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DIRICHLET SPACES OF DOMAINS BOUNDED BY QUASICIRCLES

DAVID RADNELL, ERIC SCHIPPERS, AND WOLFGANG STAUBACH

ABSTRACT. Consider a multiply-connected domain Σ in the sphere bounded by n non-intersecting quasicircles. We characterize the Dirichlet space of Σ as an isomorphic image of a direct sum of Dirichlet spaces of the disk under a generalized Faber operator. This Faber operator is constructed using a jump formula for quasicircles and certain spaces of boundary values.

Thereafter, we define a Grunsky operator on direct sums of Dirichlet spaces of the disk, and give a second characterization of the Dirichlet space of Σ as the graph of the generalized Grunsky operator in direct sums of the space $\mathcal{H}^{1/2}(\mathbb{S}^1)$ on the circle. This has an interpretation in terms of Fourier decompositions of Dirichlet space functions on the circle.

1. INTRODUCTION

Let Σ be a domain in $\overline{\mathbb{C}}$ bounded by n non-overlapping quasicircles Γ_k , $k = 1, \dots, n$. Assume that Σ contains ∞ . Let Ω_k^+ and Ω_k^- denote the bounded and unbounded components of the complements of Γ_k in the sphere $\overline{\mathbb{C}}$, so that $\Sigma = \cap_{k=1}^n \Omega_k^-$. Let $f = (f_1, \dots, f_n)$ be an n -tuple of quasiconformally extendible conformal maps $f_k : \mathbb{D}^+ \rightarrow \Omega_k^+$, where $\mathbb{D}^+ = \{z : |z| < 1\}$.

Let $\mathcal{D}_\infty(\Sigma)$ denote the Dirichlet space of holomorphic functions on Σ vanishing at ∞ and let $\mathcal{D}_\infty(\mathbb{D}^-)$ denote the Dirichlet space of holomorphic functions on $\mathbb{D}^- = \{z : |z| > 1\} \cup \{\infty\}$ vanishing at ∞ . For Cauchy projections $P(\Omega_k^-)$ from functions on Γ_k into $\mathcal{D}_\infty(\Omega_k^-)$ define

$$\begin{aligned} \mathbf{I}_f : \bigoplus_{k=1}^n \mathcal{D}_\infty(\mathbb{D}^-) &\longrightarrow \mathcal{D}_\infty(\Sigma) \\ (h_1, \dots, h_n) &\longmapsto \sum_{k=1}^n P(\Omega_k^-) \mathcal{C}_{f_k^{-1}} h_k \end{aligned}$$

where $\mathcal{C}_{f_k^{-1}} h_k = h_k \circ f_k^{-1}|_{\Gamma_k}$. We show that $\mathbf{I}_f$ is defined and is an isomorphism. As an application, we show that Dirichlet space functions have a unique multiple Faber series approximation which converges uniformly on compact subsets of Σ .

It is far from obvious that the composition and projection operators are well-defined and bounded, since f_k maps into the complement of Σ for each k . These problems were resolved by E. Schippers and W. Staubach in [19, 20] where a theory of harmonic extension and a Plemelj-Sokhotski jump decomposition for quasicircles is developed. Since quasicircles are not necessarily rectifiable, to define the projection operator, we use a limiting integral along

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curves approaching Γ . In [19] it was shown that for quasicircles, the integral can be taken over limiting curves approaching Γ from either the inside or the outside with the same result. This is a key theorem which is necessary for the results obtained in the present paper. This in turn relied on the fact that for every Dirichlet-bounded harmonic function on one side of a curve, there is a Dirichlet-bounded harmonic function on the other side with the same boundary values, which we will call the “reflection”; furthermore this correspondence is bounded.

The isomorphism $\mathbf{I}_f$ itself can be thought of as a kind of Faber operator for multiply-connected domains. The Faber operator is of general interest in approximation theory [1], [23, Chapter VII] since it determines the Faber series approximation of a given function. Different choices of regularity of the boundary and norm of the functions can be made (see the table on page 91 of [23]). The version of the Faber operator $\mathbf{I}_f$ given here is closely related, in the simply connected case, to Faber series for Dirichlet space functions [3, 20, 21]. In this setting existence of Faber series was shown by A. Çavuş [3], and existence and uniqueness was proven by Y. Shen [21].

When treated as a map between Dirichlet spaces $\mathcal{D}_\infty(\mathbb{D}^-)$ and $\mathcal{D}(\Omega^-)$, the Faber operator was shown by Schippers and Staubach [20] to be an isomorphism if and only if the curve is a quasicircle [20]. This result was first obtained by Shen [21], with a somewhat different formulation: there, the isomorphism takes ℓ^2 sequences to Faber series. Later, H.Y. Wei, M.L. Wang and Y. Hu showed that for rectifiable Jordan curves, the Faber operator on Dirichlet space is an isomorphism if and only if the curve is a rectifiable quasicircle [25]. They also disproved a conjecture of Anderson that the Faber operator is a bounded isomorphism of Besov spaces B^p for general Jordan curves. The formulation of Schippers and Staubach as a map between Dirichlet spaces for general quasicircles (in terms of the composition operator and Cauchy-type projection) required the jump formula on quasicircles as well as the existence of a bounded reflection of Dirichlet-bounded harmonic functions, obtained in [19, 20].

Dirichlet space seems to be the weakest (L^2 -type) regularity for which this is possible for quasicircles. Hardy space functions, for example, have only boundary values almost everywhere with respect to harmonic measure. Since sets of harmonic measure zero with respect to one side of a Jordan curve need not have harmonic measure zero with respect to the other side, the boundary value problem on the complement will not be well-posed. Dirichlet space functions have boundary values except on a set of capacity zero. This stronger result makes the problem well-posed.

For other recent results for Faber operators with different choices of regularity, see for example Y. E. Yıldırım and R. Çetintaş [26] for Hardy-Orlicz and Smirnov-Orlicz spaces for Dini-smooth curves, or D. Gaier [7] where it is shown that the Faber operator can be unbounded for quasicircles with respect to the supremum norm, in contrast to its behaviour with respect to the Dirichlet norm.

We also define a generalized Grunsky operator corresponding to the aforementioned n -tuple of quasiconformally extendible maps $(f_1, \dots, f_n)$. We show that our generalized Grunsky operator characterizes the Dirichlet space $\mathcal{D}(\Sigma)$ of holomorphic functions with bounded

Dirichlet energy in the following sense: up to constants, the set

$$\mathbf{W} = \{(h \circ f_1|_{\mathbb{S}^1}, \dots, h \circ f_n|_{\mathbb{S}^1}) : h \in \mathcal{D}(\Sigma)\} \subseteq \bigoplus^n \mathcal{H}(\mathbb{S}^1),$$

where $\mathcal{H}(\mathbb{S}^1)$ is the $H^{1/2}$ Sobolev space on $\mathbb{S}^1$, is the graph of our Grunsky operator. This can be thought of as a characterization of the possible Fourier series on $\mathbb{S}^1$ obtained by pulling back elements of $\mathcal{D}(\Sigma)$. This characterization relates to the so-called Segal-Wilson Grassmannian, and has applications to two-dimensional conformal field theory and to Teichmüller theory, see D. Radnell, E. Schippers and W. Staubach [16, 17].

As in the case of Faber operators, the regularity of the curve relates to the properties of the Grunsky operator in the simply connected case. As is well known [14], the Grunsky operator (formulated in terms of ℓ^2 sequences) is bounded by a constant strictly less than one, precisely for quasicircles. We show that the generalized Grunsky operator is strictly less than one in norm for multiply-connected domains bounded by quasicircles. It was shown independently by Shen [22] and L. Takhtajan and L.-P. Teo [24] that a Jordan curve is a Weil-Petersson class quasicircle if and only if the Grunsky operator is Hilbert-Schmidt and bounded by one. Shen also proved that the Grunsky operator is compact if and only if the curve corresponds to a univalent function with asymptotically conformal quasiconformal extension [21].

Here is an outline of the paper. In Section 2 we review some facts and results on the Dirichlet spaces, the harmonic reflection for quasicircles and the projection operators. In Section 3 we explain the relation of the projection operators to the Grunsky operator given in the papers [19, 20], and formulate them in a way suitable for this paper. We then prove the first main result that the Faber operator is an isomorphism of Dirichlet spaces defined in [20] to multiply-connected domains bounded by quasicircles. We apply this to show that elements of the Dirichlet space have unique multiple Faber series, which converge uniformly on compact sets. In Section 4 we define the generalized Grunsky operator, derive some integral expressions for its blocks, and show that it is bounded by some $k < 1$ for multiply-connected domains bounded by quasicircles. Finally we show that $\mathbf{W}$ is the graph of the Grunsky operator.

