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Shear deformable plate elements based on exact elasticity solution

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Abstract

The 2-D approximation functions based on a general exact 3-D plate solution are used to derive locking-free, rectangular, 4-node Mindlin (i.e., first-order plate theory), Levinson (i.e., a third-order plate theory), and Full Interior plate finite elements. The general plate solution is defined by a biharmonic mid-surface function, which is chosen for the thick plate elements to be the same polynomial as used in the formulation of the well-known nonconforming thin Kirchhoff plate element. The displacement approximation that stems from the biharmonic polynomial satisfies the static equilibrium equations of the 2-D plate theories at hand, the 3-D Navier equations of elasticity, and the Kirchhoff constraints. Weak form Galerkin method is used for the development of the finite element model, and the matrices for linear bending, buckling and dynamic analyses are obtained through analytical integration. In linear buckling problems, the 2-D Full Interior and Levinson plates perform particularly well when compared to 3-D elasticity solutions. Natural frequencies obtained suggest that the optimal value of the shear correction factor of the Mindlin plate theory depends primarily on the boundary conditions imposed on the transverse deflection of the 3-D plate used to calibrate the shear correction factor.

Keywords: Interior plate, Boundary layer, Galerkin's method, Finite element, Eigenvalues

1. Introduction

Shear deformation plate theories provide dimensionally reduced models for the structural analysis of flat solid bodies that may be moderately thick. The simplest among these theories is usually attributed to Mindlin and assumes that the transverse shear stresses are constant throughout the plate thickness [1–4]. The discrepancy between the predicted and actual shear behavior is corrected with an extrinsic shear correction factor. The Mindlin plate theory has survived decades of engineering practice showing that it is easy to use and gives accurate results for a wide range of real-life problems. The paper by Hrabok and Hrudey [5] delivers an overview on the early finite element developments that first extended the application of the Mindlin and classical Kirchhoff plate theories to modern, geometrically complex engineering structures. The Mindlin plate or, more aptly, shell finite elements that are nowadays used by swarms of engineers through software like Abaqus, Ansys and LS-DYNA are largely based on the works of Hughes et al. [6–10] and Belytschko et al. [11–14]. Another branch of shell elements that can be found, for example, in Adina, are the MITC elements by Bathe et al. [15–18].

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Regardless of the sweeping success of the Mindlin plate theory, which is also known as the first-order shear deformation plate theory (FSDT) due to a linear displacement variation through the plate thickness, it is sometimes necessary to use a plate theory that describes more accurately the actual plate displacements, strains and stresses. To this end, the third-order shear deformation plate theory (TSDT) by Reddy [19, 20] offers an alternative to the FSDT, for example, when the interlaminar stresses of a composite plate are of interest, by accommodating quadratic variations of the transverse shear stresses with respect to the plate thickness coordinate. Moreover, the TSDT does not require a shear correction factor, the determination of which can be cumbersome for composite plates, in particular. However, the total differential order of the governing equations of the TSDT is higher than that of the FSDT causing the analysis of the TSDT to be more laborious. In terms of finite elements, a typical four-node plate element based on the FSDT has three degrees of freedom at each node, whereas an element founded on the TSDT has five degrees of freedom (four rotations) at each node requiring ultimately considerably more computational effort.

We find that it would be convenient to have a plate model that both carries the benefits of Reddy’s TSDT, and retains the simple mathematical structure of Mindlin’s FSDT. It is worth noting that if the latter feature is achieved, many of the analytical and numerical methods applicable in the context of the Mindlin plate theory, including recent isogeometric developments [21–26], could be applied with little additional effort to the new plate model. Furthermore, the inclusion of the new model into commercial finite element software could be carried out following the footsteps of current implementations. We recently showed that it is possible to develop, within a well-defined interior framework, such plate models that combine the pros of the FSDT and TSDT [27]. We found that the Mindlin plate theory is in fact a special case of a general interior elasticity solution for a linearly elastic three-dimensional plate. Moreover, the solution includes third-order models, namely, Levinson and Full Interior plates, as other special cases. The equilibrium equations in terms of stress resultants are of the same form for all three theories and they do not include any higher-order stress resultants kin to those in the TSDT. In the present paper, we develop rectangular finite elements for the shear deformable plate models that are included in the 3-D interior elasticity solution.

In the remaining sections, we first present a brief overview on the interior framework for plates on the basis of our earlier works on the topic [27, 28]. This is followed by the formulation of the rectangular finite elements for Mindlin, Levinson, and Full Interior plates by Galerkin’s method of weighted residuals. The shape functions are obtained from the 3-D elasticity solution and they are the same for each 2-D element. The stiffness and consistent mass and geometric stiffness matrices are attained in closed-form through analytical integration. The shape functions satisfy the Kirchhoff constraints exactly so that each derived element reduces to a Kirchhoff element when the plate thickness tends to zero. Finally, buckling and natural frequency eigenproblems are studied using the novel, locking-free rectangular plate elements, and the performance and accuracy of the 2-D Mindlin, Levinson and Full Interior elements against each other and against 3-D plate solutions are evaluated. For further understanding on shear deformable plate theories, the physical crux of the numerical studies is to take a detailed look at the meaning of the shear correction factor of the Mindlin plate. The value of this factor is shown to depend notably on the boundary conditions of the 3-D plate which is used to calibrate the factor.

Figure 1: Rectangular interior plate with stress-free top and bottom faces. The membrane and bending behavior are not coupled in the geometrically linear case and only the solution for interior bending is studied. Integration through the plate thickness reduces the 3-D presentation into the conventional form of 2-D plate theories (Sec. 2.2).

2. Overview on interior plates

Here, we first consider a plate with stress-free top and bottom faces and then focus on the interior stress state of the plate. The general 3-D solution to the interior problem is reviewed and presented in the conventional form of 2-D plate theories. We discuss the total potential energy of plates without boundary layers.

2.1. Starting point – Stress-free faces and the interior solution

Let us consider a three-dimensional linearly elastic, isotropic, homogeneous plate of constant thickness h in Cartesian xyz -coordinate system. The stress boundary conditions on the faces of the plate read

$$\sigma_z(x, y, \pm h/2) = \tau_{xz}(x, y, \pm h/2) = \tau_{yz}(x, y, \pm h/2) = 0. \quad (1)$$

The most general state of stress within this plate can be decomposed into three parts: (1) interior state, (2) shear state, (3) Papkovitch–Fadle state [29–31]. Detailed, general 3-D elasticity solutions for plates with stress-free faces which account for all these three states have been given by several authors [32–37]. It has been proven that both the shear and Papkovitch–Fadle states are predominantly related to edge effects [29]. Our focus will be on the interior bending state, also known as the “plate theory part” [29].

