\overline{\Pi_2}) \cup \{\text{maximal elements of } \Pi_1\}$, then clearly the image also has a negative cycle. (Caveat: After applying the quotient map, the cycle might become a wedge of cycles. But since the total weight of the original cycle is the sum of the weights of the cycle in the image, at least one of these cycles in the wedge has to be negative.)

So we only need to consider the case where a negative cycle contains edges which are contained in Π_1 and in Π_2 . We can divide the cycle into a part, whose edges are entirely in $q_1^{-1}(\overline{\Pi_1}) \cup \{\text{minimal elements of } \Pi_2\}$, and a part whose edges are entirely in $q_2^{-1}(\overline{\Pi_2}) \cup \{\text{maximal elements of } \Pi_1\}$. Note that the edges between Π_1 and Π_2 appear in both parts. Therefore, the total weight w of the cycle equals

$$0 > w = w_{\overline{\Pi_1}} + w_{\overline{\Pi_2}} - w_{\Pi_1 \Pi_2},$$

where $w_{\overline{\Pi_1}}$ and $w_{\overline{\Pi_2}}$ are the weights of the parts in $q_1^{-1}(\overline{\Pi_1}) \cup \{\text{minimal elements of } \Pi_2\}$ and $q_2^{-1}(\overline{\Pi_2}) \cup \{\text{maximal elements of } \Pi_1\}$, respectively. The weight of the connecting edges between Π_1 and Π_2 is denoted $w_{\Pi_1 \Pi_2}$. This weight is 0, since there are as many up- as there are down-edges and the weights are -1 and 1 , respectively. Therefore, either $w_{\overline{\Pi_1}}$ or $w_{\overline{\Pi_2}}$ or both are negative. If $w_{\overline{\Pi_1}}$ is negative, applying the quotient map gives us a wedge of cycles in $\overline{\Pi_1}$ with negative weight. Hence it contains at least one negative cycle. The case where $w_{\overline{\Pi_2}}$ is negative is similar. Therefore, we have seen that either Π_1 or Π_2 is not level, proving the claim. $\square$

For the remainder of this section, let $\Pi = \Pi_1 \triangleleft \Pi_2$. We will first give a geometric description of the order polytope and the chain polytope of Π in terms of the order and chain polytopes of Π_1 and Π_2 , respectively.

LEMMA 6.3. *Let $\mathcal{C}(\Pi)$, $\mathcal{C}(\Pi_1)$, $\mathcal{C}(\Pi_2)$ be the chain polytopes of Π , Π_1 , and Π_2 , respectively. Then*

$$(6.1) \quad \mathcal{C}(\Pi) = \text{conv}\{\mathcal{C}(\Pi_1) \times \mathbf{0}_{\Pi_2} \cup \mathbf{0}_{\Pi_1} \times \mathcal{C}(\Pi_2)\} = \mathcal{C}(\Pi_1) \oplus \mathcal{C}(\Pi_2),$$

where $\oplus$ is the free sum of $\mathcal{C}(\Pi_1)$ and $\mathcal{C}(\Pi_2)$.

Proof. By Theorem 2.5, the vertices of the chain polytope are given by the indicator vectors of antichains. Now one notices that no antichain can contain elements from both Π_1 and Π_2 . $\square$

LEMMA 6.4. *Let $\mathcal{O}(\Pi), \mathcal{O}(\Pi_1), \mathcal{O}(\Pi_2)$ be the order polytopes of Π, Π_1, Π_2 , respectively. Then*

$$(6.2) \quad \mathcal{O}(\Pi) = \text{conv}\{\mathbf{1}_{\Pi_2} \times \mathcal{O}(\Pi_1) \cup \mathcal{O}(\Pi_2) \times \mathbf{0}_{\Pi_1}\}.$$

Proof. By Corollary 2.4, the vertices of the order polytope are given by the indicator vectors of filters. Now one notices that as soon as a filter contains an element of Π_1 , it contains *all* elements of Π_2 . $\square$

Moreover, we have the following.