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2. REFLECTION OF DIRICHLET SPACES IN QUASICIRCLES

2.1. Notation. Let $\mathbb{D}^+ = \{z : |z| < 1\}$, $\mathbb{D}^- = \{z : |z| > 1\} \cup \{\infty\}$, $\mathbb{S}^1 = \{z : |z| = 1\}$, and $\overline{\mathbb{C}}$ be the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Complements are taken in the Riemann sphere. Throughout the paper Γ will be either a Jordan curve or specifically a quasicircle, not passing through ∞ , and we denote the bounded component of its complement by Ω^+ and the unbounded component by Ω^- . We will shorten this by saying Γ bounds $\Omega^\pm$.

For domains Ω_i in $\overline{\mathbb{C}}$, $i = 1, 2$, we say that $f : \Omega_1 \rightarrow \Omega_2$ is conformal if it is a meromorphic bijection. In particular, f has at most a simple pole.

2.2. Dirichlet spaces. Let Ω be a Jordan domain in $\overline{\mathbb{C}}$. Throughout the paper, ∞ will not lie on the boundary of Ω . Using conformal invariance, the definitions below extend in an obvious way to domains whose boundaries contain ∞ , although we will not require this.

We define the harmonic Dirichlet space

$$(1) \quad \mathcal{D}_{\text{harm}}(\Omega) = \left\{ h : \Omega \rightarrow \mathbb{C} : h \text{ harmonic and } \iint_{\Omega} \left| \frac{\partial h}{\partial z} \right|^2 dA + \iint_{\Omega} \left| \frac{\partial h}{\partial \bar{z}} \right|^2 dA < \infty \right\}.$$

If $\infty \in \Omega$, then the statement that h is harmonic is taken to mean that h is harmonic on $\Omega \setminus \{\infty\}$ and $h(1/z)$ is harmonic on an open set containing 0. In a similar way, we define the notion of holomorphicity at ∞ . We define a semi-norm on $\mathcal{D}_{\text{harm}}(\Omega)$ by

$$(2) \quad \|h\|_{\mathcal{D}_{\text{harm}}(\Omega)}^2 = \iint_{\Omega} \left| \frac{\partial h}{\partial z} \right|^2 dA + \iint_{\Omega} \left| \frac{\partial h}{\partial \bar{z}} \right|^2 dA.$$

This semi-norm is invariant under composition by Möbius transformations; that is, for any Möbius transformation T one has

$$(3) \quad \|h\|_{\mathcal{D}_{\text{harm}}(T(\Omega))} = \|h \circ T\|_{\mathcal{D}_{\text{harm}}(\Omega)}.$$

In particular, we have that $h \in \mathcal{D}_{\text{harm}}(T(\Omega))$ if and only if $h \circ T \in \mathcal{D}_{\text{harm}}(\Omega)$.

We denote the holomorphic Dirichlet spaces by

$$\mathcal{D}(\Omega) = \{h \in \mathcal{D}_{\text{harm}}(\Omega) : h \text{ holomorphic}\}.$$

The antiholomorphic Dirichlet space consisting of $\bar{h}$ such that $h \in \mathcal{D}(\Omega)$ will be denoted $\overline{\mathcal{D}(\Omega)}$. The restriction of the semi-norm to $\mathcal{D}(\Omega)$ is given by

$$\|h\|_{\mathcal{D}(\Omega)}^2 = \iint_{\Omega} |h'|^2 dA.$$

When Ω contains ∞ , we will also consider the subspace of those functions in $\mathcal{D}(\Omega)$ vanishing at ∞ . We will denote this by

$$\mathcal{D}_{\infty}(\Omega) = \{h \in \mathcal{D}(\Omega) : h(\infty) = 0\}.$$

On this space the seminorm is a norm.

Finally, we observe the following elementary conformal invariance. Let Ω_1 and Ω_2 be Jordan domains not containing ∞ in their boundaries. Let $f : \Omega_1 \rightarrow \Omega_2$ be a conformal map. The composition operator

$$\begin{aligned} \mathcal{C}_f : \mathcal{D}_{\text{harm}}(\Omega_2) &\longrightarrow \mathcal{D}_{\text{harm}}(\Omega_1) \\ h &\longmapsto h \circ f \end{aligned}$$

is a linear isomorphism such that

$$(4) \quad \|\mathcal{C}_f h\|_{\mathcal{D}_{\text{harm}}(\Omega_1)} = \|h\|_{\mathcal{D}_{\text{harm}}(\Omega_2)}.$$

The restriction of the composition operator to the holomorphic Dirichlet spaces is defined in a similar way.

2.3. Boundary values of Dirichlet space and harmonic reflection. The Dirichlet spaces have boundary values in a certain conformally invariant sense originating with Osborn [13]. We summarize the results here; a full exposition and proofs can be found in [19].

Let Γ be a Jordan curve, not containing ∞ , bounding a Jordan domain Ω . Here Ω could be either the bounded or unbounded component of the complement of Γ . Fix any point $p \in \Omega$. For any point $q \in \Gamma$ let $\gamma_{p,q}$ be the hyperbolic geodesic arc connecting p to q . For any $h \in \mathcal{D}_{\text{harm}}(\Omega)$, the limiting values

$$H(q) = \lim_{z \rightarrow q} h(z), \quad z \in \gamma_{p,q}$$

exist for all q , except possibly on a set K which is the image of a closed set $I \subseteq \mathbb{S}^1$ of zero logarithmic capacity under a conformal map $f : \mathbb{D}^+ \rightarrow \Omega$. Since disk automorphisms take sets of logarithmic capacity zero in $\mathbb{S}^1$ to sets of logarithmic capacity zero, this is independent of the choice of p and f . Let $\mathcal{H}(\partial\Omega)$ temporarily denote the set of functions on Γ obtained in this way, where two functions are identified if they agree except on a set of type K . Any $h \in \mathcal{H}(\partial\Omega)$ has a unique extension to $\mathcal{D}(\Omega)$ whose boundary values agree with h except on a set of type K .

In fact $\mathcal{H}(\partial\Omega)$ is the set of boundary values of $\mathcal{D}_{\text{harm}}(\Omega)$ on the ideal boundary of Ω , which can be identified with Γ by the Carathéodory theorem. We call the set of functions obtained as boundary values of $\mathcal{D}_{\text{harm}}(\Omega)$ in this way the *Douglas-Osborn space*. We also call these boundary values in the sense of Osborn.

Now assume that Γ is a quasicircle. Let us denote the set of boundary values of $\mathcal{D}_{\text{harm}}(\Omega^\pm)$, by $\mathcal{H}_\pm(\Gamma)$. It is not clear whether $\mathcal{H}_+(\Gamma)$ and $\mathcal{H}_-(\Gamma)$ are the same in general. However, on quasidisks we have by a result in [19, Theorem 2.14] that

$$\mathcal{H}_+(\Gamma) = \mathcal{H}_-(\Gamma)$$

in the following sense. Fix conformal maps $f : \mathbb{D}^+ \rightarrow \Omega^+$ and $g : \mathbb{D}^- \rightarrow \Omega^-$. For any element $h_+ \in \mathcal{D}(\Omega^+)$, there is a unique $h_- \in \mathcal{D}(\Omega^-)$ whose boundary values agree with those of h_+ except on a set K such that $K = f(I) = g(J)$ where $I \subseteq \mathbb{S}^1$ and $J \subseteq \mathbb{S}^1$ are closed sets of logarithmic capacity zero. Thus for quasicircles Γ not containing ∞ we set

$$\mathcal{H}(\Gamma) = \mathcal{H}_+(\Gamma) = \mathcal{H}_-(\Gamma)$$

and make the following definition.