The rectangular, linearly elastic interior plate of interest to us is depicted in Fig. 1. The length and width of the plate are $2a$ and $2b$, respectively. The general interior bending solution in terms of displacements can be written as [36]

$$2G \cdot U_x(x, y, z) = -z \frac{\partial \Psi}{\partial x} - \frac{1}{4(1-\nu)} \left[h^2 z - 2(2-\nu) \frac{z^3}{3} \right] \nabla^2 \frac{\partial \Psi}{\partial x}, \quad (2)$$

$$2G \cdot U_y(x, y, z) = -z \frac{\partial \Psi}{\partial y} - \frac{1}{4(1-\nu)} \left[h^2 z - 2(2-\nu) \frac{z^3}{3} \right] \nabla^2 \frac{\partial \Psi}{\partial y}, \quad (3)$$

$$2G \cdot U_z(x, y, z) = \Psi + \frac{\nu z^2}{2(1-\nu)} \nabla^2 \Psi, \quad (4)$$

where G and ν are the shear modulus and Poisson ratio, respectively. In addition, we have

$$\nabla^4 \Psi(x, y) = 0 \quad \left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \quad (5)$$

Every term in the biharmonic mid-surface function $\Psi(x, y)$ (taken as a polynomial later) includes an arbitrary constant coefficient and these coefficients will correspond to the nodal degrees of freedom of the plate finite elements. The 3-D displacements U_x , U_y and U_z calculated from Eqs. (2)–(4) satisfy the 3-D Navier equations of elasticity.

At this point, the meaning of the “interior state” can be explained as follows. When all three parts of the general stress state are accounted for, boundary conditions at the outer edge of the boundary layer give rise to exponentially decaying edge effects. Once these edge effects have decayed entirely with distance from the edge, the interior solution prevails. In other words, the interior solution represents a plate section which has been cut out from a complete plate far enough from the actual lateral edge. The rationale for this description is well-embedded into the above solution – the third-order throughout-thickness displacement distributions of the interior plate are not suitable for the modeling of detailed boundary layer effects.

2.2. General 3-D solution in the form of 2-D plate theories

In order to present the 3-D solution (2)–(4) in the form of a 2-D plate theory, we define the transverse deflection and normal rotations on the mid-surface as

$$\begin{Bmatrix} u_z(x, y) \\ \phi_x(x, y) \\ \phi_y(x, y) \end{Bmatrix} \equiv \begin{Bmatrix} U_z(x, y, 0) \\ \frac{\partial U_x}{\partial z}(x, y, 0) \\ \frac{\partial U_y}{\partial z}(x, y, 0) \end{Bmatrix} = \frac{1}{2G} \begin{Bmatrix} \Psi \\ -\frac{\partial}{\partial x} \left[\Psi + \frac{h^2}{4(1-\nu)} \nabla^2 \Psi \right] \\ -\frac{\partial}{\partial y} \left[\Psi + \frac{h^2}{4(1-\nu)} \nabla^2 \Psi \right] \end{Bmatrix}, \quad (6)$$

respectively. Furthermore, we find the following key relations:

$$\phi_x = -\frac{\partial u_z}{\partial x} - \frac{h^2}{4(1-\nu)} \frac{\partial}{\partial x} \nabla^2 u_z, \quad \phi_y = -\frac{\partial u_z}{\partial y} - \frac{h^2}{4(1-\nu)} \frac{\partial}{\partial y} \nabla^2 u_z, \quad (7)$$

$$\frac{\partial \phi_x}{\partial y} = \frac{\partial \phi_y}{\partial x}, \quad \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} = -\nabla^2 u_z. \quad (8)$$

By using the mid-surface variables (6), the 3-D displacements (2)–(4) can be written as

$$U_x = z\phi_x - \frac{4z^3}{3h^2} \left(\phi_x + \frac{\partial u_z}{\partial x} \right) + \frac{\nu z^3}{6(1-\nu)} \frac{\partial}{\partial x} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \right), \quad (9)$$

$$U_y = z\phi_y - \frac{4z^3}{3h^2} \left(\phi_y + \frac{\partial u_z}{\partial y} \right) + \frac{\nu z^3}{6(1-\nu)} \frac{\partial}{\partial y} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \right), \quad (10)$$

$$U_z = u_z - \frac{\nu z^2}{2(1-\nu)} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \right). \quad (11)$$

The above expressions describe the kinematics of a *Full Interior* plate and are valid for any biharmonic $\Psi(x, y)$. If we neglect the Poisson effect ($\nu = 0$) in Eqs. (9)–(11), the displacement field is exactly of the same form as that used by Levinson [38] and Reddy [19]. Furthermore, if the remaining third-order contributions z^3 are neglected, the kinematic description for the Mindlin plate theory is obtained. Finally, in the limit $h \rightarrow 0$, relations (7) give $\phi_x = -\partial u_z / \partial x$ and $\phi_y = -\partial u_z / \partial y$, which bring us to the Kirchhoff plate theory. That is to say, Eqs. (7) show that the Kirchhoff constraints are satisfied in the limit $h \rightarrow 0$. We also note that by reducing displacements (9)–(11) to xz or yz plane and making the plane strain to plane stress conversion $\nu \rightarrow \nu / (1 + \nu)$, the exact interior displacement field for the bending of a narrow beam is obtained [39, 40].

The strain-displacement relations and constitutive equations read

$$\boldsymbol{\epsilon} = \mathbf{L}\mathbf{U}, \quad (12)$$

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}, \quad (13)$$

respectively, where

$$\mathbf{U} = \{U_x \quad U_y \quad U_z\}^T, \quad (14)$$

$$\boldsymbol{\epsilon} = \{\epsilon_x \quad \epsilon_y \quad \epsilon_z \quad \gamma_{yz} \quad \gamma_{xz} \quad \gamma_{xy}\}^T, \quad (15)$$

$$\boldsymbol{\sigma} = \{\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{yz} \quad \tau_{xz} \quad \tau_{xy}\}^T \quad (16)$$

and

$$\mathbf{L}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \hat{\lambda} & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \hat{\lambda} & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \hat{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix}, \quad (17)$$

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad E = 2G(1 + \nu) \quad \text{and} \quad \hat{\lambda} = \lambda + 2G. \quad (18)$$

The given 3-D constitutive relations (13) apply as such for the 2-D Full Interior plate presented in terms of the mid-surface variables, whereas for Levinson and Mindlin plates the usual plane stress relations with $\sigma_z = 0$ are employed. Once the stresses (13) have been calculated using the strains (12) associated with the displacements (9)–(11), the moments and shear forces for each theory are obtained from

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z \, dz, \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} dz. \quad (19)$$

The elasticity solution contains an implicit shear correction factor of $\kappa = 2/3$ for the Mindlin plate theory [27]. In Section 3, we modify the solution in the case of the Mindlin plate so that any shear correction factor can be used.

