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Quantum signatures in the classical limit of electromagnetic waves

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ABSTRACT

It is common both in the classical and quantum optics to describe the optical field as a superposition of plane waves. However, it is well known that optically active materials emit photons vastly dominantly in the electric dipole approximation. The photons emitted in the electric dipole transitions are not plane waves, but spherical photon states corresponding to eigenstates $J = 1$ of the total angular momentum and $M = \pm 1$ of the z component of the total angular momentum. In addition, electric dipole photons are separated from the magnetic photons by the state index $\eta = e$ for electric photons in distinction to $\eta = m$ for magnetic photons. In this work, we study the far-field that is generated when a two-dimensional matrix of atoms emits electric dipole photons and compare this far-field at large distance from the emitting atomic matrix with a plane wave. The goal of our work is to find out if the light emitted from the electric dipole transitions carry the memory of being electric dipole photons when they are far from the emitting atoms. In particular, it is well known that a plane wave includes all angular momentum components corresponding to quantum numbers $J = 1, 2, 3, \dots$, while the far field of the electric dipole photons emitted by the atomic matrix can include only the angular momentum component $J = 1$. Although the two-dimensional atomic matrix used as a light source in our simulations is certainly nontrivial to fabricate, it is nevertheless fully physical and we expect that with some modifications, the conclusions from the present simulations can be generalized to atomic light sources that are more easy to realize experimentally.

Keywords: angular momentum of light, quantum optics, classical field limit, atomic light sources

1. INTRODUCTION

In the seminal paper of Allen *et al.* on the angular momentum of light,¹ the classical light field is described using convolutions of classical plane waves. The topologically quantized values of the orbital and spin angular momentum are then obtained by using the photon number calculated as a ratio of the total field energy and the single photon energy in the approximately monochromatic field. This approach is used not only for vacuum but also in dielectrics and it is essentially included also in the recent classical formulation of the mass-polariton theory of light,^{2–4} which accounts for the optical force-mediated coupling of the field–medium dynamics. In the mass-polariton theory, the quantized value of the angular momentum is shared between the field and the medium.^{5,6}

It is evident that the classical limit of the electromagnetic field can be obtained from the space-time expectation values of quantum mechanical operators corresponding to pertinent physical variables. However, the single photon average values of physical variables of the classical field will not necessarily agree with the expectation values of the pertinent physical operators for the true quantum optical field made of electric dipole photons.

It is well known that the emission of light by typical atomic light sources used in optics is accurately described by the electric dipole approximation. The electric dipole photons are spherical photon states of light and dramatically different from plane waves. It is therefore justified to ask if the superposition of electric dipole photons will lead to a far-field equivalent to plane waves. In this work we study the classical limit of light by

computer simulations of the true microscopic quantum field emitted by a two-dimensional atomic matrix in the electric dipole approximation to find out if the far-field retains the quantum signatures of the electric dipole transitions. The goal of our work is to develop a theory of an incoherent photon beam in which every photon is in a well-defined angular momentum state. This is in contrast to the previously widely studied topologically quantized angular momentum states of light, in which case the angular momentum states of individual photons are not unambiguously known.

2. CONTROVERSY OF THE TOTAL ANGULAR MOMENTUM OF A CLASSICAL PLANE WAVE

The fundamental classical expression of the total angular momentum of light in a volume V_d in vacuum is well known to be given by⁷⁻²⁹

$$\begin{aligned}\mathbf{J} &= \int_{V_d} \varepsilon_0 \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) d^3r \\ &= \int_{V_d} \varepsilon_0 \mathbf{r} \times [\mathbf{E} \times (\nabla \times \mathbf{A})] d^3r \\ &= \int_{V_d} \varepsilon_0 \mathbf{E} \cdot (\mathbf{r} \times \nabla \mathbf{A}) d^3r + \int_{V_d} \varepsilon_0 \mathbf{E} \times \mathbf{A} d^3r - \int_{V_d} \varepsilon_0 (\mathbf{E} \cdot \nabla) (\mathbf{r} \times \mathbf{A}) d^3r \\ &= \int_{V_d} \varepsilon_0 \mathbf{E} \cdot (\mathbf{r} \times \nabla \mathbf{A}) d^3r + \int_{V_d} \varepsilon_0 \mathbf{E} \times \mathbf{A} d^3r - \oint_{\partial V_d} \varepsilon_0 (\mathbf{r} \times \mathbf{A}) \mathbf{E} \cdot d\mathbf{s}\end{aligned}\quad (1)$$

Here we have expressed the magnetic field in terms of the vector potential as $\mathbf{B} = \nabla \times \mathbf{A}$ in the second line and the mathematical identity $\{\mathbf{r} \times [\mathbf{F}_1 \times (\nabla \times \mathbf{F}_2)]\}_j = \mathbf{F}_1 \cdot [(\mathbf{r} \times \nabla)_j \mathbf{F}_2] + (\mathbf{F}_1 \times \mathbf{F}_2)_j - (\mathbf{F}_1 \cdot \nabla)(\mathbf{r} \times \mathbf{F}_2)_j$ in the third line. In the final step, we have transformed the last volume integral into a surface integral by using the integral identity $\int_{V_d} (\mathbf{F}_1 \cdot \nabla) \mathbf{F}_2 d^3r = \oint_{\partial V_d} \mathbf{F}_2 \mathbf{F}_1 \cdot d\mathbf{s} - \int_{V_d} \mathbf{F}_2 (\nabla \cdot \mathbf{F}_1) d^3r$ with Maxwell's equation $\nabla \cdot \mathbf{E} = 0$. In the last line of Eq. (1), one conventionally identifies the first term with the orbital angular momentum and the second term with the spin angular momentum.