Definition 2.1. *Let Γ be a quasicircle not containing ∞ . For $h \in \mathcal{H}(\Gamma)$ let $\mathbf{E}_\pm h$ denote the unique elements of $\mathcal{D}(\Omega^\pm)$ whose boundary values in the sense of Osborn agree with those of h .*

Remark 2.2. *In fact, the condition that $\mathcal{H}_+(\Gamma) = \mathcal{H}_-(\Gamma)$ characterizes quasidisks in a certain sense, see [19, Theorem 2.14].*

Using $\mathbf{E}_\pm$, one can define semi-norms on $\mathcal{H}_\pm(\Gamma)$ by

$$(5) \quad \|h\|_{\mathcal{H}_\pm(\Gamma)} = \|\mathbf{E}_\pm h\|_{\mathcal{D}_{\text{harm}}(\Omega^\pm)},$$

and one also has that there is a uniform constant C , depending only on Γ , such that for any $h \in \mathcal{H}(\Gamma)$,

$$\frac{1}{C} \|\mathbf{E}_+ h\|_{\mathcal{D}(\Omega^+)_{\text{harm}}} \leq \|\mathbf{E}_- h\|_{\mathcal{D}(\Omega^-)_{\text{harm}}} \leq C \|\mathbf{E}_+ h\|_{\mathcal{D}(\Omega^+)_{\text{harm}}},$$

see [19, Equation (2.5)].

Observe that $\mathcal{H}(\Gamma)$ is conformally invariant in the following sense [19, Theorem 2.21]:

Let Γ_1 and Γ_2 be quasicircles not containing ∞ , bounding domains Ω_1 and Ω_2 . A conformal map $f : \Omega_1 \rightarrow \Omega_2$ must have a quasiconformal extension to $\overline{\mathbb{C}}$ since Γ_i are quasicircles [11]. Moreover, quasiconformal maps preserve sets of zero logarithmic capacity. Thus, we have a well-defined composition operator

$$\begin{aligned} \mathcal{C}_f : \mathcal{H}(\Gamma_2) &\longrightarrow \mathcal{H}(\Gamma_1) \\ h &\longmapsto h \circ f. \end{aligned}$$

Theorem 2.3. *Let Γ_1 and Γ_2 be quasicircles not containing ∞ , bounding domains $\Omega_1^\pm$ and $\Omega_2^\pm$. If $f^\pm : \Omega_1^\pm \rightarrow \Omega_2^\pm$ are conformal maps, then $\mathcal{C}_{f^\pm}$ are linear isomorphisms such that*

$$\|\mathcal{C}_{f^\pm} h\|_{\mathcal{H}_\pm(\Gamma_1)} = \|h\|_{\mathcal{H}_\pm(\Gamma_2)}.$$

Throughout the paper, we will not specify which semi-norm is placed on $\mathcal{H}(\Gamma)$. From this point of view, Theorem 2.3 says only that the composition operator is bounded with respect to the seminorm.

Remark 2.4. *In the case that $\Gamma = \mathbb{S}^1$ we have that*

$$\mathcal{H}_+(\mathbb{S}^1) = \mathcal{H}_-(\mathbb{S}^1) = H^{1/2}(\mathbb{S}^1) = \mathcal{H}(\mathbb{S}^1)$$

where $H^{1/2}(\mathbb{S}^1)$ denotes the set of L^2 functions on $\mathbb{S}^1$ whose Fourier series $h(e^{\sqrt{-1}\theta}) = \sum_{n=-\infty}^{\infty} h_n e^{\sqrt{-1}n\theta}$ satisfy

$$\sum_{n=-\infty}^{\infty} |n| |h_n|^2 < \infty.$$

It is also well-known that functions $h \in \mathcal{H}(\mathbb{S}^1)$ have Fourier series which converge to h almost everywhere, in fact except on a set of outer logarithmic capacity zero [6]. We can assume the set is a closed set of logarithmic capacity zero.

2.4. Decompositions of Dirichlet spaces on quasidisks and the projection operators. For a fixed $p \in \Omega^+$ we define the holomorphic Dirichlet space on Ω^+ by

$$\mathcal{D}_p(\Omega^+) = \left\{ h : \Omega^+ \rightarrow \mathbb{C} : h \text{ holomorphic, } h(p) = 0, \text{ and } \iint_{\Omega^+} |h'|^2 dA < \infty \right\}$$

and similarly on Ω^- by

$$\mathcal{D}_\infty(\Omega^-) = \left\{ h : \Omega^- \rightarrow \mathbb{C} : h \text{ holomorphic, } h(\infty) = 0, \text{ and } \iint_{\Omega^-} |h'|^2 dA < \infty \right\}.$$

Since the domains $\Omega^\pm$ are simply-connected it is immediate that

$$\mathcal{D}_{\text{harm}}(\Omega^+) = \mathcal{D}_p(\Omega^+) \oplus \overline{\mathcal{D}_p(\Omega^+)} \oplus \mathbb{C}$$

and

$$\mathcal{D}_{\text{harm}}(\Omega^-) = \mathcal{D}_\infty(\Omega^-) \oplus \overline{\mathcal{D}_\infty(\Omega^-)} \oplus \mathbb{C}$$

where $\overline{\mathcal{D}_p(\Omega^+)}$ denotes the anti-holomorphic functions on Ω^+ vanishing at p and with finite Dirichlet energy (equivalently, conjugates of elements of $\mathcal{D}_p(\Omega^+)$ and similarly for Ω^-). The $\mathbb{C}$ in the decomposition corresponds to the constants $h(p)$ and $h(\infty)$ respectively.

Moreover, we have a reflection map

$$\begin{aligned}\mathbf{R} : \mathcal{D}_\infty(\mathbb{D}^-) &\longrightarrow \overline{\mathcal{D}_0(\mathbb{D}^+)} \\ h(z) &\longmapsto h(1/\bar{z}).\end{aligned}$$

and similarly $\mathbf{R} : \mathcal{D}_0(\mathbb{D}^+) \rightarrow \overline{\mathcal{D}_\infty(\mathbb{D}^-)}$. It is easily seen that $\mathbf{R}$ is an isomorphism. Furthermore $\mathbf{R}$ preserves the boundary values of h . Thus the decomposition of $\mathcal{D}_{\text{harm}}(\Omega^+)$ also agrees with the decomposition $\mathcal{H}(\mathbb{S}^1) = \mathcal{D}_0(\mathbb{D}^+) \oplus \mathcal{D}_\infty(\mathbb{D}^-) \oplus \mathbb{C}$.

Now we define the Cauchy projections. In the following definition, and the rest of the paper, we denote by γ_r the curve $|z| = r$ oriented counterclockwise.

Definition 2.5. *Let Γ be a quasicircle not containing ∞ . Let Ω^+ and Ω^- be the bounded and unbounded components of the complement respectively. Let $f : \mathbb{D}^+ \rightarrow \Omega^+$ be conformal.*

We define, for $z \in \Omega^-$,

$$\begin{aligned}P_\infty(\Omega^-) : \mathcal{H}(\Gamma) &\longrightarrow \mathcal{D}_\infty(\Omega^-) \\ h &\longmapsto -\lim_{r \nearrow 1} \frac{1}{2\pi\sqrt{-1}} \int_{f(\gamma_r)} \frac{\mathbf{E}_+ h(\zeta)}{\zeta - z} d\zeta\end{aligned}$$

and for $z \in \Omega^+$,

$$\begin{aligned}P_p(\Omega^+) : \mathcal{H}(\Gamma) &\longrightarrow \mathcal{D}_p(\Omega^+) \\ h &\longmapsto \lim_{r \nearrow 1} \frac{1}{2\pi\sqrt{-1}} \int_{f(\gamma_r)} \frac{(z-p)\mathbf{E}_+ h(\zeta)}{(\zeta-p)(\zeta-z)} d\zeta.\end{aligned}$$

These projections are well-defined and bounded.

Theorem 2.6. *Let Γ be a quasicircle not containing ∞ . Let Ω^+ and Ω^- be the bounded and unbounded components of the complement respectively. Then*

- (1) $P_p(\Omega^+)$ and $P_\infty(\Omega^-)$ are independent of the choice of f ,
- (2) $P_p(\Omega^+)$ is a bounded map into $\mathcal{D}_p(\Omega^+)$, and
- (3) $P_\infty(\Omega^-)$ is bounded map into $\mathcal{D}_\infty(\Omega^-)$.