In order to derive consistent mass and geometric stiffness matrices for buckling and vibration problems by using the static bending solution, we need the 3-D equation of motion

$$\mathbf{L}^T \boldsymbol{\sigma} = \rho \frac{\partial^2 \mathbf{U}}{\partial t^2} - \mathbf{P}, \quad (20)$$

where ρ is the mass density and

$$\mathbf{P} = \{0 \quad 0 \quad P\}^T \quad \text{with} \quad P = \frac{\partial}{\partial x} \left(\sigma_x \frac{\partial U_z}{\partial x} + \tau_{xy} \frac{\partial U_z}{\partial y} \right) + \frac{\partial}{\partial y} \left(\sigma_y \frac{\partial U_z}{\partial y} + \tau_{xy} \frac{\partial U_z}{\partial x} \right). \quad (21)$$

Equation (20) can be derived using the principle of virtual displacements (dynamic case) and the von Kármán strains that consider small strains but moderate rotations [41]. A derivation with slightly different strain terms has been given by Wittrick [42]. In the following, we assume that the

stresses in P are constant and that we have $U_z = u_z$ in P so that the geometric stiffness matrix will be the same for all interior plate theories at hand. It will be shown in Section 4 that the removal of the second-order term from U_z of the Full Interior plate [Eqs. (11) and (21)] has little effect on the buckling loads. Moreover, all geometrical nonlinearities that originate from the von Kármán strains will be omitted in the stress resultants so that they reduce to their linear forms – only linear bending, vibration and buckling problems are of interest in this paper. Keeping the above assumptions in mind, we multiply the first two equations of motion by z and then integrate equations (20) through the plate thickness to obtain

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = \rho \int_{-h/2}^{h/2} z \frac{\partial^2 U_x}{\partial t^2} dz, \quad (22)$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = \rho \int_{-h/2}^{h/2} z \frac{\partial^2 U_y}{\partial t^2} dz, \quad (23)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p + N = \rho \int_{-h/2}^{h/2} \frac{\partial^2 U_z}{\partial t^2} dz, \quad (24)$$

where $p(x, y)$ describes distributed loads and constant uniform in-plane forces N_x^0 , N_y^0 and N_{xy}^0 are included in

$$N = \frac{\partial}{\partial x} \left(N_x^0 \frac{\partial u_z}{\partial x} + N_{xy}^0 \frac{\partial u_z}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_y^0 \frac{\partial u_z}{\partial y} + N_{xy}^0 \frac{\partial u_z}{\partial x} \right). \quad (25)$$

The integration through the plate thickness reduces the 3-D equations (20) to a form suitable for 2-D plate theories. The thickness integration leads to a situation where Saint Venant's principle is applied as pointwise stresses on the plate edges are replaced by *statically equivalent* stress resultants. Note that each stress resultant and inertia term is different in the thickness-integrated equations for each interior plate theory at hand. Despite the differences, the elasticity solution (6) satisfies both the exact 2-D Full Interior, and the approximate Levinson and Mindlin ($\kappa = 2/3$) equations *in the static case without the in-plane forces* [27]. Ultimately this means that the rectangular Mindlin, Levinson and Full Interior finite elements will share the same shape functions.

2.3. Boundary layers, and the lack thereof, in plate theories

With the general elasticity solution reduced into the conventional form of 2-D plate theories, we can revisit some known boundary layer results for Mindlin and Levinson plates. In their study on boundary layers and interior equations for plates, Nosier and Reddy [43] showed that the governing equations of the Mindlin and Levinson plate theories may be presented in terms of a second-order edge-zone equation and a fourth-order interior equation. Their potential function describing the edge-zone was defined as

$$\Phi = \frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}. \quad (26)$$

Hägglblad and Bathe used a similar definition to introduce boundary layer corrections to a Mindlin plate [44]. Edge effects in Mindlin plates have also been studied by Arnold and Falk [45, 46]. In the current isotropic case we have $\partial \phi_x / \partial y = \partial \phi_y / \partial x$ according to Eq. (8)₁ and, thus, the boundary layer vanishes along with the edge-zone potential ($\Phi = 0$). To underscore, the linearly elastic, isotropic, homogeneous Kirchhoff, Mindlin and Levinson plate theories are interior theories and they do not have boundary layers. The absence of a boundary layer has a significant effect on the total potential energy of a plate, as will be discussed next.

Figure 2: Rectangular plate consisting of an interior part and a boundary layer. When only the interior part is studied, the interior stresses (positive directions shown) act as surface tractions and do work on the lateral edges of the plate.

2.4. Interior surface tractions in the total potential energy

Let us consider the rectangular plate with a boundary layer shown in Fig. 2. As a rule of thumb in isotropic cases, the boundary layer is as thick as the plate itself – the thinner the plate is, the more confined the edge effects are. In an interior plate without a boundary layer, fully-developed interior stresses are active all-over the plate at hand, including the lateral plate edges, where they act as surface tractions as depicted on the right side in Fig. 2. These surface tractions contribute to the total potential energy of the plate which can be written for the interior plate in Fig. 1 as

$$\Pi = U - W_s = U - W_s^{(+a)} - W_s^{(-a)} - W_s^{(+b)} - W_s^{(-b)}, \quad (27)$$

where the strain energy stored in the interior plate is

$$U = \frac{1}{2} \int_V (\boldsymbol{\sigma}^T \boldsymbol{\epsilon}) dV \quad (28)$$

and the work W_s due to the interior stresses on the lateral edges of the plate consists of

$$\begin{aligned} W_s^{(\pm a)} &= \pm \int_{-b}^b \int_{-h/2}^{h/2} [\sigma_x U_x + \tau_{xy} U_y + \tau_{xz} U_z] (\pm a, y, z) dz dy, \\ W_s^{(\pm b)} &= \pm \int_{-a}^a \int_{-h/2}^{h/2} [\sigma_y U_y + \tau_{xy} U_x + \tau_{yz} U_z] (x, \pm b, z) dz dx. \end{aligned} \quad (29)$$

The work term W_s has interesting consequences.

- The conventional wisdom on beam and plate stiffness matrices states that they are always symmetric in linear cases by virtue of Betti's reciprocal theorem [47, 48]. However, this statement is generally based on analyses in which work W_s , in the form given above, is absent. For example, the stiffness matrix of a Full Interior circular plate element is in fact not symmetric, nevertheless, the general interior solution (2)–(4) from which the element is derived without making any assumptions satisfies the reciprocal theorem when W_s is used [28]. The stiffness matrix of an anisotropic interior beam element is not symmetric either [40]. In both cases [28, 40], it was also noted that each nodal load was associated with multiple nodal degrees of freedom in work W_s . For example, shear force Q_1 of the circular plate was not conjugate only to $u_{z,1}$ in the conventional manner ($Q_1 \cdot u_{z,1}$), but also to all the other degrees of freedom.

- According to Clapeyron's theorem, the strain energy stored in an elastic body is equal to one-half of the work done by the surface tractions and body forces if they were to move (slowly) through their respective displacements from an unstressed state to the state of equilibrium [48, 49]. In the present case, Clapeyron's theorem leads to $2U - W_s = 0$, which can be shown to be satisfied by any choice of the biharmonic function $\Psi(x, y)$ in Eqs. (2)–(4). Clapeyron's theorem serves as a simple validity check in the interior framework.
- In commonplace variational formulations of different higher-order plate theories [50], W_s is not taken into account and, subsequently, the total differential order of the resulting governing equations is higher than it should be, as higher-order stress resultants appear in the equilibrium equations. The inclusion of W_s into a variational formulation generates natural interior boundary conditions that eliminate the higher-order stress resultants, or equivalently, they rid us of artificial, exponentially decaying edge effects [27]. If W_s is included in the formulation of Reddy's TSDT [19], the end result is in fact Levinson's plate theory [38]. A variational formulation that leads to the Mindlin plate theory will not contain any higher-order stress resultants under any circumstances because of the simple form of the assumed displacement field that includes only a linear displacement variation through the plate thickness.