PROPOSITION 6.5. *Let $h_{\Pi}^*, h_{\Pi_1}^*, h_{\Pi_2}^*$ be the h^* -polynomials of the order polytopes of Π_1, Π_2 , and Π , respectively. Then*

$$h_{\Pi}^* = h_{\Pi_1}^* h_{\Pi_2}^*.$$

Proof. By Theorem 2.6 the Ehrhart series of the chain polytope of a poset Π is the same as the Ehrhart series of the order polytope of Π . In [10, Lem. 3.2], Hibi and Higashitani show that if the free sum, $P \oplus Q$, of two lattice polytopes P, Q both containing the origin has the integer decomposition property, then $h_{P \oplus Q}^* = h_P^* h_Q^*$. Note that every chain polytope and every order polytope has a unimodular triangulation and thus has the integer decomposition property. Using Lemma 6.3 together with [10, Lem. 3.2] implies the result. $\square$

Richard Stanley [18] showed that the Gorenstein property of a Cohen–Macaulay graded domain can be characterized in terms of the h -polynomial of its Hilbert series. On the other hand, in [9], Takayuki Hibi gave an example of two Cohen–Macaulay graded domains R_1 and R_2 with the same h -polynomial, where R_1 is level and R_2 is not level. This shows that the level property cannot be characterized by the h -polynomial. Now, we give an example of two order polytopes $\mathcal{O}(\Pi_1), \mathcal{O}(\Pi_2)$ that have the same h^* -polynomial, but where Π_1 is level and where Π_2 is not level.

Example 6.6. Let Π_1 and Π_2 be the posets from Figure 4. Then Π_1 is level. On the other hand, Π_2 is not level. However, the order polytopes $\mathcal{O}(\Pi_1)$ and $\mathcal{O}(\Pi_2)$ have the same h^* -polynomial $1 + 6z + 9z^2 + 2z^3$.

With this example and the following corollary, we can create infinitely many examples of two order polytopes $\mathcal{O}(\Pi_1)$ and $\mathcal{O}(\Pi_2)$ where both have the same h^* -polynomial, but Π_1 is level and Π_2 is not level.

FIG. 4. An example of a level poset and a nonlevel poset.

COROLLARY 6.7. *Let Π_1 and Π_2 be posets such that*

- $\mathcal{O}(\Pi_1)$ and $\mathcal{O}(\Pi_2)$ have the same h^* -polynomial,
- Π_1 is level,
- Π_2 is not level.

Then for any level poset Π_3 , it follows that

- $\mathcal{O}(\Pi_1 \triangleleft \Pi_3)$ and $\mathcal{O}(\Pi_2 \triangleleft \Pi_3)$ have the same h^* -polynomial,
- $\Pi_1 \triangleleft \Pi_3$ is level,
- $\Pi_2 \triangleleft \Pi_3$ is not level.

Proof. This follows from Theorem 6.2 and Proposition 6.5. $\square$

7. Connected components of level posets. In this section, we discuss connected components of level posets. Since a poset is Gorenstein if and only if every maximal chain has the same length [8], any connected component of a Gorenstein poset is Gorenstein. This fact naturally leads us to examine whether any connected component of a level poset is level. However, this is not true in general. From the following result we know that there exists a level poset which has a connected component that is not level.

THEOREM 7.1 (see [6, Thm. 4.7]). *Let Π be a poset on d elements and let C_s be a totally ordered set with s elements. Then the poset on the set $\Pi \cup C_s$, where elements from Π and C_s are incomparable, is level for all $s \gg 0$.*

We give an explicit bound for s appearing in Theorem 7.1.

THEOREM 7.2. *Let Π be a poset on d elements and let C_s be a totally ordered set with s elements. Then the poset on the set $\Pi \cup C_s$, where elements from Π and C_s are incomparable, is level for all $s \geq d$.*

In order to prove this theorem, we consider a more general class of lattice polytopes containing order polytopes.

DEFINITION 7.3. *We say that a polytope $P \subset \mathbb{R}^d$ is alcoved if P is an intersection of some halfspaces bounded by the hyperplanes*

$$H_{i,j}^m = \{(z_1, \dots, z_d) \in \mathbb{R}^d : z_i - z_j = m\} \text{ for } 0 \leq i < j \leq d, m \in \mathbb{Z},$$

where $z_0 = 0$.

By definition, order polytopes are alcoved. After a unimodular change of coordinates, every chain polytope is alcoved, too. Furthermore, any alcoved polytope possesses the integer decomposition property. For a lattice polytope $P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\} \subset \mathbb{R}^d$, we set $P^{(1)} = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b} - 1\}$.