Furthermore, for any conformal map $g : \mathbb{D}^- \rightarrow \Omega^-$ we have the alternate integral formulas

$$P_p(\Omega^+)h(z) = \lim_{r \searrow 1} \frac{1}{2\pi\sqrt{-1}} \int_{g(\gamma_r)} \frac{(z-p)\mathbf{E}_- h(\zeta)}{(\zeta-p)(\zeta-z)} d\zeta, \quad \text{for } z \in \Omega^+,$$

and

$$P_\infty(\Omega^-)h(z) = -\lim_{r \searrow 1} \frac{1}{2\pi\sqrt{-1}} \int_{g(\gamma_r)} \frac{\mathbf{E}_- h(\zeta)}{\zeta - z} d\zeta, \quad \text{for } z \in \Omega^-.$$

Proof. Observe that the projection from the full Dirichlet space $\mathcal{D}(\Omega^+)$ to $\mathcal{D}_p(\Omega^+)$ given by $h \mapsto h - h(p)$ is bounded with respect to the norm $\|h\|_{\mathcal{D}(\Omega^+)}^2 = \|h\|_{\mathcal{D}_p(\Omega^+)}^2 + |h(p)|^2$. The integral kernel of this projection is

$$\frac{1}{2\pi\sqrt{-1}} \frac{(z-p)}{(\zeta-p)(\zeta-z)}.$$

The claim thus reduces to [19, Theorems 3.2, 3.5]. □

We remark that on the disk, the integral can be taken over the boundary: that is,

$$P_p(\mathbb{D}^+)h(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{S}^1} \frac{(z-p)\mathbf{E}_+h(\zeta)}{(\zeta-p)(\zeta-z)} d\zeta$$

and similarly for $P_\infty(\mathbb{D}^-)$. In fact this holds for any Weil-Petersson class quasidisk [15].

3. THE FABER ISOMORPHISM AND GRUNSKY OPERATORS

3.1. Faber isomorphism and Grunsky operator. Next, we define two operators. We notationally suppress a trace to the boundary in both definitions for simplicity.

Definition 3.1. *Let Γ be a quasicircle not containing ∞ , with bounded and unbounded components Ω^+ and Ω^- respectively. Let $f : \mathbb{D}^+ \rightarrow \Omega^+$ be a conformal map. Define I_f by*

$$\begin{aligned} I_f : \mathcal{D}_\infty(\mathbb{D}^-) &\longrightarrow \mathcal{D}_\infty(\Omega^-) \\ h &\longmapsto P_\infty(\Omega^-)\mathcal{C}_{f^{-1}}h, \end{aligned}$$

and define the Grunsky operator by

$$\begin{aligned} \text{Gr}_f : \mathcal{D}_\infty(\mathbb{D}^-) &\longrightarrow \mathcal{D}_0(\mathbb{D}^+) \\ h &\longmapsto P_0(\mathbb{D}^+)\mathcal{C}_f I_f h. \end{aligned}$$

As it is clear from the notations and the definitions, the maps I_f and Gr_f depend on f .

Remark 3.2. *Our definition of the Grunsky operator differs slightly from that in [20]. In that paper, the projection did not annihilate the constant as it does here.*

In [20, Theorems 3.7, 3.14] we showed the following facts.

Theorem 3.3. *Gr_f is a bounded operator and I_f is a bounded isomorphism.*

We register useful alternate expressions for I_f and Gr_f . For $h \in \mathcal{D}_\infty(\mathbb{D}^-)$,

$$(6) \quad I_f h(z) = - \lim_{r \nearrow 1} \frac{1}{2\pi\sqrt{-1}} \int_{f(\gamma_r)} \frac{(\mathbf{R}h) \circ f^{-1}(\zeta)}{\zeta - z} d\zeta.$$

This follows immediately from the observation that $(\mathbf{R}h) \circ f^{-1}$ is the harmonic extension of $\mathcal{C}_{f^{-1}}h$ to Ω^+ .

Remark 3.4. *It can be shown that if h is a polynomial in $1/z$, then $\mathbf{R}h$ can be replaced by h in the above expression [20, Theorem 3.4]. In particular, $I_f(z^{-n})$ is the n th Faber polynomial of the domain Ω^- . Although we will not need this result, it is closely related to the fact that I_f is an isomorphism.*

Theorem 3.5. *The Grunsky operator has the integral expression*

$$[\text{Gr}_f h](z) = \lim_{r \nearrow 1} \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_r} \left[\frac{z}{\zeta(\zeta - z)} - \frac{f'(\zeta)}{f(\zeta) - f(z)} + \frac{f'(\zeta)}{f(\zeta) - f(0)} \right] (\mathbf{R}h)(\zeta) d\zeta.$$

Proof. Let $P(\mathbb{D}^+)$ denote the projection onto $\mathcal{D}(\mathbb{D}^+)$, that is, the full Dirichlet space without the condition that the functions be constant at 0. In [20, Theorem 3.9], we showed that the operator $\check{\text{Gr}}_f = P(\mathbb{D}^+)\mathcal{C}_f I_f$ has the integral expression

$$\check{\text{Gr}}_f h(z) = \lim_{r \nearrow 1} \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_r} \left[\frac{1}{\zeta - z} - \frac{f'(\zeta)}{f(\zeta) - f(z)} \right] (\mathbf{R}h)(\zeta) d\zeta.$$

The claim now follows from the facts that $\text{Gr}_f h(z) = \check{\text{Gr}}_f h(z) - \check{\text{Gr}}_f h(0)$ and

$$\check{\text{Gr}}_f h(0) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_r} \left[\frac{1}{\zeta} - \frac{f'(\zeta)}{f(\zeta) - f(0)} \right] (\mathbf{R}h)(\zeta) d\zeta.$$

□

3.2. Isomorphism onto Dirichlet spaces of multiply-connected domains. In this section we generalize the isomorphism $\mathbf{I}_f$ to multiply-connected domains. Let Σ be a domain in $\overline{\mathbb{C}}$, bordered by non-intersecting quasicircles Γ_i , $i = 1, \dots, n$, such that $\infty \in \Sigma$. Let Ω_i^+ and Ω_i^- denote the bounded and unbounded components of $\overline{\mathbb{C}} \setminus \Gamma_i$ respectively. Let

$$\mathcal{D}_\infty(\Sigma) = \left\{ h : \Sigma \rightarrow \mathbb{C} : h \text{ holomorphic, } h(\infty) = 0, \text{ and } \iint_\Sigma |h'|^2 dA < \infty \right\}$$

denote the Dirichlet space of Σ . Let

$$\mathcal{D}_\infty(\mathbb{D}^-)^n = \mathcal{D}_\infty(\mathbb{D}^-) \oplus \dots \oplus \mathcal{D}_\infty(\mathbb{D}^-)$$

where there are n summands on the right hand side. Given an n -tuple $f = (f_1, \dots, f_n)$ of conformal maps $f_i : \mathbb{D}^+ \rightarrow \Omega_i^+$, we define an isomorphism

$$(7) \quad \begin{aligned} \mathbf{I}_f : \mathcal{D}_\infty(\mathbb{D}^-)^n &\longrightarrow \mathcal{D}_\infty(\Sigma) \\ (g_1, \dots, g_n) &\longmapsto \sum_{i=1}^n \mathbf{I}_{f_i} g_i|_\Sigma. \end{aligned}$$

To show that this is indeed a bounded isomorphism, we require the following theorems.

Theorem 3.6. *Let Σ be a multiply-connected domain in $\overline{\mathbb{C}}$ containing ∞ bounded by non-intersecting quasicircles Γ_i , $i = 1, \dots, n$. Any $h \in \mathcal{D}(\Sigma)$ has boundary values $\mathcal{H}(\Gamma_i)$ in the sense of Osborn for all $i = 1, \dots, n$.*

Remark 3.7. *For a multiply-connected domain of the type in Theorem 3.6, by “boundary values in the sense of Osborn” we mean the following. Let $h \in \mathcal{D}(\Sigma)$. Choose one boundary curve Γ_i and let Ω_i^- be the component of the complement containing Σ as above. For any fixed point $p \in \Omega_i^-$, let $\gamma_{p,q}$ be the hyperbolic geodesic ray beginning at p and terminating at $q \in \Gamma_i$. The limit of h exists as $z \rightarrow q$ along $\gamma_{p,q}$ for all q except possibly along a set K which is the image of a Borel set I of logarithmic capacity zero in $\mathbb{S}^1$ under a conformal map $f : \mathbb{D}^+ \rightarrow \Omega_i^-$. Note that the hyperbolic geodesic must eventually lie entirely in Σ so this makes sense. Furthermore, the boundary values in this sense are in $\mathcal{H}(\Gamma_i)$; i.e. there is a harmonic function H in $\mathcal{D}(\Omega_i^-)$ whose boundary values are the same as those of h up to a set of logarithmic capacity zero.*

Proof. It is enough to prove this in the case that $\Gamma_i = \mathbb{S}^1$ and Σ is an annulus $A_r = \{z : r < |z| < 1\}$ for $0 < r < 1$. To see this, choose a boundary curve Γ_i and a conformal map $f : \mathbb{D}^+ \rightarrow \Omega_i^-$. Choose r such that $f(A_r) \subset \Sigma$. Let H be the function in $\mathcal{D}(\mathbb{D}^+)$ whose boundary values agree with those of $h \circ f$, in the sense of Remark 3.7. Thus $H \circ f^{-1}$ is in $\mathcal{D}(\Omega_i^-)$ and has boundary values in $\mathcal{H}(\Gamma_i)$ agreeing with h .