2.5. Boundary conditions in practice

In engineering problems, plates can be modeled to a good approximation by using only the interior solution when the plate is not too thick. An inherent discrepancy of this approach is that the boundary conditions along the edges of a 2-D interior plate may be chosen only so as to imitate true, pointwise boundary conditions of a 3-D problem. If a proper boundary layer is desired, the elasticity solution for the linearly elastic, isotropic, homogeneous plate that deals with Papkovitch-Fadle-type eigenfunctions may be used as a starting point [36]. In the Mindlin plate theory the shear correction factor may be, nevertheless, calibrated according to a 3-D solution so that the effect of actual pointwise boundary conditions is also captured better. Different shear correction factors and their meaning will be studied in Section 4.

3. Rectangular interior finite elements

The shape functions are derived first after which the finite element matrices are attained through the weak form of the equations of motion (22)–(24). A Mathematica notebook in which the general shape functions for all interior elements and the finite element matrices for the Mindlin plate (as an example) are developed in analytical form is provided as an online supplementary file (PlateFE.nb).

3.1. Shape functions for finite elements

We modify the general interior solution (6) in such a way that the shape functions are valid for any shear correction factor κ of the Mindlin plate theory. By inspection, we get

$$\begin{Bmatrix} u_z(x, y) \\ \phi_x(x, y) \\ \phi_y(x, y) \end{Bmatrix} = \frac{1}{2G} \begin{Bmatrix} \Psi \\ -\frac{\partial}{\partial x} \left[\Psi + \left(\frac{2}{3\kappa} \right) \frac{h^2}{4(1-\nu)} \nabla^2 \Psi \right] \\ -\frac{\partial}{\partial y} \left[\Psi + \left(\frac{2}{3\kappa} \right) \frac{h^2}{4(1-\nu)} \nabla^2 \Psi \right] \end{Bmatrix}, \quad (30)$$

Figure 3: Set-up according to which the interior plate finite elements are developed.

where $\kappa = 2/3$ for the Levinson and Full Interior plates, and any value of κ can be used for the Mindlin plate. The biharmonic function $\Psi(x, y)$ is taken as

$$\Psi = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 + c_7x^3 + c_8y^3 + c_9x^2y + c_{10}xy^2 + c_{11}x^3y + c_{12}xy^3. \quad (31)$$

In the case of a thin plate, the polynomial approximation (31) leads to the well-known *nonconforming* Kirchhoff plate element first presented by Adini and Clough [51], which passes the patch test and produces convergent results [52]. Relations (30) enable us to use the same polynomial for shear deformable plate elements for the first time. Figure 3 presents the setting according to which the plate finite elements are developed. Each node in Fig. 3 has three degrees of freedom, namely, transverse displacement $u_{z,i}$ and rotations $\phi_{x,i}$ and $\phi_{y,i}$ ($i = 1, 2, 3, 4$). By using the above expressions (30) for the mid-surface variables, we define the FE degrees of freedom as

$$\begin{aligned} u_{z,1} &= u_z(-a, b), & u_{z,2} &= u_z(a, b), & u_{z,3} &= u_z(a, -b), & u_{z,4} &= u_z(-a, -b), \\ \phi_{x,1} &= \phi_x(-a, b), & \phi_{x,2} &= \phi_x(a, b), & \phi_{x,3} &= \phi_x(a, -b), & \phi_{x,4} &= \phi_x(-a, -b), \\ \phi_{y,1} &= \phi_y(-a, b), & \phi_{y,2} &= \phi_y(a, b), & \phi_{y,3} &= \phi_y(a, -b), & \phi_{y,4} &= \phi_y(-a, -b), \end{aligned} \quad (32)$$

In matrix form we have

$$\mathbf{u} = \mathbf{H}\mathbf{c}, \quad (33)$$

where

$$\mathbf{u} = \{u_{z,1} \ \phi_{x,1} \ \phi_{y,1} \ u_{z,2} \ \phi_{x,2} \ \phi_{y,2} \ u_{z,3} \ \phi_{x,3} \ \phi_{y,3} \ u_{z,4} \ \phi_{x,4} \ \phi_{y,4}\}^T, \quad (34)$$

$$\mathbf{c} = \{c_1 \ c_2 \ c_3 \ c_4 \ c_5 \ c_6 \ c_7 \ c_8 \ c_9 \ c_{10} \ c_{11} \ c_{12}\}^T, \quad (35)$$

and $\mathbf{H}$ is a coefficient matrix. The constants coefficients c_i are obtained in terms of the FE degrees of freedom by

$$\mathbf{c} = \mathbf{H}^{-1}\mathbf{u}. \quad (36)$$

The mid-surface variables (30) in terms of the FE degrees of freedom may then be written as

$$\begin{Bmatrix} u_z(x, y) \\ \phi_x(x, y) \\ \phi_y(x, y) \end{Bmatrix} = \mathbf{G}\mathbf{c} = \mathbf{G}\mathbf{H}^{-1}\mathbf{u} = \begin{Bmatrix} \mathbf{N}_{uz} \\ \mathbf{N}_{\phi_x} \\ \mathbf{N}_{\phi_y} \end{Bmatrix} \mathbf{u}, \quad (37)$$

where $\mathbf{G}$ is a polynomial matrix and $\mathbf{N}_{uz}$, $\mathbf{N}_{\phi x}$ and $\mathbf{N}_{\phi y}$ contain the shape functions, which satisfy the Kronecker delta property. Once the nodal displacements $\mathbf{u}$ are known, the mid-surface displacement field is acquired by substituting them into Eq. (37), after which the calculation of the total displacement field (9)–(11), the interior strains (12) and stresses (13) is straightforward.

3.2. Weak form of equations of motion

In order to obtain the weak form of the equations of motion (22), (23) and (24), we move all terms to the RHS in the equations, multiply the equations by weight functions $\hat{\phi}_x$, $\hat{\phi}_y$ and $\hat{u}_z$, respectively, sum the equations, and then integrate over the mid-surface. Finally, by employing integration by parts, we arrive at

$$\begin{aligned} & \int_{-b}^b \int_{-a}^a \left[M_x \frac{\partial \hat{\phi}_x}{\partial x} + M_y \frac{\partial \hat{\phi}_y}{\partial y} + M_{xy} \left(\frac{\partial \hat{\phi}_x}{\partial y} + \frac{\partial \hat{\phi}_y}{\partial x} \right) + Q_x \left(\hat{\phi}_x + \frac{\partial \hat{u}_z}{\partial x} \right) + Q_y \left(\hat{\phi}_y + \frac{\partial \hat{u}_z}{\partial y} \right) \right. \\ & \quad + N_x^0 \frac{\partial u_z}{\partial x} \frac{\partial \hat{u}_z}{\partial x} + N_y^0 \frac{\partial u_z}{\partial y} \frac{\partial \hat{u}_z}{\partial y} + N_{xy}^0 \frac{\partial u_z}{\partial x} \frac{\partial \hat{u}_z}{\partial y} + N_{xy}^0 \frac{\partial u_z}{\partial y} \frac{\partial \hat{u}_z}{\partial x} \\ & \quad \left. + \int_{-h/2}^{h/2} \rho \left(z \hat{\phi}_x \frac{\partial^2 U_x}{\partial t^2} + z \hat{\phi}_y \frac{\partial^2 U_y}{\partial t^2} + \hat{u}_z \frac{\partial^2 U_z}{\partial t^2} \right) dz - p \hat{u}_z \right] dx dy \\ & = \int_{-b}^b \left[M_x \hat{\phi}_x + M_{xy} \hat{\phi}_y + \left(Q_x + N_x^0 \frac{\partial u_z}{\partial x} + N_{xy}^0 \frac{\partial u_z}{\partial y} \right) \hat{u}_z \right]_{-a}^a dy \\ & \quad + \int_{-a}^a \left[M_y \hat{\phi}_y + M_{xy} \hat{\phi}_x + \left(Q_y + N_y^0 \frac{\partial u_z}{\partial y} + N_{xy}^0 \frac{\partial u_z}{\partial x} \right) \hat{u}_z \right]_{-b}^b dx, \end{aligned} \quad (38)$$

where, for example, the first row is associated with the stiffness matrix and the second row with the geometric stiffness matrix. In accordance with Galerkin's method, the weight functions $\hat{u}_z$, $\hat{\phi}_x$ and $\hat{\phi}_y$ will be approximated by the same shape functions (interpolants) as u_z , ϕ_x and ϕ_y , respectively.