Remark 7.4. If $P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\} \subset \mathbb{R}^d$ is an alcoved polytope for some $m \times d$ integer matrix A and some integer vector $\mathbf{b} \in \mathbb{Z}^m$, then A is a totally unimodular matrix, and $P^{(1)}$ is a lattice polytope. In particular, one has $P^{(1)} = \text{conv}(\text{int}(P) \cap \mathbb{Z}^d)$.

We record the following simple fact without proof.

LEMMA 7.5. *Let $P \subset \mathbb{R}^d$ be an alcoved polytope. Then for any positive integer k , kP and $P^{(1)}$ are alcoved.*

For two lattice polytopes P and Q in $\mathbb{R}^d$, set

$$\text{Cayley}(P, Q) = \text{conv}(P \times \{0\} \cup Q \times \{1\}) \subset \mathbb{R}^{d+1}.$$

We say that $\text{Cayley}(P, Q)$ is the *Cayley polytope* of P and Q .

LEMMA 7.6. *Let P and Q be alcoved polytopes in $\mathbb{R}^d$. Then $\text{Cayley}(P, Q)$ has a regular unimodular triangulation. In particular, $\text{Cayley}(P, Q)$ has the integer decomposition property.*

Proof. This is [7, Lem. 4.15], since alcoved polytopes have a type A root system. $\square$

This directly implies the following result.

COROLLARY 7.7. *If $P, Q \subset \mathbb{R}^d$ are alcoved polytopes, then the map*

$$(P \cap \mathbb{Z}^d) \times (Q \cap \mathbb{Z}^d) \rightarrow (P + Q) \cap \mathbb{Z}^d$$

is onto.

Proof. We have that

$$2 \text{Cayley}(P, Q) \cap \{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} : x_{d+1} = 1\} = (P + Q) \times \{1\}.$$

Since $\text{Cayley}(P, Q)$ has the integer decomposition property, it follows that every integer point in $2 \text{Cayley}(P, Q)$ can be written as a sum of two integer points in $\text{Cayley}(P, Q)$. However, the only way we can get an integer point at height 1 is if we add one integer point at height 0 and at height 1, i.e., one integer point belongs to P and one belongs to Q , proving the claim. $\square$

Now, we give a characterization on levelness of alcoved polytopes.

PROPOSITION 7.8. *Let $P \subset \mathbb{R}^d$ be an alcoved polytope and let $r = \text{codeg}(P)$. Then P is level if and only if for any integer $k \geq r$, it follows that $(kP)^{(1)} = (rP)^{(1)} + (k - r)P$.*

Proof. First, assume that P is level. Then from the definition of levelness, for any integer $k \geq r$, $\text{int}(kP) \cap \mathbb{Z}^d = \text{int}(rP) \cap \mathbb{Z}^d + (k - r)P \cap \mathbb{Z}^d$. Hence one has

$$\begin{aligned} (kP)^{(1)} &= \text{conv}(\text{int}(rP) \cap \mathbb{Z}^d + (k - r)P \cap \mathbb{Z}^d) \\ &= \text{conv}(\text{int}(rP) \cap \mathbb{Z}^d) + \text{conv}((k - r)P \cap \mathbb{Z}^d) \\ &= (rP)^{(1)} + (k - r)P. \end{aligned}$$

Conversely, assume that for any integer $k \geq r$, $(kP)^{(1)} = (rP)^{(1)} + (k - r)P$. By Lemma 7.5, $(kP)^{(1)}$, $(rP)^{(1)}$, and $(k - r)P$ are alcoved polytopes. Hence by Corollary 7.7, P is level. $\square$

LEMMA 7.9. *Let $P \subset \mathbb{R}^d$ be a lattice polytope and let $r' \geq \text{codeg}(P)$ be an integer. Assume that there exists an integer $k > r'$ such that $(kP)^{(1)} = (r'P)^{(1)} + (k - r')P$. Then for any integer $k > k' \geq r'$, we have $(k'P)^{(1)} = (r'P)^{(1)} + (k' - r')P$.*

Proof. We note that the containment $(k'P)^{(1)} \supset (r'P)^{(1)} + (k' - r')P$ always holds. Assume that there exists an integer $k > k' \geq r'$ such that $(k'P)^{(1)} \supsetneq (r'P)^{(1)} + (k' - r')P$. Then we have

$$(kP)^{(1)} \supset (k'P)^{(1)} + (k - k')P \supsetneq (r'P)^{(1)} + (k - r')P = (kP)^{(1)}.$$

Hence this is a contradiction. $\square$

On levelness of dilated polytopes, the following theorem is known.