Restricting therefore to the case of an annulus A_r , assume that $h \in \mathcal{D}(A_r)$. Let γ be the curve $|z| = s$ traced counterclockwise for some $s \in (r, 1)$. Let

$$G_\pm(z) = \pm \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{h(\zeta)}{\zeta - z} d\zeta$$

where the sign is chosen according to whether z is in the bounded or unbounded component of γ respectively. Since h is holomorphic in A_r , by deforming the curve we see that $G_{\pm}$ are holomorphic in $\mathbb{D}^{\pm}$ respectively. By the classical Plemelj-Sokhotski jump relation, on γ we have $h = G_+ + G_-$ (note that h , G_+ and G_- are all holomorphic on a neighbourhood of γ so equality holds literally and not just in a limiting sense). Thus $h = G_+ + G_-$ identically on A_r .

Since G_- is holomorphic in a neighbourhood of $\mathbb{D}^-$, it is in $\mathcal{D}(\mathbb{D}^-)$, and since it is continuous on $\mathbb{S}^1$ the limiting values exist everywhere in the sense of Osborn from within A_r . Fixing a $t \in (r, 1)$, since $G_+ = h - G_-$ and h and G_- are both in $\mathcal{D}(A_t)$, it follows that G_+ is in $\mathcal{D}(A_t)$ and therefore $G_+ \in \mathcal{D}(\mathbb{D}^+)$.

Since $h = G_+ + G_-$ on A_r and both G_+ and G_- have boundary values in $\mathcal{H}(\mathbb{S}^1)$ in the sense of Osborn, this completes the proof. $\square$

Theorem 3.8. *Let Σ be a multiply-connected domain in $\overline{\mathbb{C}}$ containing ∞ bounded by non-intersecting quasicircles Γ_i , $i = 1, \dots, n$. Let Ω_i^+ and Ω_i^- be the bounded and unbounded components respectively of Γ_i . Every element $h \in \mathcal{D}_{\infty}(\Sigma)$ has a unique representation*

$$h = h_1 + \dots + h_n, \quad h_i \in \mathcal{D}_{\infty}(\Omega_i^-),$$

where $h_i = P_{\infty}(\Omega_i^-)h$ (that is, we apply $P_{\infty}(\Omega_i^-)$ to the boundary values of h).

Proof. Let $g_i : \mathbb{D}^- \rightarrow \Omega_i^-$ be conformal maps such that $g_i(\infty) = \infty$. For any fixed $z \in \Sigma$, there is an $R > 1$ close enough to 1 so that z is in the unbounded component of the complement of $g_i(\gamma_R)$ for all $i = 1, \dots, n$, and furthermore that $g_i(\gamma_r)$ is in Σ for $1 < r \leq R$. Define

$$H_i(z) = -\lim_{r \searrow 1} \frac{1}{2\pi\sqrt{-1}} \int_{g_i(\gamma_r)} \frac{h(\zeta)}{\zeta - z} d\zeta$$

where $z \in \Omega_i^-$, so that H_i is holomorphic in Ω_i^- . Observe that the integral is eventually independent of r .

We first claim that $H_i = P_{\infty}(\Omega_i^-)h$. First, observe that by the Cauchy integral formula $h = H_1 + \dots + H_n$ in Σ . For any fixed i , $h - H_i = \sum_{k \neq i} H_k$ extends to a holomorphic function in Ω_i^+ . This is because every term on the right hand side is a holomorphic function in Ω_j^- for some $j \neq i$, and thus is holomorphic on Ω_i^+ since $\Omega_i^+ \subset \Omega_j^-$ for all $j \neq i$. Also note that since the closure of Ω_i^+ is compactly contained in Ω_j^- , the extension of $h - H_i$ is in $\mathcal{D}(\Omega_i^+)$. In particular, the boundary values of $h - H_i$ are in $\mathcal{H}(\Gamma_i)$. By Theorem 3.6, h has a restriction to the boundary in $\mathcal{H}(\Gamma_i)$. Thus $H_i = h - (h - H_i)$ has a restriction to the boundary in $\mathcal{H}(\Gamma_i)$.

Thus we may apply $P_{\infty}(\Omega_i^-)$ to the boundary values of both h and H_i . Since $h - H_i$ extends to a function in $\mathcal{D}(\Omega_i^+)$, by [19, Theorem 3.10], $P_{\infty}(\Omega_i^-)[h - H_i] = 0$ (where by this we mean of course the application of the projection to the boundary values of $h - H_i$). Again applying [19, Theorem 3.10] we obtain that $P(\Omega_i^-)H_i = H_i$, which proves that $P(\Omega_i^-)h = H_i$ as claimed.

Now choose s such that $1 < s < R$. By the Cauchy integral formula, since h is holomorphic on Σ ,

$$\begin{aligned} h(z) &= -\sum_{i=1}^n \frac{1}{2\pi\sqrt{-1}} \int_{g_i(\gamma_s)} \frac{h(\zeta)}{\zeta - z} d\zeta \\ &= -\sum_{i=1}^n \lim_{r \searrow 1} \frac{1}{2\pi\sqrt{-1}} \int_{g_i(\gamma_r)} \frac{h(\zeta)}{\zeta - z} d\zeta \\ &= \sum_{i=1}^n [P_\infty(\Omega_i^-)h](z) \end{aligned}$$

where in the last step we have used Theorem 2.6. Now defining $h_i = P_\infty(\Omega_i^-)h$, completes the proof of the existence of the representation.

To show that the representation is unique, let $u_1 + \dots + u_n = h$ be another representation with $u_i \in \mathcal{D}_\infty(\Omega_i^-)$ for each i . We then have that

$$-h_1 + u_1 = h_2 - u_2 + \dots + h_n - u_n.$$

Now the left hand side is holomorphic in Ω_1^- , while the right hand side is holomorphic in $\Omega_2^- \cap \dots \cap \Omega_n^-$, which contains the closure of Ω_1^+ . Since the intersection of the domains on the left and the right hand sides is non-empty (it is in fact Σ), this shows that $h_1 - u_1$ has a holomorphic extension to $\overline{\mathbb{C}}$ and therefore is a constant. Since h_1 and u_1 both vanish at ∞ , $h_1 = u_1$. Continuing in this way, we can show that $h_i = u_i$ for all $i = 1, \dots, n$. $\square$

Given this theorem, we can now prove the following.

Theorem 3.9. *Let Σ , Γ_i , Ω_i^+ , and Ω_i^- be as in Theorem 3.8. Let $f = (f_1, \dots, f_n)$ where $f_i : \mathbb{D}^+ \rightarrow \Omega_i^+$ is a conformal map for $i = 1, \dots, n$. The map $\mathbf{I}_f$ is a bounded isomorphism.*

Proof. For any $h = (h_1, \dots, h_n) \in \mathcal{D}_\infty(\mathbb{D}^-)^n$, since $\Sigma \subset \Omega_i^-$ for all i , using Minkowski's inequality

$$\|\mathbf{I}_f h\|_{\mathcal{D}_\infty(\Sigma)} \leq \sum_{i=1}^n \|\mathbf{I}_{f_i} h_i\|_{\mathcal{D}_\infty(\Sigma)} = \sum_{i=1}^n \left(\iint_{\Sigma} |(\mathbf{I}_{f_i} h_i)'|^2 dA \right)^{1/2} \leq \sum_{i=1}^n \|\mathbf{I}_{f_i} h_i\|_{\mathcal{D}_\infty(\Omega_i^-)}.$$

Since $\mathbf{I}_{f_i}$ is bounded for each i by Theorem 3.3, this proves that $\mathbf{I}_f$ is bounded. By Theorems 3.3 and 3.8, $\mathbf{I}_f$ is a bijection, so by the open mapping theorem, $\mathbf{I}_f$ is a bounded isomorphism. $\square$

We conclude the section with two remarks to place the results in context.