3.3. Interior stiffness matrices

By using the shape functions (37), the twelve-by-twelve stiffness matrix of a four-node rectangular interior plate element that stems from the weak form (38) can be written as

$$\begin{aligned} \mathbf{K} = \int_{-b}^b \int_{-a}^a & \left[\frac{\partial \mathbf{N}_{\phi x}^T}{\partial x} \mathbf{M}_x + \frac{\partial \mathbf{N}_{\phi y}^T}{\partial y} \mathbf{M}_y + \left(\frac{\partial \mathbf{N}_{\phi x}^T}{\partial y} + \frac{\partial \mathbf{N}_{\phi y}^T}{\partial x} \right) \mathbf{M}_{xy} \right. \\ & \left. + \left(\mathbf{N}_{\phi x}^T + \frac{\partial \mathbf{N}_{uz}^T}{\partial x} \right) \mathbf{Q}_x + \left(\mathbf{N}_{\phi y}^T + \frac{\partial \mathbf{N}_{uz}^T}{\partial y} \right) \mathbf{Q}_y \right] dx dy, \end{aligned} \quad (39)$$

where

$$\mathbf{Q}_x = \kappa G h \left(\mathbf{N}_{\phi x} + \frac{\partial \mathbf{N}_{uz}}{\partial x} \right) \quad \text{and} \quad \mathbf{Q}_y = \kappa G h \left(\mathbf{N}_{\phi y} + \frac{\partial \mathbf{N}_{uz}}{\partial y} \right). \quad (40)$$

Similarly to other higher-order plate theories, e.g., [53, 54], the Levinson and Full Interior plates do not require shear correction factors; for them κ is replaced here by 2/3 given by the 3-D elasticity solution (6). The moment-related terms for the interior theories are as follows

Mindlin:

$$\begin{aligned}\mathbf{M}_x &= D \left(\frac{\partial \mathbf{N}_{\phi x}}{\partial x} + \nu \frac{\partial \mathbf{N}_{\phi y}}{\partial y} \right), \\ \mathbf{M}_y &= D \left(\nu \frac{\partial \mathbf{N}_{\phi x}}{\partial x} + \frac{\partial \mathbf{N}_{\phi y}}{\partial y} \right), \\ \mathbf{M}_{xy} &= \frac{D(1-\nu)}{2} \left(\frac{\partial \mathbf{N}_{\phi x}}{\partial y} + \frac{\partial \mathbf{N}_{\phi y}}{\partial x} \right),\end{aligned}\tag{41}$$

Levinson:

$$\begin{aligned}\mathbf{M}_x &= \frac{D}{5} \left[4 \left(\frac{\partial \mathbf{N}_{\phi x}}{\partial x} + \nu \frac{\partial \mathbf{N}_{\phi y}}{\partial y} \right) - \left(\frac{\partial^2 \mathbf{N}_{uz}}{\partial x^2} + \nu \frac{\partial^2 \mathbf{N}_{uz}}{\partial y^2} \right) \right], \\ \mathbf{M}_y &= \frac{D}{5} \left[4 \left(\nu \frac{\partial \mathbf{N}_{\phi x}}{\partial x} + \frac{\partial \mathbf{N}_{\phi y}}{\partial y} \right) - \left(\nu \frac{\partial^2 \mathbf{N}_{uz}}{\partial x^2} + \frac{\partial^2 \mathbf{N}_{uz}}{\partial y^2} \right) \right], \\ \mathbf{M}_{xy} &= \frac{D}{5} \left(\frac{1-\nu}{2} \right) \left[4 \left(\frac{\partial \mathbf{N}_{\phi x}}{\partial y} + \frac{\partial \mathbf{N}_{\phi y}}{\partial x} \right) - 2 \frac{\partial^2 \mathbf{N}_{uz}}{\partial x \partial y} \right],\end{aligned}\tag{42}$$

Full Interior:

$$\begin{aligned}\mathbf{M}_x &= D \left(\frac{\partial \mathbf{N}_{\phi x}}{\partial x} + \nu \frac{\partial \mathbf{N}_{\phi y}}{\partial y} \right) - \frac{Dh^2}{40} \left(\frac{2-\nu}{1-2\nu} \right) \left[(1-\nu) \frac{\partial^2}{\partial x^2} + \nu \frac{\partial^2}{\partial y^2} \right] \left(\frac{\partial \mathbf{N}_{\phi x}}{\partial x} + \frac{\partial \mathbf{N}_{\phi y}}{\partial y} \right), \\ \mathbf{M}_y &= D \left(\nu \frac{\partial \mathbf{N}_{\phi x}}{\partial x} + \frac{\partial \mathbf{N}_{\phi y}}{\partial y} \right) - \frac{Dh^2}{40} \left(\frac{2-\nu}{1-2\nu} \right) \left[(1-\nu) \frac{\partial^2}{\partial y^2} + \nu \frac{\partial^2}{\partial x^2} \right] \left(\frac{\partial \mathbf{N}_{\phi x}}{\partial x} + \frac{\partial \mathbf{N}_{\phi y}}{\partial y} \right), \\ \mathbf{M}_{xy} &= \frac{D(1-\nu)}{2} \left(\frac{\partial \mathbf{N}_{\phi x}}{\partial y} + \frac{\partial \mathbf{N}_{\phi y}}{\partial x} \right) - \frac{Dh^2}{40} (2-\nu) \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial \mathbf{N}_{\phi x}}{\partial x} + \frac{\partial \mathbf{N}_{\phi y}}{\partial y} \right),\end{aligned}\tag{43}$$

where we have introduced the flexural rigidity

$$D = \frac{Eh^3}{12(1-\nu^2)}.\tag{44}$$

The stress resultants on which Eqs. (40)–(43) are based on can be found in Ref. [27]. The stiffness and mass matrices (below) of the rectangular Levinson and Full Interior elements are not symmetric (cf. Section 2.4). We reiterate that, while this may seem unconventional, in the static case the reciprocal theorem is satisfied when it is based on work W_s (29), see Ref. [28] for details. The consistent geometric stiffness matrix for an interior plate element reads

$$\mathbf{K}_g = \int_{-b}^b \int_{-a}^a \left[N_x^0 \frac{\partial \mathbf{N}_{uz}^T}{\partial x} \frac{\partial \mathbf{N}_{uz}}{\partial x} + N_y^0 \frac{\partial \mathbf{N}_{uz}^T}{\partial y} \frac{\partial \mathbf{N}_{uz}}{\partial y} + N_{xy}^0 \frac{\partial \mathbf{N}_{uz}^T}{\partial x} \frac{\partial \mathbf{N}_{uz}}{\partial y} + N_{xy}^0 \frac{\partial \mathbf{N}_{uz}^T}{\partial y} \frac{\partial \mathbf{N}_{uz}}{\partial x} \right] dx dy,\tag{45}$$

which is symmetric. The load vector due to a distributed load p is

$$\mathbf{p} = \int_{-b}^b \int_{-a}^a p \mathbf{N}_{uz}^T dx dy.\tag{46}$$

As a final stiffness consideration one may take $p = -k_w u_z$ for an interior plate on a Winkler foundation, where k_w is the foundation modulus, and the related stiffness matrix to be added to (39) is

$$\mathbf{K}_w = \int_{-b}^b \int_{-a}^a k_w \mathbf{N}_{uz}^T \mathbf{N}_{uz} dx dy.\tag{47}$$