From the first line of Eq. (1), it directly follows that the total angular momentum density of a plane wave is zero since it can have no component in the direction perpendicular to the plane of the electric and magnetic fields. This reasoning applies equally well for all transverse polarizations including the left and right circular polarizations that are commonly considered to carry a net total angular momentum with the electromagnetic field. This question has raised discussion over a long time in the optics literature.³⁰⁻³⁵ That the expressions of the orbital and spin angular momenta in Eq. (1) are given in terms of the vector potential has also raised discussion since these expressions are in general gauge dependent. However, for fields that are exactly transverse, gauge invariance is obtained if $\mathbf{A}$ is defined to be the gauge invariant transverse vector potential.^{12,36}

In 1936, Beth³⁷ was able to experimentally demonstrate the transfer of angular momentum between circularly polarized light and the rotational motion of a birefringent wave plate suspended on a filament. Previously, it has been suggested that this paradox is explained by changes of the electromagnetic field at the edges of the material object that interacts with the electromagnetic field. This bending of the electromagnetic field gives the fields components in the direction of propagation. This in turn would give rise to a Poynting vector that is not parallel to the direction of propagation and to an angular momentum vector that is.^{31,33,34}

3. QUANTUM MECHANICAL ANGULAR MOMENTUM EXPECTATION VALUE

Here we briefly review the quantum mechanical calculation of the orbital angular momentum and spin angular momentum expectation values in the photon wave function picture. This approach is not necessarily the most fundamental description of the angular momentum of the quantum optical field. It is plausible that a more advanced description can be developed from the second quantization states of optical fields. In the photon wave function picture, the expectation value of a general operator $\hat{O}$ is given by³⁸

$$\langle \hat{O} \rangle = \frac{1}{4\hbar\omega_k} \int_{V_d} \left[\varepsilon_0 \mathbf{E}^*(\mathbf{r}, t) \cdot \hat{O} \mathbf{E}(\mathbf{r}, t) + \frac{1}{\mu_0} \mathbf{B}^*(\mathbf{r}, t) \cdot \hat{O} \mathbf{B}(\mathbf{r}, t) \right] d^3r. \quad (2)$$

Here V_d is the volume for which the operator expectation value is calculated. The complex electric and magnetic field amplitudes act as the photon wave functions.

The orbital angular momentum operator $\hat{\mathbf{L}}$ is well known to be given in the real space as $\hat{\mathbf{L}} = -i\hbar\mathbf{r} \times \nabla$. Correspondingly, the spin angular momentum operator is given by $\hat{\mathbf{S}} = \hbar\boldsymbol{\sigma}$, where the components of the spin matrices σ_x , σ_y , and σ_z , are given in the Cartesian basis by³⁸

$$\sigma_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (3)$$

Substituting the orbital angular momentum operator $\hat{\mathbf{L}} = -i\hbar\mathbf{r} \times \nabla$ into Eq. (2), we find that the expectation value of the orbital angular momentum is given by

$$\begin{aligned} \langle \hat{\mathbf{L}} \rangle &= \frac{1}{4\hbar\omega_k} \int_{V_d} \left\{ \varepsilon_0 \mathbf{E}^*(\mathbf{r}, t) \cdot \left[-i\hbar\mathbf{r} \times \nabla \mathbf{E}(\mathbf{r}, t) \right] + \frac{1}{\mu_0} \mathbf{B}^*(\mathbf{r}, t) \cdot \left[-i\hbar\mathbf{r} \times \nabla \mathbf{B}(\mathbf{r}, t) \right] \right\} d^3r \\ &= \frac{1}{4\hbar\omega_k} \int_{V_d} \left(-i\hbar\varepsilon_0 \mathbf{r} \times \{ \mathbf{E}^*(\mathbf{r}, t) \times [\nabla \times \mathbf{E}(\mathbf{r}, t)] \} + i\hbar\varepsilon_0 \mathbf{E}^*(\mathbf{r}, t) \times \mathbf{E}(\mathbf{r}, t) - i\hbar\varepsilon_0 [\mathbf{E}^*(\mathbf{r}, t) \cdot \nabla] [\mathbf{r} \times \mathbf{E}(\mathbf{r}, t)] \right. \\ &\quad \left. - \frac{i\hbar}{\mu_0} \mathbf{r} \times \{ \mathbf{B}^*(\mathbf{r}, t) \times [\nabla \times \mathbf{B}(\mathbf{r}, t)] \} + \frac{i\hbar}{\mu_0} \mathbf{B}^*(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) - \frac{i\hbar}{\mu_0} [\mathbf{B}^*(\mathbf{r}, t) \cdot \nabla] [\mathbf{r} \times \mathbf{B}(\mathbf{r}, t)] \right) d^3r \\ &= \frac{1}{4\hbar\omega_k} \int_{V_d} \left(\hbar\omega_k\varepsilon_0 \mathbf{r} \times [\mathbf{E}^*(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)] + i\hbar\varepsilon_0 \mathbf{E}^*(\mathbf{r}, t) \times \mathbf{E}(\mathbf{r}, t) - i\hbar\varepsilon_0 [\mathbf{E}^*(\mathbf{r}, t) \cdot \nabla] [\mathbf{r} \times \mathbf{E}(\mathbf{r}, t)] \right. \\ &\quad \left. - \hbar\omega_k\varepsilon_0 \mathbf{r} \times [\mathbf{B}^*(\mathbf{r}, t) \times \mathbf{E}(\mathbf{r}, t)] + \frac{i\hbar}{\mu_0} \mathbf{B}^*(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) - \frac{i\hbar}{\mu_0} [\mathbf{B}^*(\mathbf{r}, t) \cdot \nabla] [\mathbf{r} \times \mathbf{B}(\mathbf{r}, t)] \right) d^3r \\ &= \frac{1}{2} \int_{V_d} \varepsilon_0 \mathbf{r} \times \text{Re}[\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}^*(\mathbf{r}, t)] d^3r + \frac{i}{4\omega_k} \int_{V_d} \left[\varepsilon_0 \mathbf{E}^*(\mathbf{r}, t) \times \mathbf{E}(\mathbf{r}, t) + \frac{1}{\mu_0} \mathbf{B}^*(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \right] d^3r \\ &\quad - \frac{i}{4\omega_k} \int_{V_d} \left\{ \varepsilon_0 [\mathbf{E}^*(\mathbf{r}, t) \cdot \nabla] [\mathbf{r} \times \mathbf{E}(\mathbf{r}, t)] + \frac{1}{\mu_0} [\mathbf{B}^*(\mathbf{r}, t) \cdot \nabla] [\mathbf{r} \times \mathbf{B}(\mathbf{r}, t)] \right\} d^3r \\ &= \frac{1}{2} \int_{V_d} \varepsilon_0 \mathbf{r} \times \text{Re}[\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}^*(\mathbf{r}, t)] d^3r + \frac{i}{4\omega_k} \int_{V_d} \left[\varepsilon_0 \mathbf{E}^*(\mathbf{r}, t) \times \mathbf{E}(\mathbf{r}, t) + \frac{1}{\mu_0} \mathbf{B}^*(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \right] d^3r \\ &\quad - \frac{i}{4\omega_k} \oint_{\partial V_d} \mathbf{r} \times \left[\varepsilon_0 \mathbf{E}(\mathbf{r}, t) \mathbf{E}^*(\mathbf{r}, t) + \frac{1}{\mu_0} \mathbf{B}(\mathbf{r}, t) \mathbf{B}^*(\mathbf{r}, t) \right] \cdot d\mathbf{s}. \end{aligned} \quad (4)$$

