Thermodynamics and efficiency of an autonomous on-chip Maxwell’s demon

Aki Kuvonen1, Jonne Koski2 & Tapio Ala-Nissila1,3

In his famous letter in 1870, Maxwell describes how Joule’s law can be violated “only by the intelligent action of a mere guiding agent”, later coined as Maxwell’s demon by Lord Kelvin. In this letter we study thermodynamics of information using an experimentally feasible Maxwell’s demon setup based a single electron transistor capacitively coupled to a single electron box, where both the system and the Demon can be clearly identified. Such an engineered on-chip Demon measures and performs feedback on the system, which can be observed as cooling whose efficiency can be adjusted. We present a detailed analysis of the system and the Demon, including the second law of thermodynamics for bare and coarse grained entropy production and the flow of information as well as efficiency of information production and utilization. Our results demonstrate how information thermodynamics can be used to improve functionality of modern nanoscale devices.

Recent development of stochastic thermodynamics has extended the traditional macroscopic theory to small scales and non-equilibrium processes beyond linear response1–4. Information thermodynamics5–9, which additionally considers processes that include information, measurement, and feedback, allows quantified studies on problems such as Maxwell’s demon10. The Demon is known as an object that acquires microscopic information of a system and applies feedback to decrease its entropy while, to retain the second law of thermodynamics, generates at least an equal amount of entropy. The emergence of nanotechnology has given rise to various theoretical proposals11–15 as well as experimental realizations5,16–20 of a Maxwell’s demon. The most recent studies in the field consider autonomous Demons - setups containing both the system measured and the Demon such that both the measurement and feedback are performed internally and no microscopic information needs to exit the system8,9,12,13,21,22.

Recently it has been experimentally shown that an autonomous Maxwell’s demon20 device based on single electron tunneling at low temperatures14,23–26 can produce negative entropy in form of cooling its environment. More precisely, in the setup, a single electron transistor (SET)27, acts as the system to be measured, while the measurement and feedback is performed internally based on Coulomb interaction by a capacitively coupled single electron box, which acts as the Demon. The device has a limited number of relevant degrees of freedom, clear separation of different time scales, and well defined and measurable energy scales making it particularly suitable for studying dissipation at microscopic scales. In addition the device only requires fixed external voltage sources and a sufficiently low bath temperature to produce apparent negative entropy. The tunneling rates are not controlled externally during the operation. Here we study the role of information in the operation of the device in detail and show that by adjusting the properties of the Demon, the system's performance as a nanoscale cooling machine, including its efficiency, can be analyzed and tuned with thermodynamics of information.

Results

Model. Figure 1(a) shows a schematic of the device. A metallic island is connected to two external leads via tunnel junctions, both with an equal tunneling resistance of $R_L = R_R = R$, where the indices refer to ‘left’ and the ‘right’ junctions. This forms the SET system that is measured. A detector - the actual Maxwell’s demon is a single electron box, consisting of a metallic island connected to a grounded lead by a tunnel junction with tunneling resistance $R_D$. The system and the Demon islands are capacitively coupled to each other, and the whole setup is...
coupled to a phonon bath at inverse temperature $\beta = 1/(k_B T)$. Finally, the system is biased by voltage $V$ so that the current runs from left to right, and the total Hamiltonian is given by

$$H = \frac{eV}{2} l + \frac{eV}{2} (-l - x) + E_{C}^{\text{sys}} (x - \lambda_x)^2 + E_{C}^{\text{dem}} (y - \lambda_y)^2 + \kappa (x - \lambda_x)(y - \lambda_y),$$

(1)

where $E_{C}^{\text{sys}}$ and $E_{C}^{\text{dem}}$ denote the charging energies of the system and the Demon island, respectively, $\lambda_x$ and $\lambda_y$ are external electrostatic control parameters, $x$ and $y$ denote the number of excess electrons in the system and the Demon, respectively, $l$ is the number of electrons on the left lead, and $\kappa$ is the coupling energy. The dynamics are bipartite meaning that state $(l, x, y)$ may change by consecutive single electron tunneling events through the left junction $(l, x, y) \rightarrow (l \pm 1, x \pm 1, y)$, the right junction $(l, x, y) \rightarrow (l, x \pm 1, y)$, or the Demon junction $(l, x, y) \rightarrow (l, x, y \pm 1)$. Each tunneling event $i \rightarrow f$ as a short notation of $(l_i, x_i, y_i) \rightarrow (l_f, x_f, y_f)$, has an energy cost directly given by Eq. (1) as $E_{i \rightarrow f} = H(l_f, x_f, y_f) - H(l_i, x_i, y_i)$, and the corresponding tunneling rate is given by

$$\Gamma_{i \rightarrow f} = \frac{1}{e^2 R_e} \frac{E_{i \rightarrow f}}{e^{E_{i \rightarrow f}} - 1},$$

(2)

where $v = L, R, D$ refers to the junction associated with the transition $i \rightarrow f$ (cf. Fig. 1). Higher order tunneling events are neglected, which is justified when tunneling resistances are much higher than the quantum resistance, i.e. $R, R_D \gg R_e = h/e^2$.

**Energetics of electron tunneling in the setup.** Next, we consider the operation of the setup at $\lambda_x = \lambda_y = 1/2$, $eV < \kappa$, and $k_B T < \kappa$, $E_{C}^{\text{sys}}, E_{C}^{\text{dem}}$. It is then sufficient to consider only the lowest energy states $(x, y) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. The energy cost for a tunneling event in the system is

$$\begin{align*}
L(R)E_{x=0 \rightarrow 1} &= \kappa \left( y - \frac{1}{2} \right) e^\pm \frac{eV}{2} = -L(R)E_{x=1 \rightarrow 0},
\end{align*}$$

(3)

where the $+$ and $-$ signs are used in case of tunnelling through the left (L) or right junction (R), respectively, as indicated in the superscript on the left of $E$. The energy cost for a Demon tunneling event is

$$D_xE_{x=0 \rightarrow 1} = \kappa \left( x - \frac{1}{2} \right) = -D_xE_{x=1 \rightarrow 0},$$

(4)

where $D$ denotes the Demon. Note that neither Eq. (3) nor (4) depend on $l$. The energy is minimized when the islands have a single excess electron in total. Escaping the corresponding states $(0, 1)$ and $(1, 0)$ has an energy cost $\kappa/2$ for the Demon, and $(\kappa - eV)/2$ for the system. Relaxing back from $(1, 1)$ or $(0, 0)$ has an energy cost $-\kappa/2$ for the Demon, and $-(\kappa + eV)/2$ for the system. With an appropriate choice of $R_e \ll R$ and $V$, it is possible to realize a situation, where the energetically unfavored states $(1, 1)$ and $(0, 0)$ tend to relax through the Demon tunnel junction. As a result, when a tunneling event occurs in the system, cooling it by $(\kappa - eV)/2$, the Demon rapidly reacts.
through another tunneling event, resuming the setup back to its ground state. This forms a cycle, illustrated in Fig. 1(b), where electric current flows through the SET while cooling it down by $\kappa - eV$ for each passing electron apparently violating Joule's law\(^{22}\). However, Joule's law is retained by noting the heat $\kappa$ dissipated in the Demon.

**Thermodynamics of the Demon.** The probability distribution of the state $(l_x, x_y, y)$, $p_{l_x, x_y, y} \equiv p_{l, x, y}$, follows the master equation $\dot{p}_{l, x, y} = -\sum f_{l, x, y}$, where

$$I_{l, x, y} = \Gamma_{l, x, y} p_{l, x, y} - \Gamma', $$

which is always non-negative. Further, proceeding as proposed in ref. 8, Eq. (6) splits in two non-negative contributions: One produced by tunneling events in the system and the Demon, respectively.

$$\dot{S}_{\text{tot}} = \frac{1}{2} \sum_{l, x, y} I_{l, x, y} \ln \left( \frac{p_{l+1, x, y} p_{l-1, x, y}}{p_{l, x, y} p_{l+1, x, y}} \right),$$

and another describing entropy produced by tunneling events in the Demon:

$$\dot{S}_{\text{D}} = \frac{1}{2} \sum_{x, y} I_{x, y} \ln \left( \frac{D_{x, y} p_{x, y}}{D_{x+1, y} p_{x+1, y}} \right).$$

where $I_{x, y}$ are the changes in the mutual information $I = \ln |p_{x, y}/(\sum_p p_{x, y})|$ due to the tunneling events in the Demon and the system, respectively, and $Q_{\text{S}} = Q_{\text{L}} + Q_{\text{D}}$ and $Q_{\text{D}}$ are the heat dissipation rates in the system and the Demon. The heat dissipation rate in each junction is

$$Q_{l, x, y} = -\sum_{l', x', y'} I_{l', x', y'} \ln \left( \frac{\Gamma_{l', x', y'} p_{l', x', y'}}{\Gamma_{l, x, y} p_{l, x, y}} \right),$$

where $W_{l, x, y} = \Gamma_{l, x, y} + \Gamma'$. The term $I_{x, y}$ is the rate of mutual information produced by the Demon and quantifies how much transitions in $y$ increase correlation between $x$ and $y$. In steady state the total time derivative of $I$ vanishes, but there is a flow of information $I^X = -I^Y$ between the Demon and the system. The terms $I^X$ and $I^Y$ also give the change in the Shannon entropy of the total system induced by a transition in the system and the Demon, respectively.

**Demon as a refrigerator.** In the low temperature regime, where both the system and the Demon have only two possible values of charge occupancy, the probability distribution is given by

$$p_{0,0} = p_{1,1} = \frac{1}{2 \Gamma_0 + \Gamma_1}, \quad p_{0,1} = p_{1,0} = \frac{\Gamma_1}{\Gamma_0 + \Gamma_1},$$

where $\Gamma_0 = \Gamma_{x, y, 0} = \Gamma_{x, y, 1}$ is the relaxation rate and $\Gamma_{x, y, 0} = \Gamma_{x, y, 1}$ is the excitation rate. For any $V \neq 0$, $I^Y > 0$, implying that the tunneling events over the Demon junction on average increase the correlation between $x$ and $y$. Since $I^X = -I^Y$, the mutual information produced by the Demon is consumed.
in the system. To satisfy Eq. (8) the Demon must dissipate enough heat to its environment. The negative flow of information $I^X$ allows for negative $\beta \dot{Q}_S < 0$ dissipation rate for the system without breaking the second law of Eq. (7), as shown in Fig. 2(a).

The heat dissipation rate in the system, Eq. (9), may be written as:

$$\dot{Q}_L = \dot{Q}_R = -\left(\frac{\kappa - eV}{2} \Gamma_{y=0}^{x=0} + \frac{\kappa + eV}{2} \Gamma_{x=0}^{y=0}\right) p_{0,1}$$

$$+ \left(\frac{\kappa - eV}{2} \Gamma_{y=0}^{x=0} + \frac{\kappa + eV}{2} \Gamma_{x=0}^{y=0}\right) p_{1,1},$$

where the first term is always negative, and the second term is always positive. Thus increasing the probability $p_{0,1}$ increases the cooling power. Therefore, as can be seen from Eq. (11), the maximum cooling power is obtained when the tunneling rate over the Demon junction is maximized. This is in agreement with the numerical results which show that a faster Demon ($R_D < R$) gives rise to more cooling power as shown in Fig. 2(b). The operating temperature $T$ has to be sufficiently low, less than $0.13\kappa^{-1} \kappa$, in order to obtain cooling. In addition, if $R_D < R$, the optimal temperature, where the cooling power is maximized is roughly at $0.08\kappa^{-1} \kappa$.

**Coarse grained entropy.** We next examine entropy production in the setup, but now assuming that only the states of the system and the Demon, $x$ and $y$, are observed, and focus on the information exchange between the system and the Demon similar to refs 8,13. Therefore, we only consider the change $x_i \rightarrow x_i$ but do not distinguish whether the electron tunnels through the left or the right junction. With this approach the total entropy production rate is again given by Eq. (6), but the $x$ degree of freedom changes at the effective rate $W_{x_i \rightarrow x_i}^{y} \Gamma_{x_i \rightarrow x_i}^{y} + \Gamma_{x_i \rightarrow x_i}^{y}$. The total entropy production rate of the system is (cf. Eq. (7))

$$S^c_{sys} = \dot{S}^c + \dot{I}^Y \geq 0,$$

where $\dot{S}^c = 1/2 \sum_{x,y} I_{x \rightarrow y} \sigma^x$ and $\sigma^x = \ln \left(W_{x_i \rightarrow x_i}^{y} / W_{x_i \rightarrow x_i}^{y} \right)$ defines the (coarse grained) entropy produced by the transition $x_i \rightarrow x_i$. In our setup, for non-zero bias, the entropy $\sigma^x$ is always negative and thus the device works as a Maxwell's Demon, as shown in Fig. 2(a).

**Efficiency of production and utilization of information.** As shown in Fig. 3(a), a Demon with higher reaction rate ($R_D^{-1}$) is able to produce more information $\dot{I}^Y$. The entropic cost for sustaining the flow of information is the dissipation rate in the Demon $\beta \dot{Q}_D$ through heat. We define $\epsilon_Y = \dot{I}^Y / \beta \dot{Q}_D$, that characterizes the efficiency of the Demon information production. In Fig. 3(b) we show that a faster Demon is more efficient and in the limit of extremely fast reacting Demon, the flow of information $\dot{I}^Y$ coincides with the heat dissipation rate, i.e. $\dot{I}^Y = \beta \dot{Q}_D$, corresponding the maximum efficiency of $\epsilon_Y = 1$. The same result is obtained analytically by assuming the Demon is fast enough to thermalize on a time scale faster than the transitions occur in the system.

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**Figure 2.** Entropy production rate and cooling power dependence on temperature and bias voltage. (a) Entropy production rate $\beta \dot{Q}_S$ and the coarse grained entropy production rate $\dot{S}^c$ in the fast Demon limit $(R_D = 10^{-2} \kappa)$ in different operating temperatures as a function of bias to coupling energy ratio. The coarse grained entropy is always negative and underestimates the entropy production. At low enough operating temperatures, there exists an optimal non zero bias voltage where the cooling is maximized. In higher temperatures no cooling is obtained. Temperatures used here are $T_1 = 0.05\kappa^{-1} \kappa$, $T_2 = 0.08\kappa^{-1} \kappa$ and $T_3 = 0.13\kappa^{-1} \kappa$. (b) Minimum system dissipation rate $\dot{Q}_D$ with optimal bias voltage as a function of operating temperature with three different Demon reaction rates ($R_D^{-1}$). Smaller resistance $R_D$ makes the Demon faster and more cooling is obtained. At temperatures higher than $T_3$, no cooling is obtained, while there exists an optimal operating temperature $T_i$ where the cooling power is maximized. Results are obtained by numerically solving the master equation with rates of Eq. (2).
On the system side the apparent violation of the second law ($\sigma^X < 0$) is provided by the flow of information $Y$, which the system is able to utilize with efficiency $\epsilon_X = -\sigma^X / Y$. Contrary to $\epsilon_Y$, $\epsilon_X$ increases when the Demon is slower (large $R_D$) as shown in Fig. 3(a,b). We obtain, both analytically and numerically, that in the case of a very slow Demon, we have $Y = -\sigma^X$, which corresponds to the maximum efficiency of $\epsilon_X = 1$.

Furthermore, a straightforward calculation shows that the efficiency of the whole measurement-feedback cycle, defined as $\epsilon_T = \epsilon_X \epsilon_Y = -\sigma^X / \beta Q_D$, is given by

$$\epsilon_T = \frac{2}{(\beta \kappa)} \sigma^X,$$

where $\sigma^X = \ln[\frac{W_0}{W_{0,0}}]$ is the coarse grained entropy production in the relaxation from $(0, 0)$ to $(1, 0)$ or equivalently from $(1, 1)$ to $(0, 1)$. Furthermore, this efficiency is independent of the Demon reaction rate $R_D$, and thus a better Demon performance decreases the efficiency $\epsilon_X$ of the system as shown in Fig. 3(b). The flow of mutual information in the fast and slow demon regimes is analyzed in the Supplementary material in detail.

**Relation between coarse grained and bare entropies.** We next study the relation between the entropy production rate $\beta Q_D$ and $\sigma^X$. Because the rates $W$ do not satisfy local detailed balance condition, $\sigma^X$ differs from the entropy $\beta Q_D$. However, as shown in the Supplementary material, the entropies are related as

$$\langle e^{-\beta Q_A} \rangle = e^{-\sigma^X},$$

where $\langle \rangle$ denotes averaging over the conditional probabilities $P_L = \Gamma_{\mathrm{rel}} / (\Gamma_{\mathrm{rel}} + \Gamma_{\mathrm{tr}} + \Gamma_{\mathrm{pr}})$ and $P_R = \Gamma_{\mathrm{rel}} / (\Gamma_{\mathrm{rel}} + \Gamma_{\mathrm{tr}} + \Gamma_{\mathrm{pr}})$ to tunnel over the left and right junctions, respectively. Furthermore, Eq. (15) results in an integral fluctuation theorem for the coarse graining cost $S_{cg} = \beta Q_D - \sigma^X$:

$$\langle e^{-S_{cg}} \rangle = 1,$$

which by using Jensen’s inequality gives

$$\langle S_{cg} \rangle \geq 0,$$

implying that the coarse grained entropy underestimates the bare entropy production. This can also be seen in Fig 2(a), while in the small bias $eV/\kappa \ll 1$ and at low temperature $T$ the entropy production rates $\beta Q_A$ and $\sigma^X$ coincide. By observing only the $x$ degree of freedom there can be an apparent violation of the second law, $\sigma^X < 0$, even in the regime where the bare entropy production rate $\beta Q_A$ is positive. However, as can also be seen in Fig. 3(a), the coarse grained entropy production rate including the information, $\hat{S}_{cg}^X = \sigma^X + Y$ is positive (Eq. (13)).

![Figure 3](image-url)

**Figure 3. Flow of information and the efficiency of its production and utilization.** (a) Entropy production rate in the Demon $\beta Q_D$, flow of information $Y$, and the coarse grained entropy production rate $\sigma^X$ in the system as a function of Demon tunneling resistance ($R_D$). Smaller resistance makes the Demon faster. While the apparent entropy production rate in the system $\sigma^X < 0$, the total entropy production rate $S_{cg} = \sigma^X + Y \geq 0$ (Eq. (13)). In addition, the Demon entropy production rate is always the largest of the three ensuring the inequality $S_{cg} = \beta Q_D + Y \geq 0$ (Eq. (8)). (b) The efficiency of information production $\epsilon_Y$ and utilization $\epsilon_X$, and that of the whole production-utilization, $\epsilon_T$. In the fast Demon limit ($R_D < < R$), the flow of information in the Demon equals the heat dissipation rate ($\epsilon^T = 1$), while in the slow limit the utilization of information flow becomes efficient ($\epsilon^X = 1$). Parameters in both (a,b) are those optimal for maximum cooling power, $T = 0.08ke^{-1}$ and $eV/\kappa = 0.72$, extracted from data shown in Fig. 2 of the main text.
The positivity of the coarse graining cost, Eq. (17), then also ensures positivity of the entropy production rate $\dot{S}_{\text{tot}} = \beta \dot{Q}_s + \dot{S}^X \geq 0$ (Eq. (7)).

### Discussion
To summarize, we have analyzed entropy production and flow of information in the experimentally feasible isothermal nanoscale device described in Fig. 1(a). The setup works as a Maxwell's demon device, where both the system and the Demon can be identified and where the measurement and the feedback are performed internally by the on-chip Demon. We have shown that depending on which variables are accessible for measurement, different apparent negative entropy productions result, however, the second law of thermodynamics always holds for the total combined system. Nevertheless, the performance and efficiency of the device to function as a cooler can be analyzed and adjusted by using thermodynamics of information. Thus, we conclude that information thermodynamics can be used to construct nanoscale devices with desired thermodynamic properties, e.g. to design dissipation in the device.

### References

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### Author Contributions
All authors contributed to writing and reviewing the manuscript. A.K. and J.K. wrote the first draft and A.K. prepared figures 1–3.

### Additional Information
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