3.4. Interior mass matrices

The consistent mass matrices of the Mindlin, Levinson and Full Interior plate elements obtained through the weak form (38) are

Mindlin:

$$\mathbf{M} = \int_{-b}^b \int_{-a}^a \frac{\rho h^3}{12} \left(\mathbf{N}_{\phi x}^T \mathbf{N}_{\phi x} + \mathbf{N}_{\phi y}^T \mathbf{N}_{\phi y} + \frac{12}{h^2} \mathbf{N}_{uz}^T \mathbf{N}_{uz} \right) dx dy, \quad (48)$$

Levinson:

$$\mathbf{M} = \int_{-b}^b \int_{-a}^a \frac{\rho h^3}{60} \left[\mathbf{N}_{\phi x}^T \left(4\mathbf{N}_{\phi x} - \frac{\partial \mathbf{N}_{uz}}{\partial x} \right) + \mathbf{N}_{\phi y}^T \left(4\mathbf{N}_{\phi y} - \frac{\partial \mathbf{N}_{uz}}{\partial y} \right) + \frac{60}{h^2} \mathbf{N}_{uz}^T \mathbf{N}_{uz} \right] dx dy, \quad (49)$$

Full Interior:

$$\begin{aligned} \mathbf{M} = & \int_{-b}^b \int_{-a}^a \frac{\rho h^3}{60} \mathbf{N}_{\phi x}^T \left[4\mathbf{N}_{\phi x} - \frac{\partial \mathbf{N}_{uz}}{\partial x} + \frac{\nu h^2}{8(1-\nu)} \left(\frac{\partial^2 \mathbf{N}_{\phi x}}{\partial x^2} + \frac{\partial^2 \mathbf{N}_{\phi y}}{\partial x \partial y} \right) \right], \\ & + \frac{\rho h^3}{60} \mathbf{N}_{\phi y}^T \left[4\mathbf{N}_{\phi y} - \frac{\partial \mathbf{N}_{uz}}{\partial y} + \frac{\nu h^2}{8(1-\nu)} \left(\frac{\partial^2 \mathbf{N}_{\phi x}}{\partial x \partial y} + \frac{\partial^2 \mathbf{N}_{\phi y}}{\partial y^2} \right) \right], \\ & + \rho h \mathbf{N}_{uz}^T \left[\mathbf{N}_{uz} - \frac{\nu h^2}{24(1-\nu)} \left(\frac{\partial \mathbf{N}_{\phi x}}{\partial x} + \frac{\partial \mathbf{N}_{\phi y}}{\partial y} \right) \right] dx dy. \end{aligned} \quad (50)$$

In this paper, the presented integrals are calculated analytically and, thus, need to be calculated only once for FE analyses. In other words, the use of the presented elements does not require numerical integration. Finally, we note that a very accurate 24-DOF quadrilateral plate element based on the same 3-D solution that was used here has been developed by Piltner [55]. His hybrid-Trefftz approach and the resulting element for static plate bending analysis are fundamentally different from the ones presented here, reflecting well the fact that the interior elasticity solution opens many fruitful venues for the study of thin and thick plates.

4. Numerical studies

The convergence of the rectangular 2-D interior plate elements is studied first. Then buckling loads given by the 2-D plates are compared to 3-D elasticity solutions. A 3-D FE (Abaqus) plate model used in natural frequency analyses in addition to the 2-D interior plate elements is described briefly. Finally, natural frequencies of several plates are calculated using the 2-D and 3-D models.

4.1. Convergence of the interior plate elements

For convergence purposes, we study the bending of a clamped plate subjected to a point load $P = 1$ N at its center (CCCC- P) and the bending of a simply-supported plate under a uniform pressure $p = 1$ N/m² (SSSS- p). The plates are square-shaped ($a = b$) and have a Poisson ratio of $\nu = 0.3$. The clamped plate is relatively thin with thickness $h = 0.001$ m and length $2a = 1$ m. The simply-supported plate is of the same length and its thickness is $h = 0.1$ m. For the sake of convenience, Young's modulus is chosen in both cases so that $D = 1$ Nm. The reference midpoint deflection (Kirchhoff solution) for the CCCC- P plate is $u_z = 0.00560 \times P(2a)^2/D$ [56]. In the thick SSSS- p plate case, the midpoint deflection obtained by each mesh is compared here to the

Figure 4: Convergence in terms of midpoint deflections of the current Mindlin, Levinson and Full Interior plate elements, which converge practically similarly, compared to the S4 and S4R shell elements provided by Abaqus. (a) Thin clamped square plate subjected to a point load at its center. b) Thick simply-supported square plate under uniform pressure.

deflection obtained by a 60×60 mesh for plate model. The error in the midpoint deflection is given by

$$\Delta u_z = 100 \times \frac{\text{Current solution} - \text{Reference solution}}{\text{Reference solution}} \quad [\%]. \quad (51)$$

The convergence of the Mindlin, Levinson and Full Interior plate elements is studied in Fig. 4. Results calculated by Abaqus using S4 and S4R elements are given for comparison. The standard Abaqus elements exhibit a slightly faster rate of convergence than the current interior elements, all of which practically converge in the same way. In both cases, CCCC- P and SSSS- p , the current interior elements give a too stiff overall behavior with a small number of elements. By adding elements, satisfactory results are obtained with a reasonable number of elements. The convergence of the interior elements is nearly monotonic; the coarsest possible mesh (2×2) in the CCCC- P case is peculiar as the S4 and S4R elements do not provide a reasonable result at all. In the following analyses we will use a 50×50 mesh for square-shaped 2-D interior plates.