Remark 3.10. *In the statement of Theorem 3.8, the projection operators $P_\infty(\Omega_i^-)$ are defined using Cauchy integrals of the harmonic extension of $h|_{\Gamma_i}$ to the complement Ω_i^+ of Ω_i^- . If we define the projection operator using integrals along curves inside $\Sigma \subset \Omega_i^-$, then the decomposition exists for much wider classes of functions, see e.g. [5, Theorem 10.12] for the case of Hardy spaces.*

However, the definition of Faber operator uses a pull-back under conformal maps onto the complements Ω_i^+ . Thus we need to use the fact that the projection using Cauchy integrals over curves approaching the boundary from the outside of Σ has the same result as the projection defined using curves approaching from the outside. This makes the use of Theorem 2.6

unavoidable. Furthermore, this theorem cannot be extended to more general function spaces such as Hardy spaces (see the introduction).

Remark 3.11. The proofs of Theorems 3.8 and 3.9 also require the fact that Γ is a quasicircle. This is because the statement and proof of Theorem 2.6 in turn requires the fact that any Dirichlet bounded function on Ω_i^- has a reflection, i.e. a Dirichlet-bounded harmonic function on Ω_i^+ with the same boundary values. That this is true for quasicircles is not entirely obvious, and was obtained in [19, Theorem 3.5]; recall also that for Dirichlet spaces this holds if and only if the curve is a quasicircle [20].

3.3. Multiple Faber series. As an application, we show that every function h in $\mathcal{D}_\infty(\Sigma)$ has a unique multiple Faber series. This polynomial approximation converges uniformly on compact subsets of Σ . This application also illustrates the meaning of the isomorphism $\mathbf{I}_f$.

Let Σ , $\Omega^\pm$, and $f_k : \mathbb{D}^+ \rightarrow \Omega_k^+$ be as in the previous section. There are many equivalent definitions of Faber polynomials, see [14, 23]. In our setting, for integers $m > 0$ we define the m th Faber polynomial of f_k by

$$(8) \quad \Phi_m^k(w) = -\frac{1}{2\pi\sqrt{-1}} \int_{f_k(\gamma_r)} \frac{(f_k^{-1}(\zeta))^{-m}}{\zeta - z} d\zeta$$

where γ_r is the circle $|\zeta| = r$ traced counterclockwise for some fixed $r \in (0, 1)$ such that z is in the unbounded component of the complement of $f(\gamma_r)$.

Let $p_i = f(0)$. $\Phi_m^k(z)$ is a polynomial of degree m in $1/(z - p_i)$. To see this, expand $f_k^{-1}(\zeta)^{-m}$ in a Laurent series

$$(9) \quad f_k^{-1}(\zeta)^{-m} = \frac{c_{-m}}{(\zeta - p_i)^m} + \cdots + \frac{c_1}{\zeta - p_i} + c_0 + c_{-1}(\zeta - p_i) + \cdots$$

in $0 < |\zeta - p_i| < \delta$ for $\delta > 0$ sufficiently small, and deform the curve $f_k(\gamma_r)$ to $|\zeta - p_i| = \delta/2$. We then see that Φ_m^k is the principle part of (9). By [20, Theorem 3.4], we have

$$\Phi_m^k = I_{f_k}(z^{-m})$$

(note that by z^{-m} we mean the function $z \mapsto z^{-m} \in \mathcal{D}_\infty(\mathbb{D}^-)$).

We define a multiple Faber series of a holomorphic function h on Σ to be a series of the form

$$h(z) = \sum_{m=1}^{\infty} \sum_{k=1}^n a_m^k \Phi_m^k(z)$$

for complex constants a_m^k , converging uniformly on compact subsets of Σ . We then have the following.

Theorem 3.12. Let Σ be a multiply-connected domain in $\overline{\mathbb{C}}$ containing ∞ bounded by non-intersecting quasicircles Γ_i , $i = 1, \dots, n$. Let Ω_i^+ and Ω_i^- be the bounded and unbounded components respectively of Γ_i . Let $f_i : \mathbb{D}^+ \rightarrow \Omega_i^+$ be conformal maps for $i = 1, \dots, n$. Then every $h \in \mathcal{D}(\Sigma)$ has a unique Faber series expansion

$$h(z) = \sum_{m=1}^{\infty} \sum_{k=1}^n a_m^k \Phi_m^k(z)$$

which converges uniformly on compact subsets of Σ . The coefficients a_m^k are the coefficients of the expansion

$$g_k(z) = \sum_{m=1}^{\infty} a_m^k z^{-m}$$

where

$$(g_1, \dots, g_n) = \mathbf{I}_f^{-1}(h).$$

Proof. Let $h = h_1 + \dots + h_n$ be the decomposition obtained in Theorem 3.8. Applying [20, Theorem 3.17] to each $h_k \in \mathcal{D}(\Omega_k^-)$, we see that each h_k has a unique Faber series

$$h_k(z) = \sum_{m=1}^{\infty} a_m^k \Phi_m^k(z)$$

which converges uniformly on compact subsets of Ω_i^+ . Setting $g_k(z) = \sum_{m=1}^{\infty} a_m^k z^{-m}$ we have $I_{f_k}(g_k) = h_k$ and $\mathbf{I}_f(g_1, \dots, g_n) = h_1 + \dots + h_n = h$. Since $\mathbf{I}_f$ is an isomorphism by Theorem 3.9, this proves the claim. $\square$

4. DIRICHLET SPACES AS GRAPHS OF GRUNSKY OPERATORS

4.1. The Dirichlet space of a multiply-connected domain as the graph of a Grunsky operator. In this section we describe the Dirichlet space of Σ as the graph of a certain generalized Grunsky operator. We first show this for Σ simply-connected, and then extend to the general case.

Let Γ be a quasicircle not containing ∞ , and let Ω^+ and Ω^- be the bounded and unbounded components of the complement respectively. Let $f : \mathbb{D}^+ \rightarrow \Omega^+$ be a fixed conformal map. Any element $h \in \mathcal{H}(\Gamma)$ satisfies $\mathcal{C}_f h \in \mathcal{H}(\mathbb{S}^1)$ and therefore has a Fourier series

$$h \circ f(e^{\sqrt{-1}\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{\sqrt{-1}n\theta}$$

which converges to $h \circ f$ almost everywhere (in fact, except on a closed set of logarithmic capacity zero). We will ignore the constant terms. Define

$$\mathcal{H}_0(\mathbb{S}^1) = \left\{ h(e^{\sqrt{-1}\theta}) = \sum_{n=-\infty, n \neq 0}^{\infty} a_n e^{\sqrt{-1}n\theta} \in \mathcal{H}(\mathbb{S}^1) \right\}.$$

Note that this has the decomposition $\mathcal{H}_0(\mathbb{S}^1) = \mathcal{D}_{\infty}(\mathbb{D}^-) \oplus \mathcal{D}_0(\mathbb{D}^+)$ obtained by replacing $e^{\sqrt{-1}\theta}$ by z .

Define

$$\hat{\mathcal{C}}_f h = h \circ f - h \circ f(0).$$

Note that with this definition, we have $\mathbf{I}_f = \mathbf{P}_{\infty}(\Omega^-) \hat{\mathcal{C}}_{f^{-1}}$ and $\text{Gr}_f = \mathbf{P}_{\infty}(\Omega^-) \hat{\mathcal{C}}_f \mathbf{I}_f$. We will use these facts without comment ahead.

We pose the following question:

Question (1): Which Fourier series in $\mathcal{H}_0(\mathbb{S}^1)$ arise as $\hat{\mathcal{C}}_f h$ for some $h \in \mathcal{D}_\infty(\Omega^-)$? That is, how can one characterize the set

$$\mathcal{W} = \hat{\mathcal{C}}_f \mathcal{D}_\infty(\Omega^-)?$$

The answer to this question is a characterization of the set of boundary values of $\mathcal{D}(\Omega^-)$.

The following theorem answers this question.

Theorem 4.1. *Let Γ be a quasicircle not containing ∞ , and let Ω^+ and Ω^- be the bounded and unbounded components respectively. Let $f : \mathbb{D}^+ \rightarrow \Omega^+$ be a conformal map. Then*

$$P_\infty(\mathbb{D}^-) \hat{\mathcal{C}}_f I_f = \text{Id}$$

where $\text{Id} : \mathcal{D}_\infty(\mathbb{D}^-) \rightarrow \mathcal{D}_\infty(\mathbb{D}^-)$ is the identity on $\mathcal{D}(\mathbb{D}^-)$, and

$$P_0(\mathbb{D}^+) \hat{\mathcal{C}}_f I_f = \text{Gr}_f.$$

Thus $\mathcal{W}$ is the graph of Gr_f in $\mathcal{H}_0(\mathbb{S}^1) = \mathcal{D}_\infty(\mathbb{D}^-) \oplus \mathcal{D}_0(\mathbb{D}^+)$.

Proof. The first claim is [20, Equation 3.7]. The second claim is the definition of the Grunsky operator. The fact that $\mathcal{W}$ is the graph of Gr_f follows immediately. $\square$

This also shows that for any $h \in \mathcal{W}$, the element $H \in \mathcal{D}(\Omega^-)$ such that $\hat{\mathcal{C}}_f H = h$ is uniquely determined and is given by $H = I_f P_\infty(\mathbb{D}^-) h$.

Remark 4.2. *The constant term can be included if desired, by replacing $P_\infty(\mathbb{D}^-)$ with the Cauchy integral (which does not annihilate the constant term), and altering the definition of the Grunsky operator accordingly. The matrix of the new Grunsky operator would then contain the so-called logarithmic coefficients of f . Although the constant term is significant, it does not play an important role in this paper, and furthermore introduces some notational complications in the multiply-connected case. We omit the treatment here.*

We shall now proceed by defining a generalized Grunsky operator. First we define block entries of the operator and show that they are bounded.

Theorem 4.3. *Let Σ be a multiply-connected domain bounded by n quasicircles Γ_i , let Ω_i^+ and Ω_i^- be the bounded and unbounded components respectively. Let $f = (f_1, \dots, f_n)$ where $f_i : \mathbb{D}^+ \rightarrow \Omega_i^+$ are conformal maps. The operators*

$$(10) \quad \begin{aligned} \text{Gr}_{ji}(f) : \mathcal{D}_\infty(\mathbb{D}^-) &\longrightarrow \mathcal{D}_0(\mathbb{D}^+) \\ h &\longmapsto P_0(\mathbb{D}^+) \hat{\mathcal{C}}_{f_j} I_{f_i} h. \end{aligned}$$

are bounded for all $i, j = 1, \dots, n$.

Proof. By definition $\text{Gr}_{ii}(f) = \text{Gr}_{f_i}$. Thus by Theorem 3.3 it is bounded. Now assume that $i \neq j$. Since $P(\mathbb{D}^+)$ and I_{f_i} are bounded, it is enough to show that $\hat{\mathcal{C}}_{f_j}$ is a bounded operator on $\mathcal{D}_\infty(\Omega_i^-)$. Since $\Omega_j^+ \subseteq \Omega_i^-$ when $i \neq j$, for any $h \in \mathcal{D}_\infty(\Omega_i^-)$ we have that $\hat{\mathcal{C}}_{f_j} h \in \mathcal{D}_0(\mathbb{D}^+)$ and by a change of variables (setting $p_j = f_j(0)$)

$$\|\hat{\mathcal{C}}_{f_j} h\|_{\mathcal{D}_0(\mathbb{D}^+)} = \|h\|_{\mathcal{D}_{p_j}(\Omega_j^+)} \leq \|h\|_{\mathcal{D}_\infty(\Omega_i^-)}.$$

$\square$

Remark 4.4. *Since $f_j(\mathbb{D}^+) \subseteq \Omega_i^-$ whenever $i \neq j$, the projection $P_0(\mathbb{D}^+)$ can be removed when $i \neq j$.*

Theorem 4.5. Let Γ_i , Ω_i^+ , Ω_i^- and f_i be as in Theorem 4.3 for $i = 1, \dots, n$. If $i \neq j$, then for $h \in \mathcal{D}_\infty(\mathbb{D}^-)$, $\text{Gr}_{ji}(f)$ has the integral expression

$$[\text{Gr}_{ji}(f)] h(z) = \frac{1}{\pi} \iint_{\mathbb{D}^+} \left(\frac{f'_i(\zeta)}{f_i(\zeta) - f_j(z)} - \frac{f'_i(\zeta)}{f_i(\zeta) - f_j(0)} \right) \mathbf{R}h(\zeta) dA(\zeta).$$

Observe that the integral kernel is non-singular.

Proof. Using equation (6) we have that up to the constant term the integral expression for $\text{Gr}_{ji}(f)h$ is

$$\begin{aligned} -\lim_{r \nearrow 1} \int_{f_i(\gamma_r)} \frac{(\mathbf{R}h) \circ f_i^{-1}(\zeta)}{\zeta - f_j(z)} d\zeta &= -\lim_{r \nearrow 1} \int_{\gamma_r} \frac{\mathbf{R}h(\zeta)}{f_i(\zeta) - f_j(z)} f'_i(\zeta) d\zeta \\ &= \frac{1}{\pi} \iint_{\mathbb{D}^+} \frac{\mathbf{R}h(\zeta)}{f_i(\zeta) - f_j(z)} f'_i(\zeta) dA(\zeta). \end{aligned}$$

The claim follows by subtracting the value of this integral at 0. $\square$

Generalized Grunsky operators for pairs of non-overlapping maps were considered by J. A. Hummel [8], who derived Grunsky inequalities in matrix form for such maps (the function class was formulated differently but can be seen to be the same by applying a transform and an identity for the Grunsky matrix). They were also considered in Takhtajan and Teo [24] in the equivalent L^2 setting, when the complement of the maps is of measure zero.

We can now define a generalized Grunsky operator. Let $\mathcal{D}_0(\mathbb{D}^+)^n = \mathcal{D}_0(\mathbb{D}^+) \oplus \dots \oplus \mathcal{D}_0(\mathbb{D}^+)$ where there are n summands.

Definition 4.6. For Σ , Ω_i^+ , Ω_i^- and $f = (f_1, \dots, f_n)$ as above, the generalized Grunsky operator is

$$\begin{aligned} \mathbf{Gr}(f) : \mathcal{D}_\infty(\mathbb{D}^-)^n &\longrightarrow \mathcal{D}_0(\mathbb{D}^+)^n \\ (H_1, \dots, H_n) &\longmapsto \left(\sum_{i=1}^n \text{Gr}_{1i} H_i, \dots, \sum_{i=1}^n \text{Gr}_{ni} H_i \right). \end{aligned}$$

It follows immediately from Theorem 4.3 that $\mathbf{Gr}(f)$ is bounded.

We now derive a useful identity relating $\mathbf{I}_f$ and $\mathbf{Gr}(f)$.

Theorem 4.7. Let Σ , Ω_i^+ , Ω_i^- and $f = (f_1, \dots, f_n)$ be as in Theorem 4.3. Then for any $H_1, \dots, H_n \in \mathcal{D}_\infty(\mathbb{D}^-)^n$ we have the identity

$$(11) \quad \|\mathbf{I}_f(H_1, \dots, H_n)\|_{\mathcal{D}(\Sigma)}^2 = \sum_{i=1}^n \|H_i\|_{\mathcal{D}_\infty(\mathbb{D}^-)}^2 - \sum_{i=1}^n \left\| \sum_{j=1}^n \text{Gr}(f)_{ij} H_j \right\|_{\mathcal{D}_0(\mathbb{D}^+)}^2.$$

Proof. Choose H_j to be polynomials in $1/z$ with zero constant term for $j = 1, \dots, n$. Let $(H_1, \dots, H_n) \in \mathcal{D}_\infty(\mathbb{D}^-)^n$, so that

$$\mathbf{I}_f(H_1, \dots, H_n) = \sum_{j=1}^n \mathbf{I}_{f_j} H_j.$$

For $r \in (0, 1)$ let γ_r denote the curve $|z| = r$ traced counter-clockwise. Then applying Green's identity (noting that the curves $f_i(\gamma_r)$ are negatively oriented with respect to Σ)

$$\begin{aligned} \|\mathbf{I}_f(H_1, \dots, H_n)\|_{\mathcal{D}(\Sigma)}^2 &= -\lim_{r \nearrow 1} \frac{1}{2\sqrt{-1}} \sum_{i=1}^n \int_{f_i(\gamma_r)} \left(\sum_{k=1}^n \mathbf{I}_{f_k} H_k(w) \right)' \cdot \overline{\left(\sum_{k=1}^n \mathbf{I}_{f_k} H_k(w) \right)} dw \\ &= -\lim_{r \nearrow 1} \frac{1}{2\sqrt{-1}} \sum_{i=1}^n \int_{\gamma_r} \left(\sum_{k=1}^n \hat{\mathcal{C}}_{f_i} \mathbf{I}_{f_k} H_k(z) \right)' \cdot \overline{\left(\sum_{k=1}^n \hat{\mathcal{C}}_{f_i} \mathbf{I}_{f_k} H_k(z) \right)} dz \end{aligned}$$

Here equality holds because quasicircles have measure zero [11]. Also, we can use $\hat{\mathcal{C}}_{f_i}$ in place of $\mathcal{C}_{f_i}$ in the change of variables because constants vanish in the integral. To see this, first observe that in the left factor, differentiation removes the constants. Second, we observe that since there are no simple poles in the left factor, the constants in the right factor will not contribute to the integral.

Now applying Theorem 4.1, we obtain

$$\begin{aligned} \|\mathbf{I}_f(H_1, \dots, H_n)\|_{\mathcal{D}(\Sigma)}^2 &= -\lim_{r \nearrow 1} \frac{1}{2\sqrt{-1}} \sum_{i=1}^n \int_{\gamma_r} \left(H_i(z) + \sum_{j=1}^n \text{Gr}(f)_{ij} H_j(z) \right)' \\ &\quad \cdot \overline{\left(H_i(z) + \sum_{j=1}^n \text{Gr}(f)_{ij} H_j(z) \right)} dz \\ &= -\lim_{r \nearrow 1} \frac{1}{2\sqrt{-1}} \sum_{i=1}^n \int_{\gamma_r} H_i'(z) \overline{H_i(z)} dz \\ &\quad - \lim_{r \nearrow 1} \frac{1}{2\sqrt{-1}} \sum_{i=1}^n \int_{\gamma_r} \left(\sum_{j=1}^n \text{Gr}(f)_{ij} H_j(z) \right)' \cdot \overline{\left(\sum_{j=1}^n \text{Gr}(f)_{ij} H_j(z) \right)} dz \end{aligned}$$

Here we have used the fact that cross-terms vanish. To see this, for example observe that for any j , there are constants c_l such that

$$\text{Gr}(f)_{ij} H_j(z) = \sum_{l=1}^{\infty} c_l z^l$$

and for fixed r we have (for some m)

$$\overline{H_i(z)} = b_1 \bar{z}^{-1} + \dots + b_m \bar{z}^{-m} = b_1 \frac{z}{r^2} + \dots + b_m \frac{z^m}{r^{2m}}$$

so

$$\int_{\gamma_r} \text{Gr}(f)_{ij} H_j(z) \cdot \overline{H_i(z)} dz = 0$$

for all j . The other cross-terms vanish similarly.

Another computation using the same identity $z\bar{z} = r^2$ yields

$$-\lim_{r \nearrow 1} \frac{1}{2\sqrt{-1}} \sum_{i=1}^n \int_{\gamma_r} H_i'(z) \overline{H_i(z)} dz = \|H_i\|_{\mathcal{D}_{\infty}(\mathbb{D}^-)}^2.$$

Thus applying Green's identity to the remaining terms involving $\text{Gr}(f)_{ij}$ and $\text{Gr}(f)_{ik}$ yields the identity in the case of polynomials. Since polynomials are dense in $\mathcal{D}_\infty(\mathbb{D}^-)$, this completes the proof. $\square$

The following result generalize the classical result of C. Pommerenke [14] for quasicircles in one direction.

Corollary 4.8. *With Σ , Ω_i^+ , Ω_i^- and $f = (f_1, \dots, f_n)$ as in Theorem 4.3,*

$$\|\mathbf{Gr}(f)\|_{\mathcal{D}_\infty(\mathbb{D}^-)^n \rightarrow \mathcal{D}_0(\mathbb{D}^+)^n} < 1.$$

Proof. Let $(H_1, \dots, H_n) \in \mathcal{D}_0(\mathbb{D}^-)^n$. The previous theorem immediately yields that

$$\sum_{i=1}^n \left\| \sum_{j=1}^n \text{Gr}(f)_{ik} H_k \right\|_{\mathcal{D}_0(\mathbb{D}^+)}^2 \leq \sum_{i=1}^n \|H_i\|_{\mathcal{D}_\infty(\mathbb{D}^-)}^2.$$

To improve this to strict inequality, observe that since $\mathbf{I}_f$ is an isomorphism, there is a $c \in (0, 1)$ such that

$$(1 - c) \left(\sum_{i=1}^n \|H_i\|_{\mathcal{D}_\infty(\mathbb{D}^-)}^2 \right) \leq \|\mathbf{I}_f(H_1, \dots, H_n)\|_{\mathcal{D}(\Sigma)}^2$$

independently of $(H_1, \dots, H_n)$. Inserting this in (11) completes the proof. $\square$

Remark 4.9. *It is not hard to see that the inequality continues to hold (in some sense) in the case that the mappings f_i are not necessarily quasiconformally extendible, though it no longer need be strict. This can be shown by weakening equality in the identity (11) to an inequality, and writing the norms in terms of coefficients. In the case of one mapping, this yields of course the classical Grunsky inequalities. The case of two mappings was given by Hummel [8] as mentioned above (with rather different notation than that we use here).*

Now define

$$\begin{aligned} \hat{\mathcal{C}}_f : \mathcal{D}_\infty(\Sigma) &\longrightarrow \mathcal{H}_0(\mathbb{S}^1)^n \\ H &\longmapsto (\hat{\mathcal{C}}_{f_1} H, \dots, \hat{\mathcal{C}}_{f_n} H) \end{aligned}$$

and

$$\mathbf{P}(\mathbb{D}^+) = \mathbf{P}_0(\mathbb{D}^+) \oplus \dots \oplus \mathbf{P}_0(\mathbb{D}^+)$$

and similarly

$$\mathbf{P}(\mathbb{D}^-) = \mathbf{P}_\infty(\mathbb{D}^-) \oplus \dots \oplus \mathbf{P}_\infty(\mathbb{D}^-).$$

Finally let

$$\mathbf{W} = \hat{\mathcal{C}}_f \mathcal{D}_\infty(\Sigma) = \{(h_1, \dots, h_n) \in \mathcal{H}_0(\mathbb{S}^1)^n : h_i = \hat{\mathcal{C}}_{f_i} H \text{ for some } H \in \mathcal{D}_\infty(\Sigma)\}.$$

We now ask the following question:

Question (2): Which Fourier series in $\bigoplus^n \mathcal{H}_0(\mathbb{S}^1)$ arise as $\hat{\mathcal{C}}_f h$ for some $h \in \mathcal{D}_\infty(\Sigma)$? That is, how can one characterize the set

$$\mathbf{W} = \hat{\mathcal{C}}_f \mathcal{D}_\infty(\Sigma)?$$

We now show that $\mathbf{W}$ is the graph of the generalized Grunsky operator.

Theorem 4.10. Let Σ , Ω_i^+ , Ω_i^- and $f = (f_1, \dots, f_n)$ be as in Theorem 4.3. We have that

$$\mathbf{P}(\mathbb{D}^-)\hat{\mathcal{C}}_f\mathbf{I}_f = \mathbf{Id}$$

where $\mathbf{Id} : \mathcal{D}_\infty(\mathbb{D}^-)^n \mapsto \mathcal{D}_\infty(\mathbb{D}^-)^n$ is the identity map and

$$\mathbf{P}(\mathbb{D}^+)\hat{\mathcal{C}}_f\mathbf{I}_f = \mathbf{Gr}(f).$$

Thus $\mathbf{W}$ is the graph of $\mathbf{Gr}(f)$.

Proof. As observed in the proof of Theorem 4.3, if $i \neq j$ then $\Omega_j^+ \subseteq \Omega_i^-$. Thus for all $h \in \mathcal{D}_\infty(\mathbb{D}^-)$, $\hat{\mathcal{C}}_{f_j}\mathbf{I}_{f_i}h \in \mathcal{D}_0(\mathbb{D}^+)$, and so $\mathbf{P}_\infty(\mathbb{D}^-)\hat{\mathcal{C}}_{f_j}\mathbf{I}_{f_i}h = 0$. Thus to prove the first claim it suffices to show that $\mathbf{P}_\infty(\mathbb{D}^-)\hat{\mathcal{C}}_{f_i}\mathbf{I}_{f_i}$ is the identity on $\mathcal{D}_\infty(\mathbb{D}^-)$. This follows from Theorem 4.1, and the first claim is thereby proven. The second claim is the definition of $\mathbf{Gr}(f)$. $\square$

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