Here we have used the mathematical identity $\{\mathbf{r} \times [\mathbf{F}_1 \times (\nabla \times \mathbf{F}_2)]\}_j = \mathbf{F}_1 \cdot [(\mathbf{r} \times \nabla)_j \mathbf{F}_2] + (\mathbf{F}_1 \times \mathbf{F}_2)_j - (\mathbf{F}_1 \cdot \nabla)(\mathbf{r} \times \mathbf{F}_2)_j$. In the final step, we have transformed the last volume integral into a surface integral by using the integral identity $\int_{V_d} (\mathbf{F}_1 \cdot \nabla) \mathbf{F}_2 d^3r = \oint_{\partial V_d} \mathbf{F}_2 \mathbf{F}_1 \cdot d\mathbf{s} - \int_{V_d} \mathbf{F}_2 (\nabla \cdot \mathbf{F}_1) d^3r$ with Maxwell's equations $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$. In practical applications, the last surface integral form of Eq. (4) can be used only if the volume is compact or if the fields are known to go sufficiently rapidly to zero at the infinite boundary where the surface term can be set to zero. In particular, the last form of Eq. (4) cannot be applied to a classical plane wave with an infinite volume V_d as the classical plane wave does not go to zero at infinities.

Substituting the spin angular momentum operator $\hat{\mathbf{S}} = \hbar\boldsymbol{\sigma}$ into Eq. (2), the expectation value of the spin angular momentum is given by

$$\langle \hat{\mathbf{S}} \rangle = -\frac{i}{4\omega_k} \int_{V_d} \left[\varepsilon_0 \mathbf{E}^*(\mathbf{r}, t) \times \mathbf{E}(\mathbf{r}, t) + \frac{1}{\mu_0} \mathbf{B}^*(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \right] d^3r. \quad (5)$$

The total angular momentum operator $\hat{\mathbf{J}}$ is a vector sum of the orbital and spin angular momenta $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$. Thus, from Eqs. (4) and (5), it follows that the expectation value of the total angular momentum of light is given

by

$$\begin{aligned}
\langle \hat{\mathbf{J}} \rangle &= \frac{1}{4\hbar\omega_k} \int_{V_d} \left\{ \varepsilon_0 \mathbf{E}^*(\mathbf{r}, t) \cdot \left[-i\hbar \mathbf{r} \times \nabla \mathbf{E}(\mathbf{r}, t) \right] + \frac{1}{\mu_0} \mathbf{B}^*(\mathbf{r}, t) \cdot \left[-i\hbar \mathbf{r} \times \nabla \mathbf{B}(\mathbf{r}, t) \right] \right\} d^3r \\
&\quad - \frac{i}{4\omega_k} \int_{V_d} \left[\varepsilon_0 \mathbf{E}^*(\mathbf{r}, t) \times \mathbf{E}(\mathbf{r}, t) + \frac{1}{\mu_0} \mathbf{B}^*(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \right] d^3r \\
&= \frac{1}{2} \int_{V_d} \varepsilon_0 \mathbf{r} \times \text{Re}[\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}^*(\mathbf{r}, t)] d^3r - \frac{i}{4\omega_k} \oint_{\partial V_d} \mathbf{r} \times \left[\varepsilon_0 \mathbf{E}(\mathbf{r}, t) \mathbf{E}^*(\mathbf{r}, t) + \frac{1}{\mu_0} \mathbf{B}(\mathbf{r}, t) \mathbf{B}^*(\mathbf{r}, t) \right] \cdot d\mathbf{s}. \quad (6)
\end{aligned}$$

The first term of the total angular momentum expectation value in the last line of Eq. (6) is equal to the well-known classical result of the total time-averaged angular momentum of the field in volume V_d . The surface integral term of Eq. (6) becomes zero if the fields are negligible at the boundary of V_d . However, generally the total angular momentum expectation value in Eq. (6) is obviously different from the classical Poynting vector based definition in Eq. (1). In addition, in the same way as with the case of Eq. (4), the last surface integral form of Eq. (6) cannot be applied in particular to a classical plane wave if the volume is infinite.

4. DIFFERENCE OF THE CLASSICAL PLANE WAVE AND THE FIELD EMITTED BY ATOMS

In the following, we consider a sequence of events where first a plane wave having an ideal classical Gaussian form is approaching and subsequently partly absorbed by a 2D matrix of atoms. Next, the excited atoms re-emit the radiation. This sequence is illustrated in Fig. 1. The goal is to study how the re-emitted radiation differs from the incoming Gaussian plane wave and to compare the angular momenta carried by these fields.

The incoming photons are supposed to have a fixed value of the wave vector, polarization and energy. These are the plane wave photon states that have a conventional form, given by

$$\mathbf{A}_{\mathbf{k},q}(\mathbf{r}, t) = \sqrt{\frac{\hbar}{2\omega_k V \varepsilon_0}} \mathbf{e}_{\mathbf{k}}^{(q)} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}. \quad (7)$$

Here the polarization index $q = -1, 0, +1$ and the wave vector $\mathbf{k}$ appearing as indices in the spherical polarization vector $\mathbf{e}_{\mathbf{k}}^{(q)}$ correspond to a fixed polarization and wave vector state of a photon. The normalization constant $\sqrt{\hbar/(2\omega_k V \varepsilon_0)}$ includes the reduced Planck constant $\hbar$, the angular frequency $\omega_k = kc$, where $k = |\mathbf{k}|$, the normalization volume V , and the vacuum permittivity ε_0 . We define the spherical polarization vectors as $\mathbf{e}_{\mathbf{k}}^{(0)} = \mathbf{R}_{\mathbf{k}} \cdot \hat{\mathbf{z}}$ and $\mathbf{e}_{\mathbf{k}}^{(\pm 1)} = \mathbf{R}_{\mathbf{k}} \cdot \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})$, where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors parallel to the x , y and z axes, respectively, and $\mathbf{R}_{\mathbf{k}}$ is a rotation matrix that rotates $\hat{\mathbf{z}}$ parallel to the wave vector. With these definitions, from the point of view of the source, $q = 1$ corresponds to right circular polarization, while $q = -1$ corresponds to left circular polarization. The plane wave state in Eq. (7) corresponds to well-defined value of the component of the total angular momentum in the direction of the wave vector: $\mathbf{J} \cdot \mathbf{k}/|\mathbf{k}| = q\hbar$.