4.2. Buckling analyses

Figure 5 shows the error in the nondimensional buckling loads

$$\bar{N} = \frac{N_{cr}(2b)^2}{\pi^2 D} \quad (52)$$

of the 2-D interior plates in comparison to 3-D elasticity solutions [57, 58] in two different square plate cases ($\nu = 0.3$). It can be seen that in both cases the Full Interior and Levinson plates give the best 2-D results. As for the Mindlin plate, we can see that the widely accepted shear correction

Figure 5: Non-dimensional buckling loads $\bar{N} = N_{cr}(2b)^2/(\pi^2 D)$ compared to 3D elasticity solutions [57]. (a) Simply-supported square plate under uniaxial compression. (b) Clamped square plate under biaxial compression $N_y = 1.5N_x$.

factor $\kappa = 5/6$, which is used also in commercial FE software for isotropic plates, does not lead to as accurate results as $\kappa = 2/3$, which is the value provided by the general 3-D interior plate solution. We note that for a Timoshenko beam, the value $\kappa = 5/6$ is obtained by comparing the strain energy which stems from the constant shear stress of the Timoshenko beam to a strain energy contribution due to a parabolic shear stress distribution [59]. However, the shear strain energy represents only a part of the entire shear contribution in the total potential energy [cf. Eq. (27)], which should be the subject of the comparison. Thus, the shear correction factor $\kappa = 5/6$ appears to be founded on an incomplete analysis. While the shear correction factor $\kappa = 5/(6 - \nu)$ is not of use in buckling analyses, it will turn out to be applicable in natural frequency studies.

Figure 6 displays a similar nondimensional buckling load comparison in two cases as Fig. 5 with the exception that the thickness-to-width ratio $h/(2b) = 0.1$ is now constant and the length-to-width ratio $2a/2b$ is varied. The trend remains the same, the Full Interior and Levinson plates provide very good results, as well as the Mindlin plate with the shear correction factor $\kappa = 2/3$.

The results in Figs. 5 and 6 indicate that the interior part of the general 3-D elasticity solution provides accurate estimates for the critical buckling loads. That is to say, the boundary layer solution does not need to be accounted for. We can also see that our simplification $U_z = u_z$ in Eqs. (11) and (21) to obtain a simpler geometric stiffness matrix does not deteriorate the results provided by the 2-D Full Interior plate in comparison to 3-D elasticity solutions. All in all, the 2-D interior plate results are in very good agreement with 3-D elasticity results, provided that unsuitable shear correction factors are not used. The dimensional reduction from 3-D to 2-D in the governing differential equations simplifies the analytical and computational analysis of plate buckling greatly.

Figure 6: Non-dimensional buckling loads $\bar{N} = N_{cr}(2b)^2/(\pi^2 D)$ compared to 3-D elasticity solutions [57, 58]. (a) Simply-supported plate under biaxial compression $N_x = N_y$. (b) SFSF plate subjected to uniaxial compression N_x .

4.3. 3-D FE model for natural frequency analyses

Let us consider a 3-D FE (Abaqus) model to be used for validation purposes in natural frequency analyses. As discussed in Section 2.5, the boundary conditions of a 2-D interior plate may be chosen only so as to imitate the boundary conditions of a 3-D problem that considers also the boundary layer. In the buckling problems, the boundary conditions of the 2-D interior plates and 3-D elasticity solutions were in good agreement based on the overall results. For our purposes, the main difference between buckling and vibration eigenproblems is that, in vibration, we are also interested in the higher eigenmodes. These modes may require a more intricate analysis of the boundary correspondence between the 2-D interior and 3-D FE plate models. Motivated by this, we use two kinds of boundary conditions for the transverse deflection at simply-supported and clamped edges of 3-D FE plates. For example, for simply-supported edges at $x = \pm a$ we use

$$U_y = 0, \quad \text{with} \quad U_z = 0 \quad \text{or} \quad u_z = 0 \quad (53)$$

and at $y = \pm b$

$$U_x = 0, \quad \text{with} \quad U_z = 0 \quad \text{or} \quad u_z = 0, \quad (54)$$

where $U_z = 0$ is used when an edge is fully-supported throughout the plate thickness and $u_z = 0$ is used when the transverse deflection is zero only at the mid-surface. The two different cases are never mixed here, that is, a plate has only fully-supported or mid-surface-supported edges, but not both at the same time. Hereafter, the distinction between the two options for the transverse deflection in a 3-D FE model used to validate 2-D plate results is made by assigning an upper or lower case U or u to the boundary condition definition of the case at hand, for example, we use SSSS- U and SSSS- u . Boundary conditions for displacements U_x and U_y , which are associated with rotations ϕ_x and ϕ_y , respectively, are always imposed throughout the plate thickness.

Figure 7: Non-dimensional angular frequencies $\bar{\omega}_n = \omega_n(4b^2/\pi^2)\sqrt{\rho h/D}$ of 2-D interior SSSS plates compared to 3-D FE solutions. The transverse deflection of the 3-D FE reference model is supported (a) throughout plate thickness, (b) only at the mid-surface. The analytical 3-D solution in the close-up of (a) is from [60].

In the 3-D FE (Abaqus) model, quadratic brick elements C3D20R (R stands for reduced integration) are mainly used and a typical moderately thick square plate model has the dimensions $h = 0.1$ m and $2a = 2b = 1$ m. With 8 elements in the thickness direction, we have a total of 51200 cube-shaped elements in our model. As an exception, to provide a better convergence rate, we use linear brick elements C3D8I (I stands for incompatible modes) in models with clamped edges at which the boundary condition for the transverse deflection is imposed only at the mid-surface (e.g. SCSC- u comprises solely of C3D8I elements). In such a case the mesh is denser near the mid-surface in the thickness direction and a typical model consists of approximately 320000 elements. The accuracy of the 3-D FE model is validated by known analytical 3-D solutions for natural frequencies of plates in the next section.

4.4. Natural vibration frequencies

Figures 7–9 show the error in the non-dimensional angular frequencies

$$\bar{\omega}_n = \omega_n \frac{(2b)^2}{\pi^2} \sqrt{\rho h/D} \quad (55)$$

of the bending modes of the current 2-D interior plates in comparison to 3-D FE (Abaqus) solutions ($\nu = 0.3$). In each of the Figures 7–9, two different boundary conditions are used for the 3-D FE reference model. The transverse deflection of the 3-D model is supported throughout the thickness of the plate ($U_z = 0$) in subfigures (a) and only at the mid-surface ($u_z = 0$) in subfigures (b). Changing the boundary condition alters the behavior not only at the plate edge but also in the vicinity of it, that is, in the boundary layer. Only the 3-D FE reference solution changes between cases (a) and (b). All plates are square-shaped.

It can be seen in the close-up in Fig. 7(a) that for the first six bending modes the 3-D FE model is in excellent agreement with the 3-D results found in literature [60] so the 3-D FE model

Figure 8: Non-dimensional angular frequencies $\bar{\omega}_n = \omega_n(4b^2/\pi^2)\sqrt{\rho h/D}$ of 2-D interior SCSC plates compared to 3-D FE solutions. The transverse deflection of the 3-D FE reference model is supported (a) throughout plate thickness, (b) only at the mid-surface.

is valid for our purposes. The best results by the 2-D interior plates in the SSSS- U case are obtained by using the Mindlin plate theory with the shear correction factor $\kappa = 5/(6 - \nu)$, which has also been found to be the optimal factor in several analytical studies of eigenvalue and wave propagation plate problems [42, 61, 62]. In more detail, $\kappa = 5/(6 - \nu)$ can be reached analytically through consistent truncation procedures for exact 3-D elasticity and Mindlin plate solutions, that is, by fitting the interior Mindlin plate through the shear coefficient to a full, 3-D elastodynamic solution with $U_z = 0$ as a boundary condition at all edges. That is to say, the shear correction factor compensates for the approximate kinematics and non-existent boundary layer behavior of the Mindlin plate theory.