THEOREM 7.10 (see [4, Thm. 1.3.3]). *Let P be a lattice d -polytope. Then for any integer $k \geq d + 1$, kP is level of codegree 1.*

Now, we prove the following theorem about levelness of a product of alcoved polytopes.

THEOREM 7.11. *Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be alcoved polytopes. Suppose that Q is level and $r = \text{codeg}(Q) \geq \dim P + 1$. Then $P \times Q \subset \mathbb{R}^{d+e}$ is level.*

Proof. By Theorem 7.10, rP is level. Hence since rP is alcoved and the codegree of rP is 1 for any positive integer k' , one has $(k'rP)^{(1)} = (rP)^{(1)} + (k' - 1)rP$ from Proposition 7.8. Therefore, by Lemma 7.9, it follows that for any $k \geq r$, we obtain $(kP)^{(1)} = (rP)^{(1)} + (k - r)P$. Since Q is level, for any $k \geq r$, we obtain $(kQ)^{(1)} = (rQ)^{(1)} + (k - r)Q$. Fix a positive integer $k \geq r$. Since $\text{int}(k(P \times Q)) \cap \mathbb{Z}^{d+e} = (\text{int}(kP) \cap \mathbb{Z}^d) \times (\text{int}(kQ) \cap \mathbb{Z}^e)$,

$$\begin{aligned} (k(P \times Q))^{(1)} &= (kP)^{(1)} \times (kQ)^{(1)} \\ &= ((rP)^{(1)} + (k - r)P) \times ((rQ)^{(1)} + (k - r)Q) \\ &\subseteq ((rP)^{(1)} \times (rQ)^{(1)}) + (k - r)(P \times Q) \\ &= (r(P \times Q))^{(1)} + (k - r)(P \times Q) \\ &\subseteq (k(P \times Q))^{(1)}. \end{aligned}$$

Hence $P \times Q$ is level. $\square$

Now, we prove Theorem 7.2.

Proof of Theorem 7.2. The order polytope of $\Pi \cup C_s$ is the Cartesian product of $\mathcal{O}(\Pi)$ and $\mathcal{O}(C_s)$, which is the s -dimensional unimodular simplex. This simplex has codegree $s + 1$. Hence the claim follows from Theorem 7.11. $\square$

Conversely, we consider posets all of whose connected components are level. In fact, these posets are always level.

THEOREM 7.12. *Let Π be a poset on d elements and $\Pi_1, \dots, \Pi_m$ the connected components of Π . If each Π_i is level, then Π is level.*

Theorem 7.12 follows from the following result.

THEOREM 7.13. *Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be level polytopes. If either*

1. *codeg(Q) < codeg(P) and Q has the integer decomposition property,*
2. *codeg(P) < codeg(Q) and P has the integer decomposition property,*
3. *or if codeg(Q) = codeg(P),*

then $P \times Q$ is level.

Proof. Let $r_P := \text{codeg}(P)$ and let $r_Q := \text{codeg}(Q)$. Without loss of generality, we assume $\text{codeg}(Q) \leq \text{codeg}(P)$. Then $r := \text{codeg}(P \times Q) = \max\{r_P, r_Q\} = r_P$, since $\mathbf{x} = (\mathbf{x}_P, \mathbf{x}_Q) \in \text{int}(P \times Q) \cap \mathbb{Z}^{d+e}$ implies that $\mathbf{x}_P \in \text{int}(P) \cap \mathbb{Z}^d$ and that $\mathbf{x}_Q \in \text{int}(Q) \cap \mathbb{Z}^e$. Let $(\mathbf{x}_P, \mathbf{x}_Q, h) \in \text{cone}(P \times Q) \cap \mathbb{Z}^{d+e+1}$ with $h > r$. Then this point projects to points $(\mathbf{x}_P, h) \in \text{cone}(P) \cap \mathbb{Z}^{d+1}$ and $(\mathbf{x}_Q, h) \in \text{cone}(Q) \cap \mathbb{Z}^{e+1}$. Since P is level, we have that

$$(\mathbf{x}_P, h) = (\mathbf{x}_P^\circ, r) + (\tilde{\mathbf{x}}_P, h - r),$$

where $(\mathbf{x}_P^\circ, r) \in \text{int}(\text{cone}(P)) \cap \mathbb{Z}^{d+1}$ and $(\tilde{\mathbf{x}}_P, h - r) \in \text{cone}(P) \cap \mathbb{Z}^{d+1}$. Similarly, since Q is level, we have that

$$(\mathbf{x}_Q, h) = (\mathbf{x}_Q^\circ, r_Q) + (\tilde{\mathbf{x}}_Q, h - r_Q),$$

where $(\mathbf{x}_Q^\circ, r_Q) \in \text{int}(\text{cone}(Q)) \cap \mathbb{Z}^{e+1}$ and $(\tilde{\mathbf{x}}_Q, h - r_Q) \in \text{cone}(Q) \cap \mathbb{Z}^{e+1}$. If $\text{codeg}(P) = \text{codeg}(Q)$, we now get a decomposition

$$(\mathbf{x}_P, \mathbf{x}_Q, h) = (\mathbf{x}_P^\circ, \mathbf{x}_Q^\circ, r) + (\tilde{\mathbf{x}}_P, \tilde{\mathbf{x}}_Q, h - r),$$

where $(\mathbf{x}_P^\circ, \mathbf{x}_Q^\circ) \in r \text{int}(P \times Q) \cap \mathbb{Z}^{d+e}$ and where $(\tilde{\mathbf{x}}_P, \tilde{\mathbf{x}}_Q) \in (h - r)(P \times Q) \cap \mathbb{Z}^{d+e}$ proving levelness of $P \times Q$.

So let us assume $\text{codeg}(Q) < \text{codeg}(P)$. Since Q has the integer decomposition property, we can express $(\tilde{\mathbf{x}}_Q, h - r_Q)$ as a sum of height 1 elements, and therefore $(\tilde{\mathbf{x}}_Q, h - r_Q) = (\tilde{\mathbf{x}}_Q^{(1)}, r_P - r_Q) + (\tilde{\mathbf{x}}_Q^{(2)}, h - r_P)$ and thus obtain

$$(\mathbf{x}_Q, h) = (\mathbf{x}_Q^\circ, r_Q) + (\tilde{\mathbf{x}}_Q, h - r_Q) = (\mathbf{x}_Q^\circ + \tilde{\mathbf{x}}_Q^{(1)}, r_P) + (\tilde{\mathbf{x}}_Q^{(2)}, h - r_P),$$

where $(\mathbf{x}_Q^\circ + \tilde{\mathbf{x}}_Q^{(1)}, r_P) \in \text{int}(\text{cone}(Q)) \cap \mathbb{Z}^{e+1}$ and $(\tilde{\mathbf{x}}_Q^{(2)}, h - r_P) \in \text{cone}(Q) \cap \mathbb{Z}^{e+1}$. Therefore, we can express $(\mathbf{x}_P, \mathbf{x}_Q, h)$ as

$$(\mathbf{x}_P, \mathbf{x}_Q, h) = (\mathbf{x}_P^\circ, \mathbf{x}_Q^\circ + \tilde{\mathbf{x}}_Q^{(1)}, r) + (\tilde{\mathbf{x}}_P, \tilde{\mathbf{x}}_Q^{(2)}, h - r),$$

where $(\mathbf{x}_P^\circ, \mathbf{x}_Q^\circ + \tilde{\mathbf{x}}_Q^{(1)}) \in r \text{int}(P \times Q) \cap \mathbb{Z}^{d+e}$ and where $(\tilde{\mathbf{x}}_P, \tilde{\mathbf{x}}_Q^{(2)}) \in (h - r)(P \times Q) \cap \mathbb{Z}^{d+e}$. $\square$

Remark 7.14. In Theorem 7.13, we really need the assumption that the polytope of lower codegree has the integer decomposition property. Consider the following example, where

$$P = \text{conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 2)\}$$

and where

$$Q = \text{conv}\{(0, 0), (1, 0), (0, 1)\}.$$

Then P does not have the integer decomposition property, but it is Gorenstein and thus level. Moreover, Q has the integer decomposition property and it is level, since it is Gorenstein. However, the product $P \times Q$ is not level.

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