We next study the angular momentum of the circularly polarized plane wave of the quantum field. In the Coulomb gauge, i.e., $\nabla \cdot \mathbf{A} = 0$, which is used for convenience, the spherical polarization vector is orthogonal to the wave vector, i.e., $q = \pm 1$. The vector potential operator for this transverse electromagnetic field is well known to be given by

$$\hat{\mathbf{A}}(\mathbf{r}, t) = \sum_{\mathbf{k},q} \left[\mathbf{A}_{\mathbf{k},q}(\mathbf{r}, t) \hat{a}_{\mathbf{k},q} + \mathbf{A}_{\mathbf{k},q}^*(\mathbf{r}, t) \hat{a}_{\mathbf{k},q}^\dagger \right], \quad q = \pm 1, \quad (8)$$

where $\hat{a}_{\mathbf{k},q}$ and $\hat{a}_{\mathbf{k},q}^\dagger$ are the photon annihilation and creation operators that obey the canonical commutation relations $[\hat{a}_{\mathbf{k},q}, \hat{a}_{\mathbf{k}',q'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'} \delta_{q,q'}$, $[\hat{a}_{\mathbf{k},q}, \hat{a}_{\mathbf{k}',q'}] = 0$, and $[\hat{a}_{\mathbf{k},q}^\dagger, \hat{a}_{\mathbf{k}',q'}^\dagger] = 0$

Figure 1. Schematic illustration of the simulation geometry. (a) First, a Gaussian plane wave approaches a 2D matrix of atoms. Part of the atoms get excited and the incident field goes away. (b) Second, the excited atoms spontaneously re-emit the radiation in all directions, but due to the interference, most of the radiation propagates in the forward and backward directions. The detector on the right measures the far-field of the emitted radiation.

4.1 Expansion of the plane wave in spherical photon states

In the Coulomb gauge assumption, the plane wave states $\mathbf{A}_{\mathbf{k},q}(\mathbf{r},t)$ with $q = \pm 1$ can be given as a sum over spherical photon states as^{39,40}

$$\mathbf{A}_{\mathbf{k},q}(\mathbf{r},t) = \sqrt{2\pi} e^{-i\omega_k t} \sum_{J=1}^{\infty} \sum_{M=-J}^J i^J \sqrt{2J+1} D_{M,q}^{(J)}(\phi_{\mathbf{k}}, \theta_{\mathbf{k}}, -\phi_{\mathbf{k}}) \left[\mathbf{A}_{J,M,k}^{(m)}(\mathbf{r}) + iq \mathbf{A}_{J,M,k}^{(e)}(\mathbf{r}) \right], \quad (9)$$

where $D_{M,q}^{(J)}(\phi_{\mathbf{k}}, \theta_{\mathbf{k}}, -\phi_{\mathbf{k}})$ are Wigner D-matrix elements corresponding to the three Eulerian angles that are in our case formed from the components of the wave vector $\mathbf{k}$ in the spherical coordinates, i.e., $\mathbf{k} = (k, \theta_{\mathbf{k}}, \phi_{\mathbf{k}})$. The spherical photon states $\mathbf{A}_{J,M,k}^{(\xi)}(\mathbf{r})$ in Eq. (2) are defined by

$$\begin{bmatrix} \mathbf{A}_{J,M,k}^{(e)}(\mathbf{r}) \\ \mathbf{A}_{J,M,k}^{(m)}(\mathbf{r}) \\ \mathbf{A}_{J,M,k}^{(l)}(\mathbf{r}) \end{bmatrix} = \sqrt{\frac{\hbar}{2\omega_k V \epsilon_0}} \begin{bmatrix} -\sqrt{J/(2J+1)} & 0 & \sqrt{(J+1)/(2J+1)} \\ 0 & 1 & 0 \\ \sqrt{(J+1)/(2J+1)} & 0 & \sqrt{J/(2J+1)} \end{bmatrix} \begin{bmatrix} j_{J+1}(kr) \mathbf{Y}_{J,J+1,M}(\theta_{\mathbf{r}}, \phi_{\mathbf{r}}) \\ j_J(kr) \mathbf{Y}_{J,J,M}(\theta_{\mathbf{r}}, \phi_{\mathbf{r}}) \\ j_{J-1}(kr) \mathbf{Y}_{J,J-1,M}(\theta_{\mathbf{r}}, \phi_{\mathbf{r}}) \end{bmatrix} \quad (10)$$

Here $j_J(kr)$ are the spherical Bessel functions of the first kind and $\mathbf{Y}_{J,L,M}(\theta_{\mathbf{r}}, \phi_{\mathbf{r}})$ are the vector spherical harmonic functions, defined as

$$\mathbf{Y}_{J,L,M}(\theta_{\mathbf{r}}, \phi_{\mathbf{r}}) = \sum_{q=-1}^1 \langle L, M-q, 1, q | J, M \rangle Y_{L,M-q}(\theta_{\mathbf{r}}, \phi_{\mathbf{r}}) \mathbf{u}^{(q)}. \quad (11)$$

The terms of this series are formed from the well-known Clebsch-Gordan coefficients $\langle L, M - q, 1, q | J, M \rangle$, the scalar spherical harmonic functions $Y_{L,M}(\theta_{\mathbf{r}}, \phi_{\mathbf{r}})$, and the spherical unit vector $\mathbf{u}^{(q)}$. The spherical unit vectors $\mathbf{u}^{(q)}$ are defined as $\mathbf{u}^{(0)} = \hat{\mathbf{z}}$, $\mathbf{u}^{(-1)} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}})$, and $\mathbf{u}^{(+1)} = -\frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})$. For the spherical harmonic functions, we use the definition written in terms of the associated Legendre polynomials $P_{L,M}(x)$ as

$$Y_{L,M}(\theta_{\mathbf{r}}, \phi_{\mathbf{r}}) = \sqrt{\frac{2L+1}{4\pi} \frac{(L-M)!}{(L+M)!}} P_{L,M}(\cos \theta_{\mathbf{r}}) e^{iM\phi_{\mathbf{r}}}. \quad (12)$$

For the associated Legendre polynomials, we use the Condon-Shortley phase convention. Thus, the Condon-Shortley phase is also included in our definition of the spherical harmonic functions. From these definitions, it follows that the spherical photon states obey symmetry relations, given by

$$\begin{aligned} \mathbf{A}_{J,M,k}^{(e)}(-\mathbf{r}) &= (-1)^{J+1} \mathbf{A}_{J,M,k}^{(e)}(\mathbf{r}), & \mathbf{A}_{J,M,k}^{(e)*}(\mathbf{r}) &= (-1)^M \mathbf{A}_{J,-M,k}^{(e)}(\mathbf{r}), \\ \mathbf{A}_{J,M,k}^{(m)}(-\mathbf{r}) &= (-1)^J \mathbf{A}_{J,M,k}^{(m)}(\mathbf{r}), & \mathbf{A}_{J,M,k}^{(m)*}(\mathbf{r}) &= (-1)^{M+1} \mathbf{A}_{J,-M,k}^{(m)}(\mathbf{r}), \\ \mathbf{A}_{J,M,k}^{(l)}(-\mathbf{r}) &= (-1)^{J+1} \mathbf{A}_{J,M,k}^{(l)}(\mathbf{r}), & \mathbf{A}_{J,M,k}^{(l)*}(\mathbf{r}) &= (-1)^M \mathbf{A}_{J,-M,k}^{(l)}(\mathbf{r}). \end{aligned} \quad (13)$$

The spherical photon states correspond to fixed total angular momentum, its z component and energy of the photon. For transverse fields ($\nabla \cdot \mathbf{A} = 0$) there are no longitudinal states and the parity resolves electric and magnetic spherical photon states for fixed J and M .

Note that the spherical photon states are defined with respect to the center of the atom: each atom of the 2D lattice has a manifold of states (9) of its own and the states (9) bound to different atoms are not orthogonal. The angular momentum values therefore correspond to “internal angular momentum”. The photon having a fixed wave vector and polarization has fixed probabilities obtained from the expansion coefficients of Eq. (9) to be in angular momentum states corresponding to J and M .

4.2 Photon–electron interaction and absorption of a photon by a single atom

We investigate a special case where a “classical” ideal circularly polarized plane wave is approaching, along z axis, a single layer 2D atomic lattice. First, we consider the absorption of a photon by a single atom. In this section, we assume a coordinate system whose origin is placed at the position of the atom and the z axis is oriented in the direction of the symmetry axis of the atom.

We assume that the electronic states are pure single atom states with appropriate symmetries and quantum numbers. To make our calculations as simple as possible, we choose the electron ground state to be s-symmetric $\psi_{1s}(\mathbf{r}) = \psi_{1,1,0}(\mathbf{r})$ and the excited state to be p-symmetric $\psi_{2p,M}(\mathbf{r}) = \psi_{2,1,M}(\mathbf{r})$. We also neglect the spin-orbit coupling for convenience. Without losing the depth of the insight of our analysis, based on the initial and final electronic state symmetries, we choose the atoms to be hydrogen atoms, for which the wave functions can be given analytically as

$$\psi_{N,L,M}(\mathbf{r}) = \sqrt{\left(\frac{2}{Na}\right)^3 \frac{(N-L-1)!}{2N(N+L)!}} e^{-r/(Na)} \left(\frac{2r}{Na}\right)^L L_{N-L-1}^{2L+1}\left(\frac{2r}{Na}\right) Y_{L,M}(\theta_{\mathbf{r}}, \phi_{\mathbf{r}}), \quad (14)$$

where $L_n^\alpha(x)$ are the generalized Laguerre polynomials and $a = 4\pi\epsilon_0\hbar^2/(m_e e^2)$ is the Bohr radius, in which e is the elementary charge and m_e is the electron rest mass.

The quantum state of light that most closely corresponds to the “classical” plane wave would probably be the coherent state. However, for further simplicity, we consider here the absorption of a photon from a Fock state. For the incident electromagnetic field, we choose right circular polarization with $q = 1$ and wave vector $\mathbf{k}$ of the plane wave and denote the Fock state with n photons in this mode by $|\dots, n_{\mathbf{k},q}, \dots\rangle$.

The transition rate, i.e., the transition probability per unit of time, for any atom to become excited is given by Fermi’s golden rule, written as

$$\Gamma_{fi} = \frac{2\pi}{\hbar} |\langle f | H_I | i \rangle|^2 \delta(\hbar\omega_{fi} - \hbar\omega_k). \quad (15)$$

Here i and f denote the initial and final states, ω_{fi} is the angular frequency of the transition, and the interaction Hamiltonian operator $\hat{H}_I$ is assumed to be of the so called minimal coupling form, given by⁴¹

$$\hat{H}_I(\mathbf{r}, t) = \frac{e}{m_e} \hat{\mathbf{A}}(\mathbf{r}, t) \cdot \hat{\mathbf{p}}, \quad (16)$$

where $\hat{\mathbf{p}} = -i\hbar\nabla$ is the momentum operator of the electron in the atom. Using $|i_{\mathbf{k},q}\rangle = |\psi_{1s}\rangle \otimes |\dots, n_{\mathbf{k},q}, \dots\rangle$ and $|f_{\mathbf{k},q}\rangle = |\psi_{2p,M}\rangle \otimes |\dots, n_{\mathbf{k},q} - 1, \dots\rangle$, the transition matrix element is given by

$$\begin{aligned} & \langle f_{\mathbf{k},q} | \hat{H}_I(\mathbf{r}, t) | i_{\mathbf{k},q} \rangle \\ &= \frac{e}{m_e} \langle \psi_{2p,M} | \otimes \langle \dots, n_{\mathbf{k},q} - 1, \dots | \hat{\mathbf{A}}(\mathbf{r}, t) \cdot \hat{\mathbf{p}} | \psi_{1s} \rangle \otimes |\dots, n_{\mathbf{k},q}, \dots\rangle \\ &= \frac{e}{m_e} \langle \psi_{2p,M} | \otimes \langle \dots, n_{\mathbf{k},q} - 1, \dots | \sum_{\mathbf{k}', q'} \left[\mathbf{A}_{\mathbf{k}', q'}(\mathbf{r}, t) \hat{a}_{\mathbf{k}', q'} + \mathbf{A}_{\mathbf{k}', q'}^*(\mathbf{r}, t) \hat{a}_{\mathbf{k}', q'}^\dagger \right] \cdot \hat{\mathbf{p}} | \psi_{1s} \rangle \otimes |\dots, n_{\mathbf{k},q}, \dots\rangle \\ &= \frac{e}{m_e} \sqrt{n_{\mathbf{k},q}} \langle \psi_{2p,M} | \mathbf{A}_{\mathbf{k},q}(\mathbf{r}, t) \cdot \hat{\mathbf{p}} | \psi_{1s} \rangle \\ &= \frac{e}{m_e} \sqrt{n_{\mathbf{k},q}} \langle \psi_{2p,M} | \sqrt{2\pi} e^{-i\omega_k t} \sum_{J=1}^{\infty} \sum_{M'=-J}^J i^J \sqrt{2J+1} D_{M,q}^{(J)}(\phi_{\mathbf{k}}, \theta_{\mathbf{k}}, -\phi_{\mathbf{k}}) \left[\mathbf{A}_{J,M',k}^{(m)}(\mathbf{r}) + iq \mathbf{A}_{J,M',k}^{(e)}(\mathbf{r}) \right] \cdot \hat{\mathbf{p}} | \psi_{1s} \rangle \\ &= -q \frac{e}{m_e} \sqrt{6\pi n_{\mathbf{k},q}} e^{-i\omega_k t} \langle \psi_{2p,M} | \mathbf{A}_{1,M=q,k}^{(e)}(\mathbf{r}) \cdot \hat{\mathbf{p}} | \psi_{1s} \rangle \\ &= -q \frac{16ie\hbar}{m_e a (9 + 4a^2 \omega_k^2 / c^2)^2} \sqrt{\frac{\hbar n_{\mathbf{k},q}}{\omega_k V \varepsilon_0}} e^{-i\omega_k t} \end{aligned} \quad (17)$$

In the transition amplitude in Eq. (17), we first have all electric and magnetic multipoles. However, when we calculate the terms in the series by using the wave functions of the hydrogen atom in Eq. (14), we find that only the terms $\mathbf{A}_{1,M=q,k}^{(e)}(\mathbf{r})$ and $\mathbf{A}_{1,M=-q,k}^{(e)}(\mathbf{r})$ are nonzero. However, $D_{M,q}^{(J)}(\phi_{\mathbf{k}}, \theta_{\mathbf{k}}, -\phi_{\mathbf{k}}) = \delta_{M,q}$ in our case where the wave vector is parallel to the z axis, i.e., $\theta_{\mathbf{k}} = \phi_{\mathbf{k}} = 0$. Thus, only the term $\mathbf{A}_{1,M=q,k}^{(e)}(\mathbf{r})$ remains. Therefore, there is a built-in selection rule between the values of M and q and only the related angular momentum component of the plane wave state in Eq. (9) can be absorbed through the corresponding transition.

The original plane wave state must therefore be understood as a superposition of different angular momentum states. The z component of the total angular momentum of the atom and the remaining components of the field with $n_{\mathbf{k},q} - 1$ photons in this mode is still $n_{\mathbf{k},q}\hbar$ after absorption. This can be seen by writing the field mode by using the spherical states given in Eq. (9). In the absorption, the z component of the angular momentum worth of $\hbar$ is transferred from the photon to the electron. The remaining $n_{\mathbf{k},q} - 1$ photons must be in such spherical states that the z component of their total angular momentum is $(n_{\mathbf{k},q} - 1)\hbar$. Thus, the total angular momentum is conserved. Note that if the photon angular momentum value is not equal to $\hbar$, the photon is not absorbed as the related transition matrix elements are zero. Thus, the corresponding part of the Fock space is unchanged.

Then, by substituting the transition matrix element from Eq. (17) into Fermi's golden rule in Eq. (15) and summing over wave vectors in a differential solid angle, we obtain the transition rate per solid angle for the polarization q as

$$\begin{aligned} \frac{d\Gamma_{fi,q}}{d\Omega_{\mathbf{k}_{fi}}} &= \sum_{\mathbf{k} \in d\Omega} \frac{512\pi e^2 \hbar^2 n_{\mathbf{k},q}}{m_e^2 a^2 \omega_k V \varepsilon_0 (9 + 4a^2 \omega_k^2 / c^2)^4} \delta(\hbar\omega_{fi} - \hbar\omega_k) \\ &= \frac{V}{(2\pi)^3} \int_0^\infty \frac{512\pi e^2 \hbar^2 n_{\mathbf{k},q}}{m_e^2 a^2 c k V \varepsilon_0 (9 + 4a^2 k^2)^4} \frac{\delta(k - \omega_{fi}/c)}{c\hbar} k^2 dk \\ &= \frac{64e^2 \hbar \omega_{fi} n_{\mathbf{k}_{fi},q}}{\pi^2 m_e^2 a^2 c^3 \varepsilon_0 (9 + 4a^2 \omega_{fi}^2 / c^2)^4}. \end{aligned} \quad (18)$$

Here we have replaced the sum with an integral by using $\sum_{\mathbf{k} \in d\Omega} = \frac{V}{(2\pi)^3} \int_0^\infty k^2 dk$. Note that this result applies as such only in our special case where the wave vector is parallel to the z axis.

4.3 Re-emission of radiation by the 2D atomic matrix of atoms

We next consider light that is emitted when the atoms return from the excited state $\psi_{2p,M}(\mathbf{r})$ to the atomic ground state $\psi_{1s}(\mathbf{r})$. We assume that the lifetime of the excited state is very long as compared with the pulse length, but this is not a fundamental limitation of the theory. We can then consider the absorption–emission sequence as a resonance process where the initial and final steps can be considered independently. The initial state of the photons for the final emission step is given by $|\dots, 0_{\mathbf{k}_f,q}, \dots\rangle$ while the final state is $|\dots, 1_{\mathbf{k}_f,q}, \dots\rangle$, where the wave vector $\mathbf{k}_f$ of the final state has the same length $|\mathbf{k}_f| = k$ as the original absorbed photon, if the extremely small recoil of the atom is neglected, but it can point in any direction as the direction of spontaneous emission is not determined in advance.

The amplitude for spontaneous emission from the excited state $\psi_{2p,M}(\mathbf{r})$ with $M = 1$ back to the ground state $\psi_{1s}(\mathbf{r})$ is given for $q = 1$ by

$$\begin{aligned}\langle f_{\mathbf{k}_f,q} | \hat{H}_I(\mathbf{r}, t) | i_{\mathbf{k}_f,q} \rangle &= -q \frac{e}{m_e} \sqrt{6\pi} \cos^2(\theta_{\mathbf{k}_f}/2) e^{i\omega_k t} \langle \psi_{1s} | \mathbf{A}_{1,M,k}^{(e)*}(\mathbf{r}) \cdot \hat{\mathbf{p}} | \psi_{2p,M} \rangle \\ &= q \frac{16ie\hbar}{m_e a (9 + 4a^2\omega_k^2/c^2)^2} \sqrt{\frac{\hbar}{\omega_k V \varepsilon_0}} \cos^2(\theta_{\mathbf{k}_f}/2) e^{i\omega_k t}.\end{aligned}\quad (19)$$

and for $q = -1$ by

$$\begin{aligned}\langle f_{\mathbf{k}_f,q} | \hat{H}_I(\mathbf{r}, t) | i_{\mathbf{k}_f,q} \rangle &= -q \frac{e}{m_e} \sqrt{6\pi} e^{2i\phi_{\mathbf{k}_f}} \sin^2(\theta_{\mathbf{k}_f}/2) e^{i\omega_k t} \langle \psi_{1s} | \mathbf{A}_{1,M,k}^{(e)*}(\mathbf{r}) \cdot \hat{\mathbf{p}} | \psi_{2p,M} \rangle \\ &= q \frac{16ie\hbar}{m_e a (9 + 4a^2\omega_k^2/c^2)^2} \sqrt{\frac{\hbar}{\omega_k V \varepsilon_0}} e^{2i\phi_{\mathbf{k}_f}} \sin^2(\theta_{\mathbf{k}_f}/2) e^{i\omega_k t}.\end{aligned}\quad (20)$$

Note that in the case $q = 1$ the emission probability is the largest in the direction of the positive z axis, while in the case $q = -1$ it is the largest in the direction of the negative z axis. This follows from the Wigner D-matrix elements, which lead to the sine and cosine terms in Eqs. (19) and (20).

Now the complementary angular momentum components of the field corresponding to the unabsorbed $n_{\mathbf{k},q} - 1$ photons are not any more present neither in the initial state nor in the final state of the emission process. Thus, the total angular momentum corresponding to the initial state of the emission process is equal to the angular momentum of the excited p-symmetric electronic final state. The total angular momentum of the final state is equal to the sum of the angular momenta of the final atomic state and the photon. Since the angular momentum of the final s-symmetric atomic state is equal to zero, the total angular momentum of the final state of the emission process is equal to the angular momentum of the emitted photon.

If we study many transition events, the photons are emitted in all directions – they do not correspond to a fixed wave vector, but they interfere in such a way that almost all the field energy still propagates in the z direction. The total emitted field is a superposition of spherical photons emitted by single atoms in the 2D lattice. The instantaneous field is then not necessarily orthogonal to the direction of propagation and can carry angular momentum.

Using Fermi's golden rule in the same way as in Eq. (18), we then obtain the transition rate per solid angle for $q = 1$ as

$$\frac{d\Gamma_{fi,q}}{d\Omega_{\mathbf{k}_{fi}}} = \frac{64e^2\hbar\omega_{fi}}{\pi^2 m_e^2 a^2 c^3 \varepsilon_0 (9 + 4a^2\omega_{fi}^2/c^2)^4} \cos^4(\theta_{\mathbf{k}_f}/2). \quad (21)$$

and for $q = -1$ as

$$\frac{d\Gamma_{fi,q}}{d\Omega_{\mathbf{k}_{fi}}} = \frac{64e^2\hbar\omega_{fi}}{\pi^2 m_e^2 a^2 c^3 \varepsilon_0 (9 + 4a^2\omega_{fi}^2/c^2)^4} \sin^4(\theta_{\mathbf{k}_f}/2). \quad (22)$$

The sum of the integrals of the angle-dependent transition rates in Eqs. (21) and (22) over all directions of the wave vector gives the total transition rate as

$$\Gamma_{fi} = \sum_q \int \frac{d\Gamma_{fi,q}}{d\Omega_{\mathbf{k}_{fi}}} d\Omega_{\mathbf{k}_{fi}} = \frac{512e^2\hbar\omega_{fi}}{3\pi m_e^2 a^2 c^3 \varepsilon_0 (9 + 4a^2\omega_{fi}^2/c^2)^4}. \quad (23)$$

By using the reported value for the $1s \rightarrow 2p$ transition wavelength of hydrogen equal to $\lambda_{fi} = 121.567$ nm and numerical values for all physical constants, we then obtain a numerical value for the emission rate as $\Gamma_{fi} = 6.2649 \times 10^8$ s⁻¹. This is a well-known result for this transition. The cases $q = 1$ and $q = -1$ in this transition have equal probabilities.

5. SIMULATION OF THE EMITTED FIELD

In the calculation of the total field emitted by many atoms, we make a simplifying assumption that the initial exciting field has already vanished. We then use Fermi's golden rule to the emission and forget the incoming photons. We calculate the contribution of each atom to the total emitted electromagnetic field. Since the emission is spontaneous, it happens at random times and the total emitted field amplitude is given for $q = 1$ and for $\mathbf{k}_f = \mathbf{k}$ by

$$\mathbf{A}_{\text{tot}}(\mathbf{r}, t) = \sum_j w_j \frac{2k\sqrt{2V\Delta k}}{\pi^{1/4}} \int_0^\infty \mathbf{A}_{1,M=q,k'}^{(e)}(\mathbf{r} - \mathbf{r}_j) e^{-i\omega_{k'}(t-t_j)} \varphi_{k,\Delta k}(k') dk', \quad (24)$$

where the index j ranges over all atoms, $w_j = 0, 1$ is a random variable that obtains the value 1 with the probability that the atom j has been excited, t_j are random times of the emission moments of atoms, which are assumed to follow the exponential distribution, and $\varphi_{k,\Delta k}(k')$ is the line shape function, which is assumed to be Gaussian, given by

$$\varphi_{k,\Delta k}(k') = \frac{e^{-[(k'-k)/\Delta k]^2/2}}{\sqrt{2\pi} \Delta k}. \quad (25)$$

Here Δk is the standard deviation of the wave vector length distribution. It is related to the transition rate Γ_{fi} as $\Delta k = \Gamma_{fi}/(2\sqrt{2}\ln 2c)$ and it originates from the finite life time of the excited state of atoms. The total vector potential can be calculated either semiclassically, assuming an exponential decay law or quantum mechanically using a photon walk kind of an approach. The key point in both approaches is that the photon states $\mathbf{A}_{1,q,k}^{(e)}(\mathbf{r} - \mathbf{r}_j)$ correspond to the z component of the angular momentum equal to $\hbar$ in the frame bound to atom j while they do not correspond to the fixed direction of the wave vector. The normalization factor of the vector potential in Eq. (24) is defined so that the integral of the corresponding electromagnetic energy density over all space is equal to $\hbar\omega_k$ for a single term in the sum of Eq. (24) when $\Delta k/k \ll 1$.

In the simulations, the atoms are in a rectangular lattice in area $1 \text{ mm} \times 1 \text{ mm}$ centered at the origin of the x - y plane. There are 500-nm spacings between the atoms in the lattice. Thus, the simulation contains 4 million atoms in total. At $t = 0$ s, 30% of the atoms are assumed to be in the excited state $\psi_{2p,M=1}(\mathbf{r})$. Figure 2(a) shows the time-dependence of the Poynting vector at $x = y = 0$ m, and $z = 10$ m. The Poynting vector is found to obtain its maximum value at $t = 35.75$ ns. This time is slightly longer than the time that it takes for light to travel the distance of 10 m to the point of observation. The time-dependence of the Poynting vector follows the Gaussian form as expected. Figure 2(b) presents the transverse position dependence of the Poynting vector at $y = 0$ m, $z = 10$ m, and $t = 35.75$ ns. The slight asymmetry between the left and right sides of this graph follows from the probabilistic nature of the absorption-emission sequence due to which the emission is not fully symmetric with respect to the z axis. Figure 2(c) depicts the time-dependence of the spin angular momentum density at $x = y = 0$ m, and $z = 10$ m. This graph is found to follow the Poynting vector in 2(a).

Along the z axis, the z component of the orbital angular momentum density is found to be negligibly small due to the very small z components of the fields that remind the classical plane wave in the z axis. In contrast, in the z axis, the z component of the spin angular momentum per a small cross-sectional area πR^2 , where the fields can be considered constant in the transverse direction, is given by an integral of the spin angular momentum density over time as

$$\frac{\langle \hat{S}_z \rangle}{\pi R^2} = -\frac{i}{4\omega_k} \hat{\mathbf{z}} \cdot \int_0^\infty \left[\varepsilon_0 \mathbf{E}^*(\mathbf{r}, t) \times \mathbf{E}(\mathbf{r}, t) + \frac{1}{\mu_0} \mathbf{B}^*(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \right] c dt. \quad (26)$$

Correspondingly, the energy of light per cross-sectional area is given by the integral of the Poynting vector over time as

$$\frac{\langle \hat{E} \rangle}{\pi R^2} = \frac{1}{2} \hat{\mathbf{z}} \cdot \int_0^\infty \varepsilon_0 \text{Re}[\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}^*(\mathbf{r}, t)] dt. \quad (27)$$

Figure 2. Simulation of (a) the time-dependence of the Poynting vector at $x = y = 0$ m, and $z = 10$ m, (b) the transverse position dependence of the Poynting vector at $y = 0$ m, $z = 10$ m, and $t = 35.75$ ns, and (c) the time-dependence of the spin angular momentum density at $x = y = 0$ m, and $z = 10$ m. The fields are emitted by the 2D lattice of hydrogen atoms. The atoms are in a rectangular lattice in an area of $1 \text{ mm} \times 1 \text{ mm}$ centered at the origin of the x - y plane. There are 500-nm spacings between the atoms in the lattice. At $t = 0$ s, 30% of the atoms are assumed to be in the excited state $\psi_{2p, M=1}(\mathbf{r})$. Thus, the simulation contains 4 million atoms in total. The field quantities are found to obtain their maximum values at $t = 35.75$ ns.

Using, the number of photons per cross-sectional area, given by $\langle \hat{N} \rangle / (\pi R^2) = \langle \hat{E} \rangle / (\hbar \omega_k \pi R^2)$, we then obtain the spin angular momentum per photon as

$$\frac{\langle \hat{S}_z \rangle}{\langle \hat{N} \rangle} = \frac{-\frac{i\hbar}{4} \hat{\mathbf{z}} \cdot \int_0^\infty \left[\varepsilon_0 \mathbf{E}^*(\mathbf{r}, t) \times \mathbf{E}(\mathbf{r}, t) + \frac{1}{\mu_0} \mathbf{B}^*(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \right] c dt}{\frac{1}{2} \hat{\mathbf{z}} \cdot \int_0^\infty \varepsilon_0 \text{Re}[\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}^*(\mathbf{r}, t)] dt}. \quad (28)$$

The numerical calculation of these integrals in our simulation gives $\langle \hat{S}_z \rangle / \langle \hat{N} \rangle = \hbar$ with a relative numerical accuracy of 3×10^{-10} .

If we used the classical definition of the total angular momentum density in the integrand of Eq. (1), the total angular momentum density in the z axis would be negligibly small. Thus, the angular momentum must be calculated directly from the quantum mechanical expectation values in Eqs. (4)–(6), not from the classical definition in Eq. (1). This also indicates that the surface integral term cannot be neglected in Eq. (6). The physical origin of this surface integral term is, however, not due to the interaction of the electromagnetic field with the atoms at the region of the surface only, but this term originates from the physical volume densities of the angular momenta given for the orbital angular momentum in the first row of Eq. (4) and for the spin angular momentum in Eq. (5).

6. CONCLUSIONS

In conclusion, we have studied the classical and quantum definitions of the total angular momenta of light and how it splits to the orbital and spin angular momenta. We have also considered a sequence of events where first a plane wave is absorbed and electric dipole photons are subsequently emitted by a 2D matrix of atoms. We have studied how the electric dipole radiation differs from the incoming plane wave and compared the angular momenta that these fields carry.

Using simulations, we can calculate the expectation values of the angular momentum, momentum and all field quantities in any direction, for example in the original direction of the exciting radiation. The key observation is that the expectation value of the total angular momentum, i.e., its classical limit, must be calculated directly from the quantum mechanical expectation values in Eqs. (4)–(6), not from the classical Eq. (1) which is based on the Poynting vector that is obtained by first calculating the electric and magnetic fields which presumably correspond to the classical plane wave in Eq. (7). In our example, the angular momentum calculated from the Poynting vector is zero, but the expectation value of the angular momentum calculated from Eqs. (4)–(6) is equal to the number of photons times $\hbar$.

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