Figure 7(b) shows that in the SSSS- u case with the mid-surface attachment, which changes the overall boundary layer behavior in comparison to the SSSS- U case, the most accurate results are provided by the Full Interior plate. The close-up in Fig. 7(b) highlights the fact that, with low mode numbers, the differences between the 2-D plate and 3-D FE solutions are nominal. The Full Interior plate is based on a static interior elasticity solution and, thus, may not capture all the dynamic features as effectively in the SSSS- u case as the Mindlin plate with $\kappa = 5/(6 - \nu)$ does in the SSSS- U case.

Analysis of the SCSC case in Fig. 8 reveals the same properties as found in Fig. 7. Figure 9 which presents a CCFF plate case, continues along the lines of Figs. 7 and 8. However, as a new feature, we can see that when a plate has free edges, the shear correction factor $\kappa = 5/(6 - \nu)$ starts to deviate from the nearly perfect correspondence with the 3-D results. We find that $\kappa = 5/(6 - \nu)$ works optimally only when we have $U_z = 0$ as a boundary condition at all plate edges, which is not at all surprising because the derivation of $\kappa = 5/(6 - \nu)$ is founded on such a case ($U_z = 0$). It can be seen from Figs. 7–9 that the Levinson and Mindlin ($\kappa = 2/3$) plates are a bit too flexible in comparison to the Full Interior plate. It can be further deduced from the figures that the optimal κ for a mid-surface-supported 3-D plate case is only a bit higher than $2/3$ (clearly under $5/6$). We

Figure 9: Non-dimensional angular frequencies $\bar{\omega}_n = \omega_n(4b^2/\pi^2)\sqrt{\rho h/D}$ of 2-D interior CCFF plates compared to 3-D FE solutions. The transverse deflection of the 3-D FE reference model is supported (a) throughout plate thickness, (b) only at the mid-surface.

conclude that, in dynamic analyses, the extent of adjustments to the theoretical value of $\kappa = 2/3$ to reach an optimal κ depends primarily on the actual 3-D transverse deflection boundary conditions and the resulting boundary layer behavior of the studied case.

5. Conclusions

In this study, we developed locking-free finite elements for shear deformable 2-D Mindlin, Levinson and Full Interior plates. The shape functions, which were constructed from a single biharmonic polynomial that defines the general elasticity solution to a 3-D plate bending problem, were the same for all elements at hand. Instead of a typical independent interpolation of each mid-surface variable, the general solution links the interpolation functions. The linking provides us with higher-order expansions of the displacement variables than independent interpolations would.

The presented developments were limited to linearly elastic, isotropic four-node rectangular elements for which it was possible to develop closed-form finite element matrices in a unified manner without numerical integration. The elements enabled a comparison between the different 2-D plate theories and 3-D elasticity solutions in bending, buckling and vibration problems.

While the application of the Full Interior plate is limited to isotropic plates due to its material-dependent kinematic description, the Levinson theory can be used in a wider range of applications and it carries the benefits of both Mindlin's FSDT and Reddy's TSDT plates. That is to say, in general, plate elements based on the Levinson theory have only three degrees of freedom per node like FSDT elements but, like TSDT elements, provide quadratic variations of the transverse shear stresses with respect to the thickness coordinate and do not require a shear correction factor. The developed rectangular Levinson plate finite element appears to be the first of its kind and paves way for triangular and quadrilateral elements.

References

- [1] R. D. Mindlin, Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates, *J. App. Mech.* 18 (1951) 31–38.
- [2] C. M. Wang, G. T. Lim, J. N. Reddy, K. H. Lee, Relationships between bending solutions of Reissner and Mindlin plate theories, *Eng. Struct.* 23 (7) (2001) 838–849.
- [3] R. Szilard, *Theories and applications of plate analysis: classical numerical and engineering methods*, John Wiley & Sons, 2004.
- [4] J. N. Reddy, *Theory and Analysis of Elastic Plates and Shells*, 2nd Edition, CRC press, 2007.
- [5] M. M. Hrabok, T. M. Hrudey, A review and catalogue of plate bending finite elements, *Comput. Struct.* 19 (3) (1984) 479–495.
- [6] T. J. R. Hughes, R. L. Taylor, W. Kanoknukulchai, A simple and efficient finite element for plate bending, *Int. J. Numer. Meth. Eng.* 11 (10) (1977) 1529–1543.
- [7] T. J. R. Hughes, M. Cohen, M. Haroun, Reduced and selective integration techniques in the finite element analysis of plates, *Nucl. Eng. Des.* 46 (1) (1978) 203–222.
- [8] T. J. R. Hughes, W. K. Liu, Nonlinear finite element analysis of shells: Part I. Three-dimensional shells, *Comput. Methods Appl. Mech. Engrg.* 26 (3) (1981) 331–362.
- [9] T. J. R. Hughes, W. K. Liu, Nonlinear finite element analysis of shells: Part II. Two-dimensional shells, *Comput. Methods Appl. Mech. Engrg.* 27 (2) (1981) 167–181.
- [10] T. J. R. Hughes, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Prentice-Hall, Inc., 1987.
- [11] D. P. Flanagan, T. Belytschko, A uniform strain hexahedron and quadrilateral with orthogonal hourglass control, *Int. J. Numer. Meth. Eng.* 17 (5) (1981) 679–706.
- [12] T. Belytschko, C. S. Tsay, A stabilization procedure for the quadrilateral plate element with one-point quadrature, *Int. J. Numer. Meth. Eng.* 19 (3) (1983) 405–419.
- [13] T. Belytschko, J. I. Lin, C. S. Tsay, Explicit algorithms for the nonlinear dynamics of shells, *Comput. Methods Appl. Mech. Engrg.* 42 (2) (1984) 225–251.
- [14] T. Belytschko, W. K. Liu, B. Moran, K. Elkhodary, *Nonlinear Finite Elements for Continua and Structures*, John Wiley & Sons, 2013.
- [15] K. J. Bathe, E. Dvorkin, L. W. Ho, Our discrete-Kirchhoff and isoparametric shell elements for nonlinear analysisan assessment, *Computers & Structures* 16 (1-4) (1983) 89–98.
- [16] K. J. Bathe, E. N. Dvorkin, A four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation, *Int. J. Numer. Meth. Eng.* 21 (2) (1985) 367–383.
- [17] P. S. Lee, K. J. Bathe, The quadratic MITC plate and MITC shell elements in plate bending, *Adv. Eng. Softw.* 41 (5) (2010) 712–728.
- [18] K. J. Bathe, *Finite Element Procedures*, Prentice-Hall, Inc., 2012.
- [19] J. N. Reddy, A simple higher-order theory for laminated composite plates, *J. Appl. Mech.* 51 (4) (1984) 745–752.
- [20] J. N. Reddy, *Mechanics of Laminated Composite Plates and Shells: Theory and Analysis*, 2nd Edition, CRC Press, Boca Raton, Florida, 2004.
- [21] D. J. Benson, Y. Bazilevs, M. C. Hsu, T. J. R. Hughes, Isogeometric shell analysis: the Reissner–Mindlin shell, *Comput. Methods Appl. Mech. Engrg.* 199 (5) (2010) 276–289.
- [22] C. H. Thai, H. Nguyen-Xuan, N. Nguyen-Thanh, T. H. Le, T. Nguyen-Thoi, T. Rabczuk, Static, free vibