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Electromagnetic Chirality, Part II: Macroscopic Perspective

Christophe Caloz, *Fellow, IEEE*, and Ari Sihvola, *Fellow, IEEE*

Abstract—This paper is the second part of a two-part paper presenting a bottom-up description of electromagnetic chirality, which occurs in materials composed of particles with structural handedness. This part first infers, from the microscopic study of the first part [1], the chiral constitutive relations as a subset of the most general bianisotropic relations. Then, it establishes the fundamental spatial parity conditions in chiral media upon the basis of space-reversal symmetry considerations. Next, it shows that the eigenstates of chiral media are circularly polarized waves, with polarization rotation depending on the sign of the chiral parameter, and describes the resulting polarization rotation, or (reciprocal) gyrotropy, in terms of circular birefringence. The following part presents introduces an explicit formulation of chirality based on spatial frequency dispersion or nonlocality, analyzes the temporal frequency dispersion or nonlocality of chiral media, and finally provides guidelines to design practical chiral metamaterials.

Index Terms—Chirality, optical activity, chiral media, materials and metamaterials, mirror asymmetry, polarization rotation, magnetoelectric coupling, bianisotropy, Tellegen and Pasteur media, polarizability and susceptibility dyadic tensors, spatial dispersion or nonlocality, parity conditions, temporal dispersion or nonlocality, circular birefringence and circular dichroism, reciprocal and nonreciprocal gyrotropy.

I. INTRODUCTION

This paper is the second part of a two-part paper on chiral electromagnetics and metamaterials. It completes the bottom-up description of this topic by extending its microscopic description of the first part [1] to their macroscopic description. We invite the reader to peruse [1] as the key reference to this paper¹, starting with the general introduction to chirality provided in Sec. [1].I.

This part of the paper is organized as follows. Section II infers, from the microscopic study of the first part [1], the chiral constitutive relations as a subset of the most general bianisotropic relations. Section III establishes the fundamental spatial parity conditions in chiral media upon the basis of space-reversal symmetry considerations. Section IV shows that the eigenstates of chiral media are circularly polarized waves, with polarization rotation depending on the sign of the chiral parameter. Section V describes the resulting polarization rotation, or (reciprocal) gyrotropy, in terms of circular birefringence. Finally, Secs. VI and VII respectively discuss the spatial dispersion (or spatial nonlocality) and the temporal frequency dispersion (or temporal nonlocality) aspects of chiral media, while Sec. VIII provides

guidelines to design practical chiral metamaterials. Finally, Sec. IX enumerates the main result of this part, and provide an overall conclusion to the two part-paper formed by this paper with [1].

II. CONSTITUTIVE RELATIONS

A. General Bianisotropic Relations

The behavior of an electromagnetic medium can generally be expressed by the relations [2], [3], [4]

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}_e \quad \text{and} \quad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{P}_m, \quad (1)$$

where $\epsilon_0 \mathbf{E}$ and $\mu_0 \mathbf{H}$ respectively represent the electric and magnetic responses of free space (spacings between the molecules or metaparticles), while $\mathbf{P}_e$ and $\mathbf{P}_m$ represent the response of the particle forming the medium, given by [1].(21). Equations (1) correspond to the most usual electromagnetics convention of $[\mathbf{D}; \mathbf{B}]$ being expressed in terms of $[\mathbf{E}; \mathbf{H}]$, where $\mathbf{E}$ and $\mathbf{H}$ are considered as the excitations while $\mathbf{B}$ and $\mathbf{D}$ are considered as the medium responses [3], [4]².

We now relate the field responses to the medium susceptibilities by inserting [1].(21) into (1), which yields

$$\mathbf{D} = \epsilon_0 \left(\bar{\bar{\mathbf{I}}} + \bar{\bar{\chi}}_{ee} \right) \cdot \mathbf{E} + \sqrt{\epsilon_0 \mu_0} \bar{\bar{\chi}}_{em} \cdot \mathbf{H}, \quad (2a)$$

$$\mathbf{B} = \sqrt{\epsilon_0 \mu_0} \bar{\bar{\chi}}_{me} \cdot \mathbf{E} + \mu_0 \left(\bar{\bar{\mathbf{I}}} + \bar{\bar{\chi}}_{mm} \right) \cdot \mathbf{H}. \quad (2b)$$

Defining

$$\begin{pmatrix} \bar{\bar{\epsilon}} & \bar{\bar{\xi}} \\ \bar{\bar{\zeta}} & \bar{\bar{\mu}} \end{pmatrix} = \begin{pmatrix} \epsilon_0 \left(\bar{\bar{\mathbf{I}}} + \bar{\bar{\chi}}_{ee} \right) & \sqrt{\epsilon_0 \mu_0} \bar{\bar{\chi}}_{em} \\ \sqrt{\epsilon_0 \mu_0} \bar{\bar{\chi}}_{me} & \mu_0 \left(\bar{\bar{\mathbf{I}}} + \bar{\bar{\chi}}_{mm} \right) \end{pmatrix} \quad (3)$$

transforms (2) into the conventional *bianisotropic*³ relations [2], [6], [7]

$$\mathbf{D} = \bar{\bar{\epsilon}} \cdot \mathbf{E} + \bar{\bar{\xi}} \cdot \mathbf{H} \quad \text{and} \quad \mathbf{B} = \bar{\bar{\zeta}} \cdot \mathbf{E} + \bar{\bar{\mu}} \cdot \mathbf{H}, \quad (4)$$

where $\bar{\bar{\epsilon}}$, $\bar{\bar{\mu}}$, $\bar{\bar{\xi}}$ and $\bar{\bar{\zeta}}$ are the permittivity, permeability, magnetic-to-electric coupling and electric-to-magnetic coupling dyadic tensors, respectively, which are measured in

²The book [2] also includes the alternative formulations $[\mathbf{D}; \mathbf{H}]$ versus $[\mathbf{E}; \mathbf{B}]$, based on the motivation that $\mathbf{E}$ and $\mathbf{B}$ are the fields that are directly measurable experimentally via the Lorentz force ($\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$; q : charge, $\mathbf{v}$: charge velocity), and $[\mathbf{E}; \mathbf{H}]$ versus $[\mathbf{D}; \mathbf{B}]$, based on the motivation that $\mathbf{D}$ and $\mathbf{B}$ form with the spectral spatial wavevector $\mathbf{k}$ an electromagnetic basis that does not depend on the nature of the medium.

³In the term ‘bianisotropic’, coined by Kong in [5], ‘bi’ refers to the fact that each (electric and magnetic) response depends on both the electric excitation and the magnetic excitation, while ‘anisotropic’ refers to the fact the responses are not parallel to their excitations or, equivalently, are characterized by tensorial parameters. From [1].(20), it is clear that this term applies to the polarizabilities as well as to the susceptibilities.

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¹This paper uses the conventions [1].N, [1].n and [1].(n) to respectively denote section N, figure n and equation (n) in [1] [e.g. Sec. [1].V-D, Fig. [1].9 and Eq. [1].(18)].

As/Vm, Vs/Am and s/m (see Appendix A). The four medium tensors are generally of dimension 3×3 , and involve thus 36 complex parameters overall.

The coupled equations [1].(20) – where $\bar{\mathbf{D}}$ and $\bar{\mathbf{B}}$ are inter-dependent through the coupling tensors $\bar{\xi}$ and $\bar{\zeta}$ – represent the most general explicit constitutive relations for an LTI (linear time-invariant) medium. The LTI [Assumption 1] in Sec. [1].III] is indeed a necessary condition for such relations to hold, since nonlinearity would involve powers of $\mathbf{E}$ and $\mathbf{H}$ [8], while time variance would require a more complex treatment [9].

As we have seen in Sec. [1].V-C, the straight Omega particle [Fig. [1].5(a) or [1].8(a)], whether unique or combined with copies of itself in different directions of space, is always characterized by non-reducible bianisotropic relations [Eqs. [1].(13), (14) and (15)]. Therefore, it corresponds to the most general relations (4), whose tensorial susceptibilities are obtained from [1].(13), (14) or (15) via [1].(20).

B. Biisotropic Chiral Media

In the particular case where the tensors $\bar{\epsilon}$, $\bar{\mu}$, $\bar{\xi}$ and $\bar{\zeta}$ in (4) reduce to $\epsilon\bar{\mathbf{I}}$, $\mu\bar{\mathbf{I}}$, $\xi\bar{\mathbf{I}}$ and $\zeta\bar{\mathbf{I}}$, and hence ultimately to scalars, the medium is called *biisotropic*. This is for instance the case of the medium formed by the triatomic helix particle shown in Fig. [1].10 and characterized by the polarizability tensor [1].(18). Equations (4) reduce then to

$$\mathbf{D} = \epsilon\mathbf{E} + \xi\mathbf{H} \quad \text{and} \quad \mathbf{B} = \zeta\mathbf{E} + \mu\mathbf{H}, \quad (5)$$

and a biisotropic medium is thus characterized only by the 4 complex parameters ϵ , ξ , ζ and μ .

We have established in Sec. [1].V-D that the cross coupling parameters of a chiral structure were opposite to each other [Eq. [1].(16)] and in quadrature with the direct coupling parameters (Fig. [1].9). In terms of the constitutive parameters in (5), this translates into

$$\zeta = -\xi \quad \text{and} \quad \angle\zeta, \angle\xi = \angle\epsilon \pm \pi/2 = \angle\mu \pm \pi/2, \quad (6)$$

where the last equality expresses the fact the electric and magnetic responses of a plane wave in a simple biisotropic medium – such as free space – are in phase, as seen by inserting [1].(4) into [1].(5)⁴. In order to make sure to select the correct signs in (6), let us examine the behavior of the RH helix particle described in Fig. [1].9 in some more details.

For this purpose, Fig. 1 compares the electric and magnetic responses to the electric and magnetic fields. First, the panel at the center of the figure recalls the result established in conjunction with Fig. [1].9 that the RH helix particle rotates the field polarization in the transverse plane in the direction corresponding to the left hand with the thumb pointing in the propagation direction, or in the y to x direction, corresponding to phasor multiplication by the

factor $(-i)$, or yet to the delay $T/4 = (2\pi/\omega)/4 = \pi/(2\omega)$. Comparing the excitations associated to the responses $\mathbf{D}_E$ and $\mathbf{D}_H$ shows then that the latter are $\pi/2$ (or $+T/4$) ahead of the former; it must therefore be rotated backward in time or in the positive (x to y) direction, for synchronization with the electric reference, i.e., multiplied by the factor $+i$. Similarly comparing the responses $\mathbf{B}_H$ and $\mathbf{B}_E$ shows that the latter is $\pi/2$ (or $-T/4$) behind the former; it must therefore be rotated forward in time, in the negative (y to x) direction, for synchronization with the magnetic reference, or multiplied by the factor $-i$. Opposite signs would of course be found in the case of a LH helix particle.

Fig. 1. Electric responses ($\mathbf{D}$) and magnetic responses ($\mathbf{B}$) of the RH helix particle in Fig. [1].5(b) to the electric excitation ($\mathbf{E}$) and magnetic excitation ($\mathbf{H}$). The direct responses $\mathbf{D}_E$ and $\mathbf{B}_H$ are synchronized with $\mathbf{E}$ and $\mathbf{H}$, respectively, whereas the cross responses $\mathbf{D}_H$ and $\mathbf{B}_E$ are respectively advanced ($t = +T/4$) and delayed ($t = -T/4$) with respect to them.

We have thus, with respect to the references of ϵ and μ , that $\xi = +i\chi$ and $\zeta = -i\chi$. This reformulates the chiral constitutive relations (5) as

$$\mathbf{D} = \epsilon\mathbf{E} + i\chi\mathbf{H} \quad \text{and} \quad \mathbf{B} = -i\chi\mathbf{E} + \mu\mathbf{H}, \quad (7)$$

where χ is called the chiral parameter. A chiral medium, characterized by the relations (7), is also called a *Pasteur medium*. Inserting (7) into [1].(5) provides then the following *sourceless* explicit form of Maxwell equations for a chiral or Pasteur medium:

$$\nabla \times \mathbf{E} = i\omega(-i\chi\mathbf{E} + \mu\mathbf{H}) = \omega(\chi\mathbf{E} + i\mu\mathbf{H}), \quad (8a)$$

$$\nabla \times \mathbf{H} = -i\omega(\epsilon\mathbf{E} + i\chi\mathbf{H}) = \omega(-i\epsilon\mathbf{E} + \chi\mathbf{H}). \quad (8b)$$

Note the Greek symbol χ is now associated with both the susceptibility, from [1].(21), and with the chirality factor, from (7). However, the former will always appear with the double subscript ee, em, me or mm, whereas the latter is always on its own. So, this coincidence should not pose any notational ambiguity.

The *reciprocity* of a chiral medium, which is imposed by the assumed absence of an external force or of nonlinearity combined with spatial asymmetry [10], is expressed in the

⁴The conditions (6) assume losslessness (and passivity), in addition to reciprocity. In the case of a *bianisotropic* medium, with the susceptibility tensors in (3), they generalize to $\bar{\chi}_{me}^T = \bar{\chi}_{em}^*$, $\bar{\chi}_{ee}^T = \bar{\chi}_{ee}^*$ and $\bar{\chi}_{mm}^T = \bar{\chi}_{mm}^*$. These relations also approximately hold in the presence of small loss, but break down at higher loss if the electric and magnetic losses are unbalanced.

constitutive relations (7) by the fact that $\bar{\bar{\xi}} = i\chi\bar{\bar{I}} = -\bar{\bar{\zeta}}$ satisfies the reciprocity condition

$$\bar{\bar{\xi}} = -\bar{\bar{\zeta}}^T, \quad (9)$$

following from the Lorentz nonreciprocity relations [2], [11], which is the macroscopic counterpart of the microscopic relation of Eq. [1].(12)

As we have seen in Sec. [1].V-D, the metamaterial constituted by the triatomic helix metaparticle shown in Fig. [1].10 is isotropic [Eq. [1].(18)]. Therefore, it indeed corresponds to the biisotropic chiral relations (7). Such a medium is represented in Fig. 2.

Fig. 2. Chiral metamaterial made of a 3D array of twisted Omega or helix metaparticles [Fig. [1].5(b)] oriented in the x , y and z directions of space (Fig. [1].10).

C. Generalized Biisotropic Media

The chiral relations (7) do not represent the only form of biisotropy. The magnetoelectric responses can also be in phase with their responses. In this case, we can write $\xi = \tau$ and $\zeta = \tau$, which particularizes (5) to

$$\mathbf{D} = \epsilon\mathbf{E} + \tau\mathbf{H} \quad \text{and} \quad \mathbf{B} = \tau\mathbf{E} + \mu\mathbf{H}. \quad (10)$$

These equations (10) are the constitutive relations of a *Tellegen medium*. Such medium is nonreciprocal since $\bar{\bar{\xi}} = \bar{\bar{\tau}}\bar{\bar{I}} = \bar{\bar{\zeta}} = \bar{\bar{\tau}}^T$ violates the reciprocity condition (9). Therefore, it must involve an external force, specifically permanent or induced electric and magnetic dipole moments parallel or antiparallel to each other [7].

Thus, the coupling parameters ξ and ζ of a biisotropic medium [Eq. (5)] may generally be written as [2], [7]

$$\xi = \tau + i\chi \quad \text{and} \quad \zeta = \tau - i\chi. \quad (11)$$

Inserting (11) into (5) provides the alternative general biisotropic constitutive relations

$$\mathbf{D} = \epsilon\mathbf{E} + (\tau + i\chi)\mathbf{H} \quad \text{and} \quad \mathbf{B} = (\tau - i\chi)\mathbf{E} + \mu\mathbf{H}, \quad (12)$$

which correspond to a chiral or Pasteur medium if $\tau = 0$ and $\chi \neq 0$ and to a Tellegen medium if $\tau \neq 0$ and $\chi = 0$.

It is useful to relate the three parameters ϵ , μ and χ in (7) to the corresponding susceptibilities, which are themselves obtainable from the metaparticle polarizabilities via [1].(20). These relations may be found by reducing the tensors to scalars in (3) and using (11) with $\tau = 0$, i.e., $\xi = i\chi$ and $\zeta = -i\chi$, which yields

$$\begin{pmatrix} \epsilon & \chi \\ \chi & \mu \end{pmatrix} = \begin{pmatrix} \epsilon_0(1 + \chi_{ee}) & -i\sqrt{\epsilon_0\mu_0}\chi_{em} \\ i\sqrt{\epsilon_0\mu_0}\chi_{me} & \mu_0(1 + \chi_{mm}) \end{pmatrix}, \quad (13)$$

implying that $\chi_{me} = -\chi_{em}$, consistently with the previously found results for the twisted Omega or helix particle (Sec. [1].V-D).

D. Loss

According to the Poynting theorem, a medium is lossless if the divergence of the Poynting vector in it is purely imaginary [3], [4], i.e., $\text{Re}\{\nabla \cdot \mathbf{S}\} = 0$, with $\mathbf{S} = \mathbf{E} \times \mathbf{H}^*$, where the $*$ superscripts denotes the complex conjugate operation. The divergence of $\mathbf{S}$ can be expanded as follows:

$$\begin{aligned} \nabla \cdot \mathbf{S} &= \frac{1}{2} \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2} [\mathbf{H}^* \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}^*)] \\ &= \frac{\omega}{2} [-i\epsilon^* |\mathbf{E}|^2 + i\mu |\mathbf{H}|^2 + (\chi - \chi^*) \mathbf{E} \cdot \mathbf{H}^*] \end{aligned} \quad (14)$$

where we have applied the identity $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$ in the second equality, and replaced $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}$ by their Maxwell expressions (8). Nullifying the real part of the last expression in (14) yields then the lossless conditions

$$\text{Re}\{-i\epsilon^*, i\mu\} = 0 \quad \text{or} \quad \epsilon, \mu \in \mathbb{R} \quad \text{and} \quad \chi - \chi^* = 0 \quad \text{or} \quad \chi \in \mathbb{R}. \quad (15)$$

It is easy to verify that in a Tellegen medium τ is also purely real in the lossless case and complex in the lossy case.

In the presence of loss, the parameters ϵ , μ and χ may be complex, i.e.,

$$\epsilon = \epsilon' + i\epsilon'', \quad \mu = \mu' + i\mu'', \quad \chi = \chi' + i\chi'', \quad (16)$$

where the primed and unprimed quantities are real numbers denoting the real and imaginary parts. Therefore, the RH and LH eigenwavenumbers, which will be derived in Sec. IV-A as $\beta_{\pm}^+ = \omega(\sqrt{\epsilon\mu} \pm \chi)$ [Eq. (21a)], generally do not have only different real parts but also different imaginary parts, and then different absorptions. This difference in absorption for the RH and LH waves results in transforming the lossless circularly polarized states into elliptical polarized states, as will become clear in Sec. IV-B. This phenomenon is referred to as *circular dichroism* [12].

Finally, note that the imaginary parts of (ϵ, χ_{ee}) and (μ, χ_{mm}) are purely positive in a lossy (or dissipative) medium with the assumed $e^{-i\omega t}$ harmonic time dependence while the passive lossy condition on $(\chi, \chi_{em, me})$ is less restrictive [13] (see Appendix B).

III. SPATIAL PARITY CONDITIONS

The operation of mirror reflection, which substantiates the definition of chirality (Sec. [1].I), is equivalent to the operation of *space reversal* [3]. Space reversal is defined via the operator $\mathcal{S}$ (called the parity operator – $\mathcal{P}$ – in quantum mechanics) as

$$\mathcal{S}\{\mathbf{r}\} = \mathbf{r}' = -\mathbf{r} \quad \text{or} \quad \mathcal{S}:\mathbf{r} \mapsto \mathbf{r}' = -\mathbf{r} \quad (17a)$$

when trivially applied to the space variable, $\mathbf{r}$, and generally, when applied to any other physical quantity $\Psi(\mathbf{r})$, as

$$\mathcal{S}\{\Psi(\mathbf{r})\} = \Psi'(\mathbf{r}') = \Psi(-\mathbf{r}) = \pm\Psi(-\mathbf{r}), \quad (17b)$$

where positive and negative signs correspond to even parity and odd parity, respectively. This section recalls the fundamental rules of electromagnetic space-reversal symmetries [3], and deduces from them useful parity properties for the electromagnetic constitutive parameters introduced in Sec. II. Note that the space reversal operation (17) corresponds to mirror reflection if it is applied once or an odd number of times, since $\mathcal{S}^m(\mathbf{r}) = (-1)^m\mathbf{r}$ only if m is an odd integer⁵.

According to (17), $\mathbf{r}$ is odd under space reversal, or space-reversal odd or antisymmetric, by definition. It follows that the vectorial spatial derivative operator $\nabla = \nabla_{\mathbf{r}}$, involving only spatial quantities of single multiplicity, is also space-reversal odd. In contrast, the charge and charge density, q and ρ , are space-reversal even, since reversing the coordinate system leaves charges unaffected. The space-reversal symmetry properties of ∇ and ρ imply then, via Gauss law ($\nabla \cdot \mathbf{D} = \rho$) that $\mathbf{D}$ is space-reversal odd, which also implies space-reversal odd parity for $\mathbf{E}$, considering for instance the simple case of vacuum. Combining this last result with Maxwell-Faraday law ($\nabla \times \mathbf{E} = i\omega\mathbf{B}$) indicates then that $\mathbf{B}$ is space-reversal even, and so is then also $\mathbf{H}$. We have thus

$$[\mathbf{D}, \mathbf{E}](-\mathbf{r}) = -[\mathbf{D}, \mathbf{E}](\mathbf{r}) \quad \text{and} \quad [\mathbf{B}, \mathbf{H}](-\mathbf{r}) = +[\mathbf{B}, \mathbf{H}](\mathbf{r}). \quad (18)$$

Applying these field space-reversal parities to the constitutive relations in Sec. II provides then that the parity properties of the different constitutive parameters: $\bar{\epsilon}, \epsilon$ and $\bar{\mu}, \mu$ are even, while $\bar{\xi}, \xi, \bar{\zeta}, \zeta$ and τ, χ are odd. Specifically, enforcing parity compatibility in (7) reveals that ϵ and μ are even functions of space while χ is an odd function of space (mirror antisymmetry), or⁶

$$\epsilon(-\mathbf{r}) = +\epsilon(\mathbf{r}), \quad \mu(-\mathbf{r}) = +\mu(\mathbf{r}) \quad \text{and} \quad \chi(-\mathbf{r}) = -\chi(\mathbf{r}). \quad (19)$$

These relations may serve as a precious sanity check in the modeling of chiral materials. Incidentally, note that the Tellegen parameter [Eq. (10)] is odd, as χ , i.e., $\tau(-\mathbf{r}) = -\tau(\mathbf{r})$.

⁵We have $\mathcal{S}^1(\mathbf{r}) = \mathcal{S}(\mathbf{r}) = -\mathbf{r}$ by definition [Eq. (17)] (mirror symmetry), $\mathcal{S}^2(\mathbf{r}) = \mathcal{S}[\mathcal{S}(\mathbf{r})] = \mathcal{S}(-\mathbf{r}) = \mathbf{r}$ (return to original spatial frame), $\mathcal{S}^3(\mathbf{r}) = \mathcal{S}[\mathcal{S}^2(\mathbf{r})] = \mathcal{S}(\mathbf{r}) = -\mathbf{r}$ (mirror symmetry), etc., which is rather trivial. The situation is more subtle when rotations in different directions are involved, as in Sec. [1].V-A.

⁶Note the last of these relations seem to imply that $\chi = 0$, which would absurdly imply that chiral effects cannot occur! However, this is only a *local* condition, whereas a chiral medium is fundamentally *nonlocal*, as will be discussed in Sec. VI-A.

IV. CHIRAL EIGENSTATES

A. Modal Solutions

Consider a $+z$ -propagating plane-wave excitation. Such a wave has the phasor form [1].(4), corresponding in a Cartesian coordinate system to the fields

$$\mathbf{E}^+(z) = (\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y)e^{+i\beta^+z}, \quad \mathbf{H}^+(z) = (\hat{\mathbf{x}}H_x + \hat{\mathbf{y}}H_y)e^{+i\beta^+z}, \quad (20)$$

where the $+$ superscript has been introduced to denote $+z$ propagation. Substituting (20) into the coupled system of equations (8) yields the modal wavenumbers and modal fields (see Appendix C)

$$\beta_{\pm}^+ = \omega(\sqrt{\epsilon\mu} \pm \chi), \quad (21a)$$

$$\mathbf{E}_{\pm}^+(z) = E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{+i\beta_{\pm}^+z}, \quad (21b)$$

$$\mathbf{H}_{\pm}^+(z) = \frac{\hat{\mathbf{z}} \times \mathbf{E}_{\pm}^+(z)}{\eta} = \frac{E_0}{\eta}(\hat{\mathbf{y}} \mp i\hat{\mathbf{x}})e^{+i\beta_{\pm}^+z} = \mp \frac{i}{\eta}\mathbf{E}_{\pm}^+(z), \quad (21c)$$

where E_0 is assumed (choice of phase origin) to be real, and $\eta = \sqrt{\mu/\epsilon}$ is the medium intrinsic impedance. In these relations, the $\pm$ signs correspond to RH/LH handedness, respectively, according to the observations in Sec. [1].V-D.

B. Circular Polarization

The physical waves corresponding to the chiral modes (21) are found from [1].(3) as

$$\begin{aligned} \mathcal{E}_{\pm}^+(z, t) &= \text{Re}\left\{\mathbf{E}_{\pm}^+(z)e^{-i\omega t}\right\} = \text{Re}\left\{E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{-i(\omega t - \beta_{\pm}^+z)}\right\} \\ &= E_0 \text{Re}\left\{(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})\left[\cos(\omega t - \beta_{\pm}^+z) - i\sin(\omega t - \beta_{\pm}^+z)\right]\right\} \\ &= E_0\left[\hat{\mathbf{x}}\cos(\omega t - \beta_{\pm}^+z) \pm \hat{\mathbf{y}}\sin(\omega t - \beta_{\pm}^+z)\right]. \end{aligned} \quad (22)$$

Figures 3(a) and (b) show the temporal evolution of the electric field vector $\mathcal{E}_{\pm}^+(z, t)$ in a given plane transverse to the propagation direction (here $z = 0$). The tip of $\mathcal{E}_{+}^+(0, t)$ traces a circle following the fingers of the right hand with the thumb pointing in the direction of wave propagation, as shown in Fig. 3(a); it is thus right-handed (RH) circularly polarized⁷. In contrast, the tip of the vector $\mathcal{E}_{-}^+(z, t)$ traces a circle following the fingers of the left hand with the thumb pointing in the direction of wave propagation, as shown in Fig. 3(b); it is thus left-handed (LH) circularly polarized. Thus, the modes of a chiral medium are *RH and LH circularly polarized* waves.

It should be emphasized that the notion of circular polarization refers here to the temporal variation of the field, and not to its spatial evolution, which is opposite. This easily seen by setting $t = 0$ (instead of $z = 0$) in (22). The handedness of the spatial spiral formed by the tip

⁷This corresponds to the polarization handedness convention generally used in the electrical engineering community (e.g. [2], [14], [15], [16]), for which all the results following the time-harmonic convention $e^{-i\omega t}$ (used in this paper) apply upon substituting $-i \rightarrow j$ for transformation to the time-harmonic $e^{+j\omega t}$ convention (e.g., Eq. (22): the resulting opposite signs in both the complex vector and the complex phase eventually cancel out to yield the same real result.). The physics and optics community generally follows a convention leading to the opposite handedness, referring to clock direction rather than handedness, and considering that when the field rotation is counterclockwise (resp. clockwise) with the observer facing the oncoming wave, the wave is left-handed (resp. right-handed) (e.g. [3], [17]). This is all quite confusing, but important to note!

Fig. 3. Circular polarization (CP) of the chiral modes in the plane $z=0$ [Eqs. (21) and (23)]. (a) FWD-prop. (+) and $(\hat{x} + i\hat{y})$ phasor (+), leading to RH-CP. (b) FWD-prop. (+) and $(\hat{x} - i\hat{y})$ phasor (-), leading to LH-CP. (c) BWD-prop. (-) and $(\hat{x} - i\hat{y})$ (-) phasor, leading to RH-CP. (d) BWD-prop. (-) and $(\hat{x} + i\hat{y})$ (+) phasor, leading to LH-CP.

of the field in space is always opposite to the temporal handedness, as properly illustrated in [16].

The fact that the medium presents different wavenumbers (or refractive indices) to RH-CP waves (β_+^+) and LH-CP waves (β_-^+) is referred to as *circular birefringence*, and is the fundamental cause of polarization rotation, as will be seen in Sec. V.

C. Opposite Propagation Direction

Reversing the direction of wave propagation from $+z$ to $-z$ implies reversing the sign of the argument of the spatial exponentials – and corresponding superscripts – in (20), without altering the complex vector $(\hat{x} \pm i\hat{y})$. This operation transforms (21) to⁸

$$\beta_{\pm}^- = -\omega(\sqrt{\epsilon\mu} \mp \chi) = \omega(-\sqrt{\epsilon\mu} \pm \chi), \quad (23a)$$

$$\mathbf{E}_{\pm}^-(z) = E_0(\hat{x} \pm i\hat{y}) e^{+i\beta_{\pm}^- z}, \quad (23b)$$

$$\mathbf{H}_{\pm}^-(z) = -\frac{\hat{z} \times \mathbf{E}_{\pm}^-(z)}{\eta} = \pm \frac{i}{\eta} \mathbf{E}_{\pm}^-(z), \quad (23c)$$

which results in reversing the circular-polarization handedness, as shown in Figs. 3(c) and 3(d), since the thumb is now pointing in the opposite direction whereas the signs of the complex vector have not changed.

One must be very careful to avoid confusion in determining the (temporal) wave handedness. In (21) and (23), the superscripts $\pm$ unambiguously indicate the direction of wave propagation, while the subscripts $\pm$ unambiguously indicate the sign of the complex vector $\hat{x} \pm i\hat{y}$. But neither the superscripts nor the subscripts indicate handedness. For instance, as shown in Fig. 3, the vector $(\hat{x} + i\hat{y})$ is associated

⁸Here, we chose to reverse the sign of β while leaving the sign of z unchanged. Doing the opposite would transform (23a) to the equivalent result $\beta_{\pm}^- = \omega(\sqrt{\epsilon\mu} \mp \chi)$ in the reversed coordinate system, where the reversal of the sign of χ compared to (21a) is a manifestation of this space-reversal odd symmetry of chirality [Eq. (21a)].

with right-handedness in positive propagation [Fig. 3(a)], but left-handedness in backward propagation [Fig. 3(d)].

D. Eigenstate and Modal Isotropy Perspective

Substituting the chiral modes (21) and (23) into the constitutive relations (7) respectively yields for the forward and backward propagation directions

$$\mathbf{D}_{\pm}^+ = \epsilon \mathbf{E}_{\pm}^+ + i\chi \mathbf{H}_{\pm}^+ = \left(\epsilon \pm \frac{\chi}{\eta} \right) \mathbf{E}_{\pm}^+ = \epsilon_{\pm}^+ \mathbf{E}_{\pm}^+, \quad (24a)$$

$$\mathbf{B}_{\pm}^+ = -i\chi \mathbf{E}_{\pm}^+ + \mu \mathbf{H}_{\pm}^+ = (\mu \pm \chi\eta) \mathbf{H}_{\pm}^+ = \mu_{\pm}^+ \mathbf{H}_{\pm}^+, \quad (24b)$$

and

$$\mathbf{D}_{\pm}^- = \epsilon \mathbf{E}_{\pm}^- + i\chi \mathbf{H}_{\pm}^- = \left(\epsilon \mp \frac{\chi}{\eta} \right) \mathbf{E}_{\pm}^- = \epsilon_{\pm}^- \mathbf{E}_{\pm}^-, \quad (25a)$$

$$\mathbf{B}_{\pm}^- = -i\chi \mathbf{E}_{\pm}^- + \mu \mathbf{H}_{\pm}^- = (\mu \mp \chi\eta) \mathbf{H}_{\pm}^- = \mu_{\pm}^- \mathbf{H}_{\pm}^-. \quad (25b)$$

Equations (24) and (25) reveal that $\mathbf{D}_{\pm}^{\pm} \parallel \mathbf{E}_{\pm}^{\pm}$ with amplitude ratio $\epsilon_{\pm}^{\pm}$ and $\mathbf{B}_{\pm}^{\pm} \parallel \mathbf{H}_{\pm}^{\pm}$ with amplitude ratio $\mu_{\pm}^{\pm}$, i.e., that the CP waves $(\mathbf{E}_{\pm}^{\pm}, \mathbf{H}_{\pm}^{\pm})$ and parameters $(\epsilon_{\pm}^{\pm}, \mu_{\pm}^{\pm})$ are the eigenvectors and eigenvalues of the chiral medium. One may alternatively say that, although generally biisotropic [Eq. (7)], a chiral medium is seen as *monoisotropic* ($\chi_{\pm}^{\pm} = \zeta_{\pm}^{\pm} = 0$) to CP waves, and hence preserves the polarization and handedness of these waves, altering only their phase and magnitude.

V. POLARIZATION ROTATION OR GYROTROPY⁹

How does such a chiral medium affect linearly polarized (LP) waves? The most natural way to address this question is to decompose a test LP wave into the medium (RH-CP and LH-CP) eigenstates, since these states form a simple, isotropic basis with well-defined eigenwavenumbers. Consider for instance the $\hat{x}$ -polarized $+z$ -propagating plane wave in the plane $z=0$, which reads, according to [1].(4), $\mathbf{E}_{\text{LP}}^+(z=0) = \hat{x}E_0$. The chiral medium resolves such a wave into its RH-CP and a LH-CP components as $\mathbf{E}_{\text{LP}}^+(z=0) = \frac{E_0}{2}(\hat{x} + i\hat{y}) + \frac{E_0}{2}(\hat{x} - i\hat{y})$. The propagation along the medium is then found by assigning to the RH-CP and LH-CP waves composing the LP wave their respective eigenwavenumbers, β_+^+ and β_-^+ , given by (21a), which results in

$$\mathbf{E}_{\text{LP}}^+(z) = \left[\frac{E_0}{2}(\hat{x} + i\hat{y}) e^{i\beta_+^+ z} + \frac{E_0}{2}(\hat{x} - i\hat{y}) e^{i\beta_-^+ z} \right]. \quad (26)$$

where multiplication by $e^{-i\omega t}$ shows that the two CP-wave vectors composing $\mathbf{E}_{\text{LP}}^+(z)$ rotate at different spatial rates, since $\beta_+^+ \neq \beta_-^+$, resulting in a net rotation of the (linear) polarization, or *gyrotropy*, as illustrated in Fig. 4. This is the *circular birefringence* explanation of Fresnel [18], where different medium properties are seen by the two CP waves.

⁹The terminology “gyrotropy” has been traditionally restricted to magnetized plasmas and ferrites, which are respectively characterized by permittivity and permeability tensors that involve antisymmetric terms [4]. We take here the liberty to prefer etymological logics ‘gyro-’ (γῠρος) meaning ‘circle’ or ‘spinning’ in Geek) to tradition, and hence apply the term “gyrotropy” also to chiral media, which rotate the polarization just as magnetized plasmas and ferrites, with the only difference that they do it in a reciprocal rather than a nonreciprocal fashion, as explained in this section.

Fig. 4. Chiral rotation of an LP wave (figure inspired by [19]). (a) Monoisotropic medium ($\chi = 0$), where the RH/LH-CP phasors rotate by the same amounts $\theta_{\pm}^{\pm} = \pm\omega\sqrt{\epsilon\mu}z/2$, resulting in no LP rotation. (b) Chiral medium ($\chi \neq 0$), where the RH/LH-CP phasors rotate by the different amounts $\theta_{\pm}^{\pm} = \omega(\sqrt{\epsilon\mu} \pm \chi)z/2$, resulting in a chiral rotation angle of $\theta_C^+ = \theta_+^+ - \theta_-^+ = \omega\chi z$. Note that this graph also holds for Faraday rotation [19] upon replacing the electric phasor and quantities by their magnetic counterparts, and χ by the magnetization saturation M_s [14], [20].

This may be mathematically shown by grouping the terms of same polarization in (26), and next factoring out the resulting exponential sums and difference to form sines and cosines, i.e.,

$$\begin{aligned} E_{LP}^+(z) &= E_0 \left[\hat{\mathbf{x}} \frac{e^{i\beta_+^+ z} + e^{i\beta_-^+ z}}{2} + i\hat{\mathbf{y}} \frac{e^{i\beta_+^+ z} - e^{i\beta_-^+ z}}{2} \right] \\ &= e^{i\beta_e^+ z} E_0 \{ \hat{\mathbf{x}} \cos(\beta_o^+ z) - \hat{\mathbf{y}} \sin(\beta_o^+ z) \}, \end{aligned} \quad (27a)$$

$$\text{where } \beta_e^+ = \frac{\beta_+^+ + \beta_-^+}{2} = \omega\sqrt{\epsilon\mu}, \quad \beta_o^+ = \frac{\beta_+^+ - \beta_-^+}{2} = \omega\chi. \quad (27b)$$

The last expression of (27a) reveals that the chiral medium rotates the polarization of the LP wave in space as it propagates a distance z by the angle

$$\begin{aligned} \theta_C(z; \beta_o^+) &= \tan^{-1} \left(\frac{E_y}{E_x} \right) = \tan^{-1} \left[\frac{-\sin(\beta_o^+ z)}{\cos(\beta_o^+ z)} \right] \\ &= -\tan^{-1} [\tan(\beta_o^+ z)] = -\beta_o^+ z = -\omega\chi z \end{aligned} \quad (28a)$$

and with the phase shift

$$\phi_C(z; \beta_e^+) = \beta_e^+ z = \omega\sqrt{\epsilon\mu}z. \quad (28b)$$

So a chiral medium preserves the LP nature of an LP wave, but rotates its polarization as it propagates, by an amount that is proportional to the *difference* of the RH-CP and LH-CP wavenumbers, or refractive indices $n_{\pm}^{\pm} = \beta_{\pm}^{\pm}/k_0 = c\beta_{\pm}^{\pm}/\omega$, and with a phase shift that is proportional to the *sum* of the same RH-CP and LH-CP wavenumbers.

This spatial rotation effect for an LP wave seems essentially identical to Faraday rotation [21] in magnetized plasmas [22] and magnetized ferrites [20] or transistor-loaded magnetless metaferrites [23], despite the fact that none of these media involve chiral particles (proof of ④ in Fig. [1].3) and are purely monoanisotropic (non magnetoelectric coupling), with anisotropy $\bar{\epsilon}$ for plasmas and anisotropy $\bar{\mu}$ for ferrites or metaferrites (proof $\rightarrow$ of ⑥ in Fig. [1].3). However, there is a fundamental difference between chiral rotation and Faraday rotation [21]: the former is reciprocal, as shown from the medium reciprocity condition in Sec. II, whereas the latter breaks that reciprocity condition, and is hence nonreciprocal [11]. The two types of gyrotropies are illustrated in Fig. 5.

Fig. 5. Comparison of chirality gyrotropy and Faraday gyrotropy using direct-time (left) and time-reversed (right) experiments. (a) Chiral medium (reciprocal), where $\theta_C(z) = \omega\chi z$. (b) Faraday medium (nonreciprocal), where μ and κ form the permeability tensor as $\theta_F(z) = \omega(\sqrt{\epsilon(v+\kappa)} - \sqrt{\epsilon(v-\kappa)})z/2$ with $\bar{\mu} = [v, -i\kappa, 0; i\kappa, v, 0; 0, 0, \mu_0]$ [14], [20].

The reciprocity of chiral rotation is readily apparent in (28). Since reversing the direction of wave propagation is equivalent to reversing the sign of the argument of the spatial phase, i.e., of either β or z , consider reversing the sign of β_o while maintaining the coordinate system fixed in (28). This results in

$$\theta_C(z; -\beta_o^+) = -(-\beta_o^+)z = \beta_o^+ z = -\theta_C(z; \beta_o^+), \quad (29)$$

which shows that that if the LP wave has accumulated a rotation angle of θ as propagating in the $+z$ direction, this angle is undone as the wave returns, propagating in the $-z$ direction, so that the initial polarization state is retrieved, as shown in Fig. 5(a). Such a medium is therefore time-reversal symmetric, and hence reciprocal, as it should be due to the absence of external force and asymmetric nonlinearity [11].

At the microscopic scale of the helix chiral particle [Fig. [1].5(b)], reciprocity may be understood by reversing the direction of propagation from $+z$ to $-z$ in Fig. [1].9, which corresponds to reversing the direction of either $\mathbf{E}$ or $\mathbf{H}$ so as to preserve the handedness of the triad $(\mathbf{E}, \mathbf{H}, \mathbf{k})$. First, consider the top of Fig. [1].9, with $\mathbf{E}$ still pointing towards $+x$, and hence $\mathbf{H}$ pointing now towards $-y$; since the induced dipolar moments are unchanged, $\mathbf{H}$ rotates from the $-y$ direction to the $+x$ direction. Similarly, consider now the bottom of Fig. [1].9, with $\mathbf{H}$ still pointing towards $+x$, and hence $\mathbf{E}$ pointing now towards $+y$; given the again unchanged dipolar moments, $\mathbf{E}$ rotates from the $+y$ direction to the $-x$ direction. The electromagnetic field has therefore rotated about the z axis in the direction corresponding to the right-hand with the thumb pointing in the propagation direction ($-z$), which is the opposite rotation direction than that found for $+z$ propagation. Note that with the chosen convention for rotation handedness, depicted in Fig. 3, a RH helices (as in Fig. [1].9) induce LH CP (and RH spatial spirals), and LH helices induce RH CP

(and LH spatial spirals), which corresponds to (29) and the rotation rewinding in Fig. 5.

In contrast to chiral rotation, in Faraday rotation, the angle acquired by the wave propagating along the $+z$ direction, θ , keeps accumulating in the same absolute direction, dictated by the fixed bias field $\mathbf{B}_0$, when propagating back in the $-z$ direction. This leads to a total rotation angle of 2θ for the round trip, as shown in Fig. 5(b). So, the initial state of the system is not retrieved upon time reversal, and the system is nonreciprocal [11], which allows the realization of devices such as isolators, circulators and nonreciprocal phase shifters.

VI. SPATIAL-FREQUENCY PROPERTIES

A. Spatial Dispersion (or Spatial Nonlocality)

At frequencies where the size of the particles and the distance between them are not both much smaller than the wavelength (say not smaller than $\lambda/4$), the polarization response of the medium at a given point depends on the field scattered at surrounding points, i.e.,

$$\mathbf{P}_{\{\mathbf{e}\}}(\mathbf{r}) = \int_{\Omega} \bar{\chi}_{ab}(\mathbf{r}, \mathbf{r}') \cdot \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix}(\mathbf{r}') d\mathbf{r}', \quad (30)$$

where $ab = ee, em, me$ or mm , and Ω is the volume of the medium over which this *neighboring effect* extends¹⁰. This effect is called *spatial dispersion* (or *spatial nonlocality*) [3], [24], [25], [26], and the expansion of (30) in Fourier integral with respect to space ($\mathbf{r}$), for easier treatment in terms of a set of plane waves, leads to $\mathbf{k}$ -dependent susceptibilities, namely, $\bar{\chi}_{ab} \rightarrow \bar{\chi}_{ab}(\mathbf{k})$ [24], [25], [26], [27]. Spatial dispersion is very common in metamaterials, where the condition $a \leq p \ll \lambda$ [Eq. [1].(6)] is only mildly satisfied at the edge of the homogenization range [25], [27]. Chirality can be shown to be an effect of *first-order* spatial dispersion, where the field response is associated with the first spatial derivative of the field excitation (see Appendix D).

B. Microscopic Origin of Spatial Dispersion

As mentioned in Sec. VI-A, spatial dispersion macroscopically involves a dependence of the medium properties on the wave direction ($\mathbf{k}$). However, this effect has naturally a microscopic origin. In the case of chiral metamaterials, this origin can be easily understood from the microscopic analysis of the Omega and helix particles in [1], where the local and nonlocal responses have been indicated by the respective labels 'L' and 'NL' in Figs. [1].8 and 9. First, consider the planar Omega particle. The direct responses p_{e,E_x}^x , corresponding to α_{ee}^{xx} , and p_{m,H_y}^{-y} , corresponding to α_{mm}^{xy} are produced exactly at the points of excitation, namely at the straight sections and at the looped section, respectively; they are thus *local*. In contrast, the cross responses p_{m,E_x}^y and p_{e,H_y}^{-x} , respectively corresponding to α_{me}^{yx} and α_{em}^{xy} , are produced away from the points of excitation,

¹⁰If the medium is operated in its *homogeneous* regime, then its response depends only on the *distance* between the source ($\mathbf{r}'$) and observation ($\mathbf{r}$) points, and not on their absolute positions. Therefore, we can substitute $\bar{\chi}_{ab}(\mathbf{r}, \mathbf{r}') \rightarrow \bar{\chi}_{ab}(\mathbf{r} - \mathbf{r}')$ in (30), which becomes then a convolution integral.

namely at the straight sections and at the looped section, respectively; they are thus *nonlocal*. The medium is thus globally nonlocal. The nonlocality of the twisted Omega or helix particle may be understood from the same current continuity thought experiment, with local polarizabilities α_{ee}^{xx} and α_{mm}^{xx} , and nonlocal polarizabilities α_{me}^{xx} and α_{em}^{xx} .

C. Plane-Wave Description

It is instructive to consider the particular case of *plane-wave* propagation in the chiral medium. In this case, the source may be considered to be at infinity and the current at any point of the medium may hence be set to zero, which leads to insightful relations between the response and excitation fields.

We first solve the sourceless relations (8) for $\mathbf{E}$ and for $\mathbf{H}$ in terms of $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}$, which yields

$$\mathbf{E} = \frac{1}{\omega\gamma} (i\mu\nabla \times \mathbf{H} - \chi\nabla \times \mathbf{E}) \quad \text{and} \quad \mathbf{H} = -\frac{1}{\omega\gamma} (i\epsilon\nabla \times \mathbf{E} + \chi\nabla \times \mathbf{H}), \quad (31)$$

with $\gamma = \epsilon\mu - \chi^2$. Next, we substitute these relations into the expressions of (7) that are associated with chirality, i.e., having the coefficient χ . Specifically, inserting the second equation of (31) into the first equation of (7) and the first equation of (31) into the second equation of (7) provides the alternative constitutive relations

$$\mathbf{D} = \epsilon\mathbf{E} + \frac{\chi}{\omega\gamma} (\epsilon\nabla \times \mathbf{E} - i\chi\nabla \times \mathbf{H}), \quad (32a)$$

$$\mathbf{B} = \mu\mathbf{H} + \frac{\chi}{\omega\gamma} (\mu\nabla \times \mathbf{H} + i\chi\nabla \times \mathbf{E}), \quad (32b)$$

which seem to be presented here for the first time.

Given the assumed linearity, one may then Fourier-transform (32), which leads for each plane wave of the corresponding angular spectrum expansion [28], $\tilde{\Psi}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$ ($\Psi = \mathbf{D}, \mathbf{B}, \mathbf{E}, \mathbf{H}$), the relations

$$\tilde{\mathbf{D}} = \epsilon\tilde{\mathbf{E}} + \frac{\chi}{\omega\gamma} (i\epsilon\mathbf{k} \times \tilde{\mathbf{E}} + \chi\mathbf{k} \times \tilde{\mathbf{H}}), \quad (33a)$$

$$\tilde{\mathbf{B}} = \mu\tilde{\mathbf{H}} + \frac{\chi}{\omega\gamma} (i\mu\mathbf{k} \times \tilde{\mathbf{H}} - \chi\mathbf{k} \times \tilde{\mathbf{E}}). \quad (33b)$$

Projecting these equations onto the x , y and z directions finally yields

$$\begin{pmatrix} \tilde{\mathbf{D}} \\ \tilde{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} \bar{\bar{\epsilon}}(\mathbf{k}) & \bar{\bar{\xi}}(\mathbf{k}) \\ \bar{\bar{\zeta}}(\mathbf{k}) & \bar{\bar{\mu}}(\mathbf{k}) \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{pmatrix}, \quad (34a)$$

$$\text{where} \quad \begin{pmatrix} \bar{\bar{\epsilon}}(\mathbf{k}) & \bar{\bar{\xi}}(\mathbf{k}) \\ \bar{\bar{\zeta}}(\mathbf{k}) & \bar{\bar{\mu}}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} \epsilon\bar{\bar{I}} + a_c\mathbf{k} \times \bar{\bar{I}} & a_\chi\mathbf{k} \times \bar{\bar{I}} \\ -a_\chi\mathbf{k} \times \bar{\bar{I}} & \mu\bar{\bar{I}} + a_\mu\mathbf{k} \times \bar{\bar{I}} \end{pmatrix}, \quad (34b)$$

which involves the skewon (or antisymmetric) dyadic $\mathbf{k} \times \bar{\bar{I}}$ that characterizes gyrotropic media [29]. The relations represent the plane-wave description of the chiral medium

In the general bianisotropic case, the relations (32) and (33) respectively generalize to (see Appendix E)

$$\mathbf{D} = \bar{\bar{\epsilon}} \cdot \mathbf{E} - \frac{i}{\omega} \bar{\bar{\xi}} \cdot \left(\bar{\bar{\mu}} - \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \bar{\bar{\zeta}} \right)^{-1} \cdot \left(\nabla \times \mathbf{E} + \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H} \right), \quad (35a)$$

$$\mathbf{B} = \frac{i}{\omega} \bar{\bar{\zeta}} \cdot \left(\bar{\bar{\epsilon}} - \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{-1} \cdot \bar{\bar{\zeta}} \right)^{-1} \cdot \left(\nabla \times \mathbf{H} + \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{-1} \cdot \nabla \times \mathbf{E} \right) + \bar{\bar{\mu}} \cdot \mathbf{H}, \quad (35b)$$

$$\tilde{\mathbf{D}} = (\bar{\bar{\epsilon}} + \bar{\bar{T}}_D \cdot \mathbf{k} \times \bar{\bar{I}}) \cdot \tilde{\mathbf{E}} + (\bar{\bar{T}}_D \cdot \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \mathbf{k} \times \bar{\bar{I}}) \cdot \tilde{\mathbf{H}}, \quad (36a)$$

$$\text{with } \bar{\bar{T}}_D = \frac{1}{\omega} \bar{\bar{\zeta}} \cdot (\bar{\bar{\mu}} - \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \bar{\bar{\zeta}})^{-1} \quad (36b)$$

$$\tilde{\mathbf{B}} = (\bar{\bar{T}}_B \cdot \bar{\bar{\zeta}} \cdot \bar{\bar{\mu}}^{-1} \cdot \mathbf{k} \times \bar{\bar{I}}) \cdot \tilde{\mathbf{E}} + (\bar{\bar{\mu}} + \bar{\bar{T}}_B \cdot \mathbf{k} \times \bar{\bar{I}}) \cdot \tilde{\mathbf{H}} \quad (36c)$$

$$\text{with } \bar{\bar{T}}_B = -\frac{1}{\omega} \bar{\bar{\zeta}} \cdot (\bar{\bar{\epsilon}} - \bar{\bar{\zeta}} \cdot \bar{\bar{\mu}}^{-1} \cdot \bar{\bar{\zeta}})^{-1}, \quad (36d)$$

which leads to more complex tensorial relations [33].

D. Application to the Omega Particles

This section applies the plane-wave formulation of chirality presented in Sec. VI-C to better relate the microscopic behavior of Omega-type particles (Sec. [1].V) to their macroscopic response (Secs. II).

Let us start with the planar Omega particle. Applying (36) to the tensorial structure [1].(13) yields, after translating [1].(13) into its macroscopic form with (3) and setting $\tilde{E}_y = \tilde{E}_z = \tilde{H}_x = \tilde{H}_z = k_x = k_y = 0$ ($k_z = k_0$) for the $(\tilde{E}_x, \tilde{H}_y, k_z)$ polarization in Fig. [1].8,

$$\tilde{D}_x = \epsilon_0 \epsilon_r^{xx} \left(1 + \frac{\chi_{em}^{xy}}{\varphi} \right) \tilde{E}_x - \frac{\sqrt{\epsilon_0 \mu_0} (\chi_{em}^{xy})^2}{\varphi} \tilde{H}_y, \quad (37a)$$

$$\tilde{B}_y = -\frac{\sqrt{\epsilon_0 \mu_0} (\chi_{em}^{xy})^2}{\varphi} \tilde{E}_x + \mu_0 \mu_r^{yy} \left(1 - \frac{\chi_{em}^{xy}}{\varphi} \right) \tilde{H}_y, \quad (37b)$$

$$\tilde{D}_y = \tilde{D}_z = \tilde{B}_x = \tilde{B}_z = 0, \quad (37c)$$

$$\text{with } \varphi = 1 + \chi_{ee}^{xx} + (\chi_{em}^{xy})^2 + \chi_{mm}^{yy} + \chi_{ee}^{xx} \chi_{mm}^{yy} \quad (37d)$$

where the term $(\chi_{em}^{xy})^2 = -\chi_{em}^{xy} \chi_{me}^{yx}$ from [1].(12), and

$$\epsilon_r^{xx} = 1 + \chi_{ee}^{xx}, \quad \mu_r^{yy} = 1 + \chi_{mm}^{yy}, \quad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}. \quad (38)$$

The relations (37) are in agreement with the observations made in Sec. [1].V-C, and provide in addition the exact quantitative response of the related medium. Note that, applying (36) to the perpendicular polarization $(\tilde{E}_y, -\tilde{H}_x, k_z)$ where the wave does not interact with the particle, properly retrieves the free-space solution $\tilde{D}_y = \epsilon_0 \tilde{E}_y$ and $\tilde{B}_x = -\mu_0 \tilde{H}_x$ (with $\tilde{D}_x = \tilde{D}_z = \tilde{B}_y = \tilde{B}_z = 0$). In the case of the oblique incident polarization $[(\tilde{E}_x, \tilde{E}_y), (\tilde{H}_y, -\tilde{H}_x), k_z]$, applying (36) naturally adds, by superposition, $\tilde{D}_y = \epsilon_0 \tilde{E}_y$ and $\tilde{B}_x = -\mu_0 \tilde{H}_x$ to the response in (37), leading to the polarization angle

$$\theta_D = \tan^{-1} \left(\frac{\tilde{D}_y}{\tilde{D}_x} \right) = \tan^{-1} \left(\frac{\epsilon_0 \tilde{E}_y}{\epsilon^{xx} \tilde{E}_x + \xi^{xy} \tilde{H}_y} \right), \quad (39a)$$

$$\text{with } \epsilon^{xx} = \epsilon_r^{xx} \left(\epsilon_0 + \frac{k_z \chi_{em}^{xy}}{\eta_0 \omega \varphi} \right), \quad \xi^{xy} = -\frac{k_z (\chi_{em}^{xy})^2}{\omega \varphi}, \quad (39b)$$

consistently with the waveplate effects already noted in the microscopic analysis of Sec. [1].V-C.

Let us now analyze the medium composed of twisted Omega or helix particle. Analogously to the case of the straight Omega particle, we shall apply (36) to the tensorial structure [1].(17), after translating [1].(17) into its macroscopic form with (3) and for the appropriate polarization. However, since this structure induces polarization rotation

(Sec. [1].V-D) and since its eigenstates are consequently CP modes (Sec. IV), the medium resolves LP incident waves into their two CP states (RH and LH) (Sec. V), given by (21) and (23). This means that the $\mathbf{k}$ in (36) corresponds to the eigenstates $\mathbf{k}_\pm^\pm$ in reference to (21) and (23). Let us consider, as in Fig. [1].9, a forward LH-CP wave propagating along the +z direction, which corresponds to

$$\mathbf{k}_-^+ = \beta_-^+ \hat{\mathbf{z}} = \omega (\sqrt{\epsilon \mu} - \chi) \hat{\mathbf{z}}, \quad (40a)$$

$$\begin{Bmatrix} \mathbf{E}_-^+(z) \\ \mathbf{H}_-^+(z) \end{Bmatrix} = E_0 \begin{Bmatrix} 1 \\ i/\eta \end{Bmatrix} (\hat{\mathbf{x}} - i\hat{\mathbf{y}}) e^{i\beta_-^+ z}. \quad (40b)$$

For this wave, we have to set $\mathbf{k} = \mathbf{k}_-^+ = \beta_-^+ \hat{\mathbf{z}}$ ($k_x = k_y = 0$, $k_z = k_0$), $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_-^+ = E_0 (\hat{\mathbf{x}} - i\hat{\mathbf{y}})$ and $\tilde{\mathbf{H}} = \tilde{\mathbf{H}}_-^+ = i(E_0/\eta) (\hat{\mathbf{x}} - i\hat{\mathbf{y}})$ in (36). The result is

$$\tilde{D}_x = \epsilon_0 \left[\epsilon_r^{xx} + i \frac{\chi_{em}^{xx}}{\varphi} (\epsilon_r^{xx} + \chi_{me}^{xx}) \right] E_0, \quad (41a)$$

$$\tilde{B}_x = \mu_0 \left[\mu_r^{xx} - i \frac{\chi_{me}^{xx}}{\varphi} (\chi_{em}^{xx} + \mu_r^{xx}) \right] \frac{E_0}{\eta_0}, \quad (41b)$$

$$\tilde{D}_y = -i\epsilon_0 E_0, \quad \tilde{B}_y = -i \frac{\mu_0}{\eta_0} E_0, \quad \tilde{D}_z = \tilde{B}_z = 0, \quad (41c)$$

$$\text{with } \varphi = 1 + \chi_{ee}^{xx} + (\chi_{em}^{xx})^2 + \chi_{mm}^{xx} + \chi_{ee}^{xx} \chi_{mm}^{xx}. \quad (41d)$$

where $(\chi_{em}^{xx})^2 = -\chi_{em}^{xx} \chi_{me}^{xx}$ following from [1].(16). These relations are harder to decipher than their planar-Omega counterparts, but certainly include complementary information on the bianisotropic properties of the chiral medium. If the monoatomic helical particle corresponding this medium would be transformed into the triatomic particle in Fig. [1].10 to form the biisotropic chiral metamaterial shown in Fig. 2, the complicated bianisotropic relations (41) would naturally reduce to more fundamental and practical relations (34).

VII. TEMPORAL-FREQUENCY PROPERTIES

A. Temporal Dispersion (or Temporal Nonlocality)

In general, due to the inertia of matter to elastic and damping forces, the response of a medium at a given time depends not only on the excitation at that time but also at previous times, i.e.,

$$\mathbf{P}_{\{m\}}^{\{e\}}(t) = \int_T \bar{\bar{\chi}}_{ab}(t, t') \cdot \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix}(t') dt', \quad (42)$$

where $ab = ee, em, me$ or mm , and T is the duration of this memory effect. This effect is called *temporal dispersion* (or *temporal dispersion*) – which is naturally the temporal counterpart of spatial dispersion [Eq. (30)]¹¹ –, and the expansion of (42) in Fourier integral with respect to time (t), for easier treatment in terms of a set of harmonic waves, leads to ω -dependent susceptibilities, namely, $\bar{\bar{\chi}}_{ab} \rightarrow \bar{\bar{\chi}}_{ab}(\omega)$ [2], [3], [4].

¹¹Here, if the medium is operated in a *time-invariant* regime, then its response depends only on the *difference* between the excitation (t') and response (t) times, and not on their absolute positions. Therefore, we can substitute $\bar{\bar{\chi}}_{ab}(t, t') \rightarrow \bar{\bar{\chi}}_{ab}(t - t')$ in (42), which becomes then a *convolution integral*.

B. Lorentz Response

The medium susceptibilities must exhibit temporal frequency (ω) dependences that obey causality or, mathematically, that follows the Kramers-Kronig relations [3]. For composites made of relatively low-loss particles, such as the straight and looped sections of the Omega-type particles considered in this paper (Fig. [1].5), dispersion follows the Lorentz model [2], [3], [4] (also see Appendix F)

$$\chi_{ab}(\omega) = -\frac{\omega_{p,ab}\omega_x}{\omega^2 - \omega_{0,ab}^2 + i\omega\nu_{ab}}, \omega_x = \begin{cases} \omega_{p,ab} & \text{if } ab = ee, mm, \\ -i\omega & \text{if } ab = em, me, \end{cases} \quad (43)$$

where $\omega_{0,ab}$, $\omega_{p,ab}$ and ν_{ab} denote the resonance frequencies, plasma frequencies and damping factors, respectively¹².

According to (13), the dispersive susceptibility relations (43) correspond to the relative chiral parameters are

$$\epsilon_r(\omega) = \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \chi_{ee}(\omega) = \frac{\omega^2 - \omega_{p,\epsilon}^2 - \omega_{0,\epsilon}^2 + i\omega\nu_\epsilon}{\omega^2 - \omega_{0,\epsilon}^2 + i\omega\nu_\epsilon}, \quad (44a)$$

$$\mu_r(\omega) = \frac{\mu(\omega)}{\mu_0} = 1 + \chi_{mm}(\omega) = \frac{\omega^2 - \omega_{p,\mu}^2 - \omega_{0,\mu}^2 + i\omega\nu_\mu}{\omega^2 - \omega_{0,\mu}^2 + i\omega\nu_\mu}, \quad (44b)$$

$$\chi_r(\omega) = \frac{\chi(\omega)}{\sqrt{\epsilon_0\mu_0}} = -i\chi_{em}(\omega) = \frac{\omega_{p,\chi}\omega}{\omega^2 - \omega_{0,\chi}^2 + i\omega\nu_\chi}, \quad (44c)$$

where we have replaced wherever appropriate the subscripts ee, mm and em by the subscripts ϵ , μ and χ , respectively, for notational convenience. In these relations, $\omega_{0,\epsilon}$, $\omega_{0,\mu}$ and $\omega_{0,\chi}$ are usually interdependent, as will be shown in Sec. VII-C.

Using low-loss materials, we typically have $\omega\nu_{\epsilon,\mu,\chi} \ll \omega^2$, and the damping terms $\omega\nu_{\epsilon,\mu,\chi}$ can then be neglected to the first order. Moreover, the resonance frequency is often considerably lower than the plasma and operating frequencies, i.e., $\omega_0 \ll \omega_{p,\epsilon,\mu,\chi}, \omega$. Under such conditions, Eqs. (44) can be approximated by the Drude lossless forms

$$\left\{ \begin{matrix} \epsilon_r(\omega) \\ \mu_r(\omega) \end{matrix} \right\} = 1 + \left\{ \begin{matrix} \chi_{ee}(\omega) \\ \chi_{mm}(\omega) \end{matrix} \right\} \approx \frac{\omega^2 - \left\{ \begin{matrix} \omega_{p,\epsilon}^2 \\ \omega_{p,\mu}^2 \end{matrix} \right\}}{\omega^2}, \quad (45a)$$

$$\chi_r(\omega) \approx \frac{\omega_{p,\chi}}{\omega}. \quad (45b)$$

These relations provide precious insight into the dispersion response of a low-loss chiral medium. They most importantly reveal that in such media $\epsilon(\omega)$ and $\mu(\omega)$ tend to be respectively positive and negative above and below their plasma frequency while $\chi(\omega)$ follow the opposite polarities.

¹²The odd factor ω in $\chi_{em,me}$ ensures that these susceptibilities have the opposite parity of $\chi_{ee,mm}$, as shown in Appendix F to be required, while the factor $-i$ ensures that χ and χ_{em} have the same polarity, as required from considering that $\chi = -i\sqrt{\epsilon_0\mu_0}\chi_{em}$ according to (13) while $\chi \in \mathbb{R}$ in the lossless case according to (15).

C. Metamaterial Resonance and Plasma Frequencies

A metamaterial maximally interacts with electromagnetic energy near its lowest resonance (Sec. [1].IV-B), which occurs at the frequency where the incoming wave best matches the boundary conditions of the metaparticles forming the metamaterial, while fulfilling the subwavelength dimensional constraint [1].(6). In the case of single-block particles, such as the Omega-type particles, this corresponds to the situation where the size of the unfolded structure, ℓ , is about half the wavelength at resonance ($\ell \approx \lambda_{\text{res}}/2$) if it is highly conductive, or somewhat less ($\lambda_{\text{res}}/10 < \ell < \lambda_{\text{res}}/2$) if it is dielectric or plasmonic, due to field penetration, and not to the separate sizes of the straight-pair section and looped section. There is thus a unique and common resonance frequency for the three parameters in (44), i.e.,

$$\omega_0 = \omega_{0,\epsilon} = \omega_{0,\mu} = \omega_{0,\chi}. \quad (46)$$

Contrarily to the resonance frequency, the plasma frequencies may differ from each other. Indeed, the plasma frequencies in (44) are proportional to the density of the corresponding dipole moments¹³, which are generally different for the electric ($\mathbf{p}$) and magnetic ($\mathbf{m}$) responses. For instance, in the case of the Omega particles, we have seen in Sec. [1].IV that most of the $\mathbf{p}$ response is related to the straight section while most of the $\mathbf{m}$ response is related to the looped section, which leads to particularizing [1].(9) as

$$\mathbf{p}_e = \int_{V_{\text{straight}}} \mathbf{r}' \rho(\mathbf{r}') d\mathbf{r}', \quad \mathbf{p}_m = \frac{\mu_0}{2} \int_{V_{\text{looped}}} \mathbf{r}' \times \mathbf{J}(\mathbf{r}') d\mathbf{r}', \quad (47)$$

where the relevant integration volumes have been explicitated. Omega particles with a much larger (resp. smaller) straight section and a much smaller (resp. larger) looped section, following the antagonistic dimensional constraint [1].(10), will then be characterized by larger densities of electric (resp. magnetic) dipole moments, and hence by larger electric (resp. magnetic) plasma frequency, while the cross-coupling frequency, involving the same geometrical parts, will be intermediate, i.e.,

$$2d \gg s \iff \omega_{p,\epsilon} \gtrless \omega_{p,\chi} \gtrless \omega_{p,\mu}. \quad (48)$$

D. Parametric Study

Figure 6 plots the dispersive responses (44) for a chiral medium with equal permittivity, permeability and chiral factor plasma frequencies, $\omega_{p,\epsilon} = \omega_{p,\mu} = \omega_{p,\chi} = \omega_p$, occurring when the straight and looped sections of the helix particle have comparable dimensions. As predicted by (45), the real parts of the permittivity and permeability are positive (positive refractive index) and negative (negative refractive index) above and below that frequency, respectively [10], while χ is positive in that frequency range. Moreover, the imaginary parts of ϵ , μ and χ are always positive, consistently with the implicit assumption of passivity (Sec. II-D and Appendix B).

¹³This is well-known to be the case in regular plasmas, where the plasma frequency of the conduction electrons is given by $\omega_p = \sqrt{Ne^2/(m^*\epsilon_0)}$, where N , e and m^* are the density, charge and effective mass of these electron [3], but this is also true in metamaterials, as shown in [30] for split-ring resonators upon safely defining the effective permeability as the ratio of the $\mathbf{B}$ and $\mathbf{D}$ fields.

Fig. 6. Dispersion responses (44) for the parameters $\omega_{p,\epsilon} = \omega_{p,\mu} = \omega_{p,\chi} = \omega_p = 3\omega_0$, corresponding to helix particle with similar straight-section and looped-section dimensions, $v_\epsilon = v_\mu = 0.8\omega_0$ and $v_\chi = 0.3\omega_0$. (a) Full scale. (b) Zoom about the abscissa. The exact poles are $\omega_{0,\chi}^*/\omega_0 \approx 1.000$ and $\omega_{0,\epsilon}^*/\omega_0 = \omega_{0,\mu}^*/\omega_0 \approx 1.038$.

Interestingly, at the common – or *balanced* – plasma frequency, we have $\epsilon = \mu = 0$ – and hence zero refractive index [10] – and $\chi \neq 0$. Equations (7) reduce then to

$$\mathbf{D}(\omega_p) = i\chi(\omega_p)\mathbf{H}(\omega_p) \quad \text{and} \quad \mathbf{B}(\omega_p) = -i\chi(\omega_p)\mathbf{E}(\omega_p), \quad (49)$$

which correspond to a *purely chiral* medium, also referred to as *chiral nihility* medium [31]. In this case, an LP wave is still rotated by the angle $\theta_C(z) = \mp\omega\chi z$ [Eq. (28a), where $\theta_C(z)$ does not depend on (ϵ, μ)], but it does not undergo any phase shift, since $\phi_C(z) = \pm\omega\sqrt{\epsilon\mu}z = 0$ [Eq. (77b)].

At the resonance frequency, the real parts of all the parameters change sign, which specifically means for $\chi(\omega)$ a change of the polarization-rotation direction, according to (28a), but this region of the spectrum is highly lossy and should therefore be avoided unless really needed.

Figure 7 compares chiral media with different plasma frequencies, with Fig. 7(a) and Fig. 7(b) respectively corresponding to a helix particle with a straight section that is much longer and much shorter than the looped section, where the plasma frequencies follow the two sequences in (48). Here, the difference in the plasma frequencies leads to the opening of forbidden bands, or stopbands, corresponding imaginary phase shifts in (77b). These stopbands correspond to the frequency range extending between the double-negative (or negative refractive index) and double-positive (or positive refractive index) bands. The same comment as above applies for the common resonance frequency.

VIII. DESIGN GUIDELINES

A chiral metamaterial may be designed as follows:

- 1) Choose the balanced design (Fig. 6) rather than an unbalanced design (Fig. 7) to avoid permittivity or permeability stopbands. We have then $\omega_{p,\epsilon} = \omega_{p,\mu} = \omega_{p,\chi} = \omega_p$, corresponding to a particle with balanced straight-section and looped-section dimensions.
- 2) If one targets a purely chiral response (polarization rotation without phase shift), corresponding to (49),

Fig. 7. Dispersion responses (44) for $\omega_{p,\chi} = 3\omega_0$ and $v_\epsilon = v_\mu = v_\chi = 0.1$ ($\omega_{0,\epsilon}^*/\omega_0 \approx \omega_{0,\mu}^*/\omega_0 \approx \omega_{0,\chi}^*/\omega_0 \approx 1.00$). (a) Helix with straight section much larger than looped section ($2d \gg s$), and $\omega_{p,\epsilon} = 4.5\omega_0$ and $\omega_{p,\mu} = 2\omega_0$ ($\omega_{p,\epsilon}^*/\omega_0 = 4.609$, $\omega_{p,\mu}^*/\omega_0 = 2.333$). (b) Helix with looped section much larger than straight section ($s \gg 2d$), and $\omega_{p,\epsilon} = 2\omega_0$ and $\omega_{p,\mu} = 4.5\omega_0$ ($\omega_{p,\epsilon}^*/\omega_0 = 2.333$, $\omega_{p,\mu}^*/\omega_0 = 4.609$).

then the medium should be operated at the common plasma frequency, ω_p ; so, set the plasma frequency equal to the operation frequency, ω_{op} , i.e., $\omega_p = \omega_{op}$. Otherwise, set $\omega_p = r\omega_{op}$ with $r > 0$ for negative-index response or $r < 0$ for positive-index response; $|r|$ might initially be in the order of 1.1 or 0.9 for operation at 10% off the plasma frequency.

- 3) Select a particle with a 90° twist with between its straight and looped parts, e.g., a helix.
- 4) Since the operation frequency must be larger than the resonance frequency, $\omega_{op} > \omega_0$, and since the unfolded length of the particle is about half the wavelength at resonance ($\ell \approx \lambda_{res}/2$), that length much be somewhat greater at the operation frequency, i.e., $\ell > c/(2f_{op}) = \pi c/\omega_{op}$. Set an initial guess for it, possibly using some analytical formulas [32].
- 5) Simulate the scattering parameters of the particle as indicated in Fig. 8, and extract the (frequency-dependent) parameters χ , ϵ and μ from the formulas (see Appendix G)

$$\chi = -\frac{\tan^{-1}(|S_{21}^{yx}|/|S_{21}^{xx}|)}{\omega p}, \quad (50a)$$

$$\epsilon = \frac{1}{Z_{port}} \left(\frac{\angle S_{21}^{yx}}{\omega p} \right) \left(\frac{1 - S_{11}^{xx}}{1 + S_{11}^{xx}} \right), \quad (50b)$$

$$\mu = Z_{port} \left(\frac{\angle S_{21}^{yx}}{\omega p} \right) \left(\frac{1 + S_{11}^{xx}}{1 - S_{11}^{xx}} \right), \quad (50c)$$

where p is the period of the metastructure. Note that this technique automatically accounts for interparticle coupling in case the dilute-medium condition is not satisfied, thanks to the utilization of periodic boundary conditions (infinite periodicity) in the x and y directions and to the utilization of a sufficient number of periods in the z direction for simulation convergence to the periodic regime.

- 6) Fine-tune the structure parameters using a full-wave optimization tool until satisfaction.

Fig. 8. Full-wave simulation set up of the chiral particle for synthesis of the corresponding chiral metamaterial by iterative analysis. The computational box is a z -oriented cylinder with rectangular cross section of dimension $p \times p$ and length Np (i.e., N periods, $N \in [5, 10]$ – here $N = 7$ – for convergence to a periodic response upon washing out the port-edge aperiodic effects), whose two sides perpendicular to $\hat{z}$ are defined as input and output ports while the four parallel sides parallel to $\hat{z}$ are defined as periodic boundary conditions. The ports 1 are P_1^x and P_1^y while the ports 2 are P_2^x and P_2^y (e.g. S_{21}^{yx} represents the transmission from the port P_1^x to the port P_2^y).

IX. CONCLUSIONS

In the second part of this two-part paper, we have presented a macroscopic description of electromagnetic chirality and materials. The main conclusions and results of this part may be summarized as follows:

- 1) A biisotropic medium is not necessarily chiral. It is chiral if its electric and magnetic responses are respectively in quadrature with their magnetic and electric counterpart excitations. In this case, the medium is indeed chiral, and is also called a Pasteur medium, and such a medium is reciprocal. If these responses and excitations are in phase, the biisotropic medium is called a Tellegen medium; such a medium is non-reciprocal, and therefore requires magnetization.
- 2) Whereas the permittivity and permeability functions are even under space reversal, the chirality parameter function is odd under space reversal.
- 3) The fundamental modes, or eigenstates, of a chiral medium are circularly polarized left-handed or right-handed waves, whose handedness changes with the direction of propagation. The chiral eigenstates ‘see’ the chiral medium as monoisotropic.
- 4) The polarization rotation in a chiral medium is a direct consequence of its circularly polarized eigenstates: the chiral medium resolves an incident linearly polarized wave in its right-handed and left-handed components and assigns them the corresponding different wavenumbers (or refractive indices), which results in a net polarization rotation, with rotation direction depending on the handedness of the chiral particles forming the medium. This is called circular birefringence, not to be confused with phase birefringence (e.g. case of the planar Omega particle or standard birefringent crystals) leading to waveplate-type polarization transformation.

- 5) In contrast to Faraday rotation, which is nonreciprocal (and monoanisotropic), chiral or Pasteur polarization rotation (reciprocally) unwinds to its original state upon time reversal.
- 6) Different right-handed and left-handed eigenstate absorptions in the presence of loss distort circular polarization into elliptic polarization, a phenomenon called circular dichroism.
- 7) A chiral medium is spatially dispersive (or spatially nonlocal), i.e., its response at a given point of space depends also on the excitation in the vicinity of that point. Spatial dispersion leads to an alternative of the standard bianisotropic or biisotropic constitutive relations where the component of the response fields can be explicitly written in terms of the components of the excitation fields. This formulation of chirality is explicit but tensorial, whereas the standard formulation is scalar but implicit.
- 8) A chiral medium is also temporally dispersive (or temporally nonlocal), due to causality, typically with Lorentz dispersion in its three constitutive parameters (permittivity, permeability and chirality factor). In the lossless case, the three parameters are purely real, and all are complex in the general, lossy case. The three parameters share the same resonance frequency, corresponding to the unfolded length of the particle, but have different and antagonistic electric and magnetic plasma frequencies, depending on the densities of the straight and folded parts of the chiral particle. In the balanced regime, where the electric and magnetic plasma frequencies are merged, the medium is purely chiral at the common plasma frequency, with double negative (or negative index) and double positive (or positive index) below and above that frequency, respectively. In the unbalanced regime, a stopband exists between the two plasma frequencies.
- 9) A chiral metamaterial can be designed following a given procedure that leverages the concepts developed in the paper and uses full-wave simulation extraction in a setup involving periodic boundary conditions and orthogonal ports.

Although discovered more than 200 years ago, chirality has been relatively little applied in electromagnetics (microwave, terahertz and optical) to date, probably because of implementation difficulties and theoretical complexity. We expect that current developments of nanotechnologies and modern metamaterial/metamaterial concepts will spur a diversity of developments in this area, and we hope that this paper will contribute to such developments.

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APPENDIX A

UNITS OF THE CONSTITUTIVE PARAMETERS

The units of the medium parameters in the bianisotropic constitutive relations (4) may be found by first isolating the parameter of interest in the same relations, next replacing $\mathbf{D}$ and $\mathbf{B}$ by their respective expressions in Maxwell equations [Eqs. [1].(5)], and finally remembering that the units of ∇ , $\mathbf{E}$, $\mathbf{H}$ and ω are rad/m^{14} , V/m , A/m and rad/s , respectively. This yields

$$\left[\bar{\epsilon}\right] = \left[\frac{\mathbf{D}}{\mathbf{E}}\right] = \left[\frac{\frac{\nabla \times \mathbf{H}}{-i\omega}}{\mathbf{E}}\right] = \frac{\frac{(\text{rad/m})(\text{A/m})}{\text{rad/s}}}{\text{V/m}} = \frac{\text{As}}{\text{Vm}}, \quad (51a)$$

$$\left[\bar{\mu}\right] = \left[\frac{\mathbf{B}}{\mathbf{H}}\right] = \left[\frac{\frac{\nabla \times \mathbf{E}}{i\omega}}{\mathbf{H}}\right] = \frac{\frac{(\text{rad/m})(\text{V/m})}{\text{rad/s}}}{\text{A/m}} = \frac{\text{Vs}}{\text{Am}}, \quad (51b)$$

$$\left[\bar{\xi}\right] = \left[\frac{\mathbf{D}}{\mathbf{H}}\right] = \left[\frac{\frac{\nabla \times \mathbf{H}}{-i\omega}}{\mathbf{H}}\right] = \frac{\frac{(\text{rad/m})(\text{A/m})}{\text{rad/s}}}{\text{A/m}} = \frac{\text{s}}{\text{m}}, \quad (51c)$$

or

$$\left[\bar{\zeta}\right] = \left[\frac{\mathbf{B}}{\mathbf{E}}\right] = \left[\frac{\frac{\nabla \times \mathbf{E}}{i\omega}}{\mathbf{E}}\right] = \frac{\frac{(\text{rad/m})(\text{V/m})}{\text{rad/s}}}{\text{V/m}} = \frac{\text{s}}{\text{m}}. \quad (51d)$$

APPENDIX B

SIGNS OF THE IMAGINARY PARTS OF THE CHIRAL CONSTITUTIVE PARAMETERS IN THE LOSSY CASE

Let us first consider the case of ϵ . According to (13),

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = \epsilon'_r + i\epsilon''_r = 1 + \chi_{ee} = (1 + \chi'_{ee}) + i\chi''_{ee}, \quad (52a)$$

i.e.,

$$\epsilon'_r = 1 + \chi'_{ee} \quad \text{and} \quad \epsilon''_r = \chi''_{ee}, \quad (52b)$$

where the primed and double-primed quantities denote the real and imaginary parts, respectively.

Given the assumed plane-wave forward (+ r -direction propagating) spacetime dependence $\psi \propto e^{ikr} = e^{i(k' + ik'')r} = e^{ik'r} e^{-k''r}$ [Eq. [1].(4)], one must have $k'' > 0$ for exponential decay in the + r direction if the medium is lossy. Assuming first a nonmagnetic ($\mu_r = 1$) and nonchiral ($\chi = 0$) medium, we have

$$k = k_0 \sqrt{\epsilon_r} = k_0 \sqrt{\epsilon'_r + i\epsilon''_r} = k_0 \sqrt{\epsilon'_r} \sqrt{1 + i \frac{\epsilon''_r}{\epsilon'_r}} \approx k_0 \sqrt{\epsilon'_r} \left(1 + i \frac{\epsilon''_r}{2\epsilon'_r}\right), \quad (53)$$

where we have made the low-loss assumption $\epsilon'' \ll \epsilon'$ in the last approximation. We have thus found that $k'' = k_0 \epsilon''_r / (2\sqrt{\epsilon'_r})$ and, assuming $\epsilon' > 0$, the k'' positivity required for lossy absorption translates into ϵ'' positivity, i.e., $\epsilon'' > 0$. An analogous argument leads to the conclusion that we have also $\mu'' > 0$.

The case of χ is more subtle. The eigenstates, whose wavenumbers are given by (21a) as $\beta_{\pm}^+ = \omega(\sqrt{\epsilon\mu} \pm \chi)$, must satisfy the same condition as k above, i.e., $\text{Im}\{\beta_{\pm}^+\} > 0$. At the resonance frequency [see (46)], we have $\epsilon' = \mu' = \chi' = 0$,

¹⁴In Maxwell equations, the differential operator ∇ refers to space, i.e., $\nabla = \nabla_r$, and applies to electromagnetic waves, which, assuming linearity, decompose into plane waves, $\Psi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$ [28]. We have then that $\nabla_r \times = i\mathbf{k} \times$ or, more generally, $\nabla_r = i\mathbf{k}$, whose unit is clearly rad/m .

so that this condition reduces to $|\chi''(\omega_0)| < \sqrt{\epsilon''(\omega_0)\mu''(\omega_0)}$. The absolute value in this relation indicates that there is no general constraint on the sign of χ'' , except that, mathematically, it must change sign together with χ' , i.e., with the medium handedness.

APPENDIX C DERIVATION OF THE CHIRAL EIGENSTATES

Inserting the z -forward plane-wave fields (20) into the chiral-explicit Maxwell equations (8), using the z -propagating plane-wave conditions $\partial/\partial x = \partial/\partial y = 0$, and projecting onto the x, y, z axes, yields

$$-i\beta E_y = \omega\chi E_x + i\omega\mu H_x, \quad (54a)$$

$$i\beta E_x = \omega\chi E_y + i\omega\mu H_y, \quad (54b)$$

$$-i\beta H_y = -i\omega\epsilon E_x + \omega\chi H_x, \quad (54c)$$

$$i\beta H_x = -i\omega\epsilon E_y + \omega\chi H_y, \quad (54d)$$

which form a 4×4 homogeneous linear system of equations in E_x, E_y, H_x and H_y . Solving now (54a) and (54b) for H_x and H_y , respectively, yields

$$H_x = i\frac{\chi}{\mu}E_x - \frac{\beta}{\omega\mu}E_y, \quad (55a)$$

$$H_y = \frac{\beta}{\omega\mu}E_x + i\frac{\chi}{\mu}E_y, \quad (55b)$$

and inserting these results into (54c) and (54d) reduces (54) to the 2×2 system

$$[\beta^2 - \omega^2(\epsilon\mu - \chi^2)]E_x + 2i\beta\chi\omega E_y = 0, \quad (56a)$$

$$-2i\beta\chi\omega E_x + [\beta^2 - \omega^2(\epsilon\mu - \chi^2)]E_y = 0. \quad (56b)$$

Finally, nullifying the determinant of this system for a nontrivial solution leads to the quartic equation

$$[\beta^2 + \omega^2(\chi^2 - \epsilon\mu)]^2 - 4\beta^2\omega^2\chi^2 = 0. \quad (57)$$

The resolution of this equation provides the four modal wavenumbers

$$\beta_{\pm}^{\pm} = \omega(\pm\sqrt{\epsilon\mu} \pm \chi), \quad (58)$$

while inserting these wavenumbers into (56a) or (56b), and subsequently comparing the coefficients of E_x and E_y , provides the four modal electric fields

$$\mathbf{E}_+^+ = E_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{i\beta_+^+z}, \quad (59a)$$

$$\mathbf{E}_-^+ = E_0(\hat{\mathbf{x}} - i\hat{\mathbf{y}})e^{i\beta_-^+z}, \quad (59b)$$

$$\mathbf{E}_+^- = E_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{i\beta_+^-z}, \quad (59c)$$

$$\mathbf{E}_-^- = E_0(\hat{\mathbf{x}} - i\hat{\mathbf{y}})e^{i\beta_-^-z}. \quad (59d)$$

The corresponding four modal magnetic field are then easily found by substituting (59) into (55), or by considering that, for plane-wave excitation, the simple relation $\mathbf{H} = \hat{\mathbf{z}} \times \mathbf{E}/\eta$ ($\eta = \sqrt{\mu/\epsilon}$) holds.

APPENDIX D SPATIAL DISPERSION

Spatial dispersion is a complex effect and its mathematical description is quite involved. We only enumerate here the main steps of the related developments that concern chirality, following [24], [25], [26].

The derivations traditionally [24] treat the medium as a purely dielectric one, embedding the magnetic effects in the permittivity. In the model of (1), this reverts to setting $\mathbf{P}_m = 0$ and replacing $\mathbf{P}_e$ by a generalized polarization $\mathbf{P} = -\mathbf{J}_{\text{pol}}/(i\omega)$, where $\mathbf{J}_{\text{pol}}$ is the global polarization current density. Equations (1) reduce then to

$$\mathbf{D} = \epsilon_0\mathbf{E} - \mathbf{J}_{\text{pol}}/(i\omega) \quad \text{and} \quad \mathbf{B} = \mu_0\mathbf{H}, \quad (60)$$

where $\mathbf{J}_{\text{pol}}$ is computed, similarly to (30), as

$$\mathbf{J}_{\text{pol}}(\mathbf{r}) = \int_{\Omega} \bar{\bar{\mathbf{K}}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') d\mathbf{r}', \quad (61)$$

with the kernel $\bar{\bar{\mathbf{K}}}(\mathbf{r}, \mathbf{r}')$ accounting, according to the definition of $\mathbf{J}_{\text{pol}}$, for both electric and magnetic effects.

Next, substituting $\mathbf{E}(\mathbf{r}')$ in this relation by its three-dimensional Taylor expansion, performing the integral, inserting the result into (60), and assuming isotropy, leads to a relation of the form

$$\mathbf{D} = \epsilon\mathbf{E} + \alpha\nabla \times \mathbf{E} + \beta\nabla\nabla \cdot \mathbf{E} + \gamma\nabla \times \nabla \times \mathbf{E}, \quad (62)$$

which explicitly shows the first orders – 0 (ϵ), 1 (α) and 2 (β and γ) – of spatial dispersion. Combining this equation with Maxwell equations eventually leads to

$$\mathbf{D} = \epsilon\mathbf{E} + i\frac{\omega\alpha}{2}\mathbf{B} + \beta\nabla\nabla \cdot \mathbf{E}, \quad \mathbf{H} = \left(\frac{1}{\mu_0} - \omega^2\gamma\right)\mathbf{B} + i\frac{\omega\alpha}{2}\mathbf{E}. \quad (63)$$

Comparing (63) with (5) indicates that the chiral terms are related to the $\alpha\nabla \times \mathbf{E}$ term in (62), which reveals that chirality is a first-order spatial dispersion effect.

APPENDIX E BIANISOTROPIC PLANE-WAVE RELATIONS

Inserting (4) into [1].(4) yields

$$\nabla \times \mathbf{E} = i\omega\bar{\bar{\boldsymbol{\zeta}}} \cdot \mathbf{E} + i\omega\bar{\bar{\boldsymbol{\mu}}} \cdot \mathbf{H}, \quad (64a)$$

$$\nabla \times \mathbf{H} = -i\omega\bar{\bar{\boldsymbol{\epsilon}}} \cdot \mathbf{E} - i\omega\bar{\bar{\boldsymbol{\xi}}} \cdot \mathbf{H}. \quad (64b)$$

This system of equations can be solved for $\mathbf{E}$ in terms of $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}$ by pre-dotmultiplying (64a) by $\bar{\bar{\boldsymbol{\xi}}} \cdot \bar{\bar{\boldsymbol{\mu}}}^{-1}$ and summing the resulting equation with (64b) so as to eliminate $\mathbf{H}$, and finally isolating $\mathbf{E}$. Similarly, it can be solved for $\mathbf{H}$ in terms of $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}$ by pre-dotmultiplying (64b) by $\bar{\bar{\boldsymbol{\z}}} \cdot \bar{\bar{\boldsymbol{\epsilon}}}^{-1}$ and summing the resulting equation with (64a) so as to eliminate $\mathbf{E}$, and finally isolating $\mathbf{H}$, where we have avoided to involve the inverses of $\bar{\bar{\boldsymbol{\xi}}}$ and $\bar{\bar{\boldsymbol{\z}}}$ because, in contrast to $\bar{\bar{\boldsymbol{\epsilon}}}$ and $\bar{\bar{\boldsymbol{\mu}}}$, these tensors may be non invertible. The result is

$$\mathbf{E} = \frac{i}{\omega} \left(\bar{\bar{\boldsymbol{\epsilon}}} - \bar{\bar{\boldsymbol{\xi}}} \cdot \bar{\bar{\boldsymbol{\mu}}}^{-1} \cdot \bar{\bar{\boldsymbol{\z}}} \right)^{-1} \cdot \left(\nabla \times \mathbf{H} + \bar{\bar{\boldsymbol{\xi}}} \cdot \bar{\bar{\boldsymbol{\mu}}}^{-1} \cdot \nabla \times \mathbf{E} \right), \quad (65a)$$

$$\mathbf{H} = -\frac{i}{\omega} \left(\bar{\bar{\boldsymbol{\mu}}} - \bar{\bar{\boldsymbol{\z}}} \cdot \bar{\bar{\boldsymbol{\epsilon}}}^{-1} \cdot \bar{\bar{\boldsymbol{\xi}}} \right)^{-1} \cdot \left(\nabla \times \mathbf{E} + \bar{\bar{\boldsymbol{\z}}} \cdot \bar{\bar{\boldsymbol{\epsilon}}}^{-1} \cdot \nabla \times \mathbf{H} \right). \quad (65b)$$

We next substitute these relations into the expressions of (4) that are associated with bianisotropy, i.e., having the coefficients $\bar{\bar{\xi}}$ and $\bar{\bar{\zeta}}$. Specifically, inserting (65b) into the first equation of (4) and (65a) into the first equation of (4) provides the alternative constitutive relations

$$\begin{aligned}\mathbf{D} &= \bar{\bar{\epsilon}} \cdot \mathbf{E} - \frac{i}{\omega} \bar{\bar{\xi}} \cdot \left(\bar{\bar{\mu}} - \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \bar{\bar{\xi}} \right)^{-1} \cdot \left(\nabla \times \mathbf{E} + \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H} \right), \quad (66a) \\ \mathbf{B} &= \frac{i}{\omega} \bar{\bar{\zeta}} \cdot \left(\bar{\bar{\epsilon}} - \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{-1} \cdot \bar{\bar{\zeta}} \right)^{-1} \cdot \left(\nabla \times \mathbf{H} + \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{-1} \cdot \nabla \times \mathbf{E} \right) + \bar{\bar{\mu}} \cdot \mathbf{H}, \quad (66b)\end{aligned}$$

which upon angular spectrum decomposition reduce to

$$\begin{aligned}\tilde{\mathbf{D}} &= \bar{\bar{\epsilon}} \cdot \tilde{\mathbf{E}} + \frac{1}{\omega} \bar{\bar{\xi}} \cdot \left(\bar{\bar{\mu}} - \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \bar{\bar{\xi}} \right)^{-1} \cdot \left(\mathbf{k} \times \tilde{\mathbf{E}} + \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \mathbf{k} \times \tilde{\mathbf{H}} \right), \quad (67a) \\ \tilde{\mathbf{B}} &= -\frac{1}{\omega} \bar{\bar{\zeta}} \cdot \left(\bar{\bar{\epsilon}} - \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{-1} \cdot \bar{\bar{\zeta}} \right)^{-1} \cdot \left(\mathbf{k} \times \tilde{\mathbf{H}} + \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{-1} \cdot \mathbf{k} \times \tilde{\mathbf{E}} \right) + \bar{\bar{\mu}} \cdot \tilde{\mathbf{H}}, \quad (67b)\end{aligned}$$

or, using the identity $\mathbf{k} \times \mathbf{a} = (\mathbf{k} \times \bar{\mathbf{l}}) \cdot \mathbf{a}$,

$$\tilde{\mathbf{D}} = \left(\bar{\bar{\epsilon}} + \bar{\bar{T}}_D \cdot \mathbf{k} \times \bar{\mathbf{l}} \right) \cdot \tilde{\mathbf{E}} + \left(\bar{\bar{T}}_D \cdot \bar{\bar{\xi}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \mathbf{k} \times \bar{\mathbf{l}} \right) \cdot \tilde{\mathbf{H}}, \quad (68a)$$

$$\text{with } \bar{\bar{T}}_D = \frac{1}{\omega} \bar{\bar{\xi}} \cdot \left(\bar{\bar{\mu}} - \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \bar{\bar{\xi}} \right)^{-1} \quad (68b)$$

$$\tilde{\mathbf{B}} = \left(\bar{\bar{T}}_B \cdot \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{-1} \cdot \mathbf{k} \times \bar{\mathbf{l}} \right) \cdot \tilde{\mathbf{E}} + \left(\bar{\bar{\mu}} + \bar{\bar{T}}_B \cdot \mathbf{k} \times \bar{\mathbf{l}} \right) \cdot \tilde{\mathbf{H}} \quad (68c)$$

$$\text{with } \bar{\bar{T}}_B = -\frac{1}{\omega} \bar{\bar{\zeta}} \cdot \left(\bar{\bar{\epsilon}} - \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{-1} \cdot \bar{\bar{\zeta}} \right)^{-1}. \quad (68d)$$

For a biisotropic medium (Sec. II-C), transforming the tensors into scalars reduces (35) to

$$\begin{aligned}\mathbf{D} &= \epsilon \mathbf{E} - \frac{i}{\omega} \left(\frac{\xi}{\mu - \xi \xi / \epsilon} \right) \left(\nabla \times \mathbf{E} + \frac{\zeta \nabla \times \mathbf{H}}{\epsilon} \right) \\ &= \epsilon \mathbf{E} - \frac{i}{\omega} \frac{\xi}{\gamma} (\epsilon \nabla \times \mathbf{E} + \zeta \nabla \times \mathbf{H}), \quad (69a)\end{aligned}$$

$$\begin{aligned}\mathbf{B} &= \frac{i}{\omega} \left(\frac{\zeta}{\epsilon - \xi \xi / \mu} \right) \left(\nabla \times \mathbf{H} + \frac{\xi \nabla \times \mathbf{E}}{\mu} \right) + \mu \mathbf{H} \\ &= \frac{i}{\omega} \frac{\zeta}{\gamma} (\xi \nabla \times \mathbf{E} + \mu \nabla \times \mathbf{H}) + \mu \mathbf{H}, \quad (69b)\end{aligned}$$

where

$$\gamma = \epsilon \mu - \xi \zeta, \quad (69c)$$

which further reduce to (32) in the chiral case ($\xi = i\chi$ and $\zeta = -i\chi$).

APPENDIX F

TEMPORAL FREQUENCY PARITY OF THE CHIRAL PARAMETER

Let us refer, for conciseness, to the electromagnetic field functions $\mathcal{E}(t)$, $\mathcal{H}(t)$, $\mathcal{D}(t)$ and $\mathcal{B}(t)$ by the generic function $\mathcal{C}(t)$. This function is related to its Fourier transform, $\tilde{\mathcal{C}}(\omega)$, by the Fourier transform pair [34]

$$\mathcal{C}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{\mathcal{C}}(\omega) e^{-i\omega t} d\omega = \mathcal{F}^{-1} \{ \tilde{\mathcal{C}}(\omega) \}, \quad (70a)$$

$$\tilde{\mathcal{C}}(\omega) = \int_{-\infty}^{+\infty} \mathcal{C}(t) e^{+i\omega t} dt = \mathcal{F} \{ \mathcal{C}(t) \}. \quad (70b)$$

Since it represents a physical quantity, $\mathcal{C}(t)$ is purely real, while $\tilde{\mathcal{C}}(\omega)$ is complex. Expanding (70b) into its real and imaginary parts yields then

$$\begin{aligned}\tilde{\mathcal{C}}'(\omega) + i\tilde{\mathcal{C}}''(\omega) &= \int_{-\infty}^{+\infty} \mathcal{C}(t) [\cos(\omega t) + i\sin(\omega t)] dt. \\ &= \int_{-\infty}^{+\infty} \mathcal{C}(t) \cos(\omega t) dt \\ &\quad + i \int_{-\infty}^{+\infty} \mathcal{C}(t) \sin(\omega t) dt,\end{aligned} \quad (71)$$

which separates as

$$\tilde{\mathcal{C}}'(\omega) = \int_{-\infty}^{+\infty} \mathcal{C}(t) \cos(\omega t) dt = \tilde{\mathcal{C}}'(-\omega), \quad (72a)$$

$$\tilde{\mathcal{C}}_I(\omega) = \int_{-\infty}^{+\infty} \mathcal{C}(t) \sin(\omega t) dt = -\tilde{\mathcal{C}}_I(-\omega), \quad (72b)$$

where the last equalities, revealing the parity of the real and imaginary Fourier-transform functions, straightforwardly follow from the parities of the cosine and sine functions.

Let us now decompose the Fourier-transform version of Eqs. (7), i.e.,

$$\tilde{\mathbf{D}} = \epsilon \tilde{\mathbf{E}} + i\chi \tilde{\mathbf{H}} \quad \text{and} \quad \tilde{\mathbf{B}} = -i\chi \tilde{\mathbf{E}} + \mu \tilde{\mathbf{H}}, \quad (73)$$

into their real and imaginary parts, while remembering the parities of $\epsilon(\omega)$ and $\mu(\omega)$ from their Lorentzian form [2], [3] to be¹⁵

$$\epsilon'(\omega), \mu'(\omega) \text{ are even (e),} \quad (74a)$$

$$\epsilon''(\omega), \mu''(\omega) \text{ are odd (o),} \quad (74b)$$

and analyze the parities of the resulting equations. We get

$$\stackrel{(e)}{\tilde{\mathbf{D}}'} = \epsilon' \stackrel{(e)}{\tilde{\mathbf{E}}'} - \epsilon'' \stackrel{(o)}{\tilde{\mathbf{E}}''} - \chi' \stackrel{(o)}{\tilde{\mathbf{H}}'} - \chi'' \stackrel{(e)}{\tilde{\mathbf{H}}''}, \quad (75a)$$

$$\stackrel{(o)}{\tilde{\mathbf{D}}''} = \epsilon' \stackrel{(e)}{\tilde{\mathbf{E}}''} + \epsilon'' \stackrel{(o)}{\tilde{\mathbf{E}}'} + \chi' \stackrel{(o)}{\tilde{\mathbf{H}}'} - \chi'' \stackrel{(e)}{\tilde{\mathbf{H}}''}, \quad (75b)$$

$$\stackrel{(e)}{\tilde{\mathbf{B}}'} = \mu' \stackrel{(e)}{\tilde{\mathbf{H}}'} - \mu'' \stackrel{(o)}{\tilde{\mathbf{H}}''} + \chi' \stackrel{(o)}{\tilde{\mathbf{E}}'} + \chi'' \stackrel{(e)}{\tilde{\mathbf{E}}''}, \quad (75c)$$

$$\stackrel{(o)}{\tilde{\mathbf{B}}''} = \mu' \stackrel{(e)}{\tilde{\mathbf{H}}''} + \mu'' \stackrel{(o)}{\tilde{\mathbf{H}}'} - \chi' \stackrel{(o)}{\tilde{\mathbf{E}}'} + \chi'' \stackrel{(e)}{\tilde{\mathbf{E}}''}, \quad (75d)$$

where the known parities have been indicated in brackets and the parities of χ have been deduced. We have thus found, with consistency within and redundancy across the four equations, that

$$\chi'(\omega) \text{ is odd,} \quad (76a)$$

$$\chi''(\omega) \text{ is even.} \quad (76b)$$

So, the direct and cross constitutive parameters have opposite temporal frequency parities.

¹⁵For consistency with the rest of the paper, we drop here the tildes corresponding to the Fourier transforms of the permittivity and permeability.

APPENDIX G
PARAMETER EXTRACTION FORMULAS

We consider a linearly polarized (LP) incident wave. Solving Eqs. (28) respectively for χ and $\sqrt{\epsilon\mu}$, with $z = p$ representing the lattice period (Fig. [1].4), yields

$$\chi = -\frac{\theta_C}{\omega p} = -\frac{\tan^{-1}(E_y/E_x)}{\omega p} = -\frac{\tan^{-1}(|S_{21}^{yx}|/|S_{21}^{xx}|)}{\omega p}, \quad (77a)$$

$$\sqrt{\epsilon\mu} = \frac{\phi_C}{\omega p} = \frac{\angle S_{21}^{xx}}{\omega p} = \frac{\angle S_{21}^{yx}}{\omega p}, \quad (77b)$$

where the (frequency-dependent) scattering transmission parameters S_{21}^{uv} ($u, v = x, y$) correspond to the simulation setup in Fig. 8.

The chiral parameter, χ , is readily provided by (77a), but more information is required to discriminate ϵ and μ in (77b). Such information may be accessed via the (frequency-dependent) scattering reflection parameters S_{11}^{uv} ($u, v = x, y$). Due to reciprocity, where any rotation incurred to the wave in one direction must be undone in the opposite direction, the polarization of the wave reflected by the particle must be identical to the incident wave¹⁶. Therefore, one needs to consider only the incident polarization, where the reflection parameter is simply given by the normal-incidence Fresnel coefficient $S_{11}^{xx} = (\eta - Z_{\text{port}})/(\eta + Z_{\text{port}})$, with $\eta = \sqrt{\mu/\epsilon}$. Solving this relation for η yields

$$\eta = Z_{\text{port}} \frac{1 + S_{11}^{xx}}{1 - S_{11}^{xx}} = \sqrt{\frac{\mu}{\epsilon}}. \quad (78)$$

Finally, taking successfully the product and the ratio of Eqs. (77b) and (78) provides the missing parameters:

$$\mu = Z_{\text{port}} \left(\frac{\angle S_{21}^{yx}}{\omega p} \right) \left(\frac{1 + S_{11}^{xx}}{1 - S_{11}^{xx}} \right), \quad (79a)$$

$$\epsilon = \frac{1}{Z_{\text{port}}} \left(\frac{\angle S_{21}^{yx}}{\omega p} \right) \left(\frac{1 - S_{11}^{xx}}{1 + S_{11}^{xx}} \right). \quad (79b)$$

¹⁶This may be demonstrated ad absurdum as follows [35]. If we had $S_{11}^{yx} \neq 0$ (reflection rotation), then we would also have, from isotropy, $S_{11}^{xy} \neq 0$ and $S_{11}^{xy} = -S_{11}^{yx}$. The latter result following from the fact that the wave would rotate in the same direction for the two experiments (given the same particle chirality), so that $S_{11}^{yx} = E_y^{\text{refl},a}/E_x^{\text{inc},a}$ and $S_{11}^{xy} = E_x^{\text{refl},b}/E_y^{\text{inc},b} = (-|E_x^{\text{refl},b}|)/E_y^{\text{inc},b} = -S_{11}^{yx}$. But reciprocity (or time-reversal symmetry) requires $S_{11}^{xy} = S_{11}^{yx}$. Therefore, one must have $S_{11}^{yx} = S_{11}^{xy} = 0$, i.e., no reflection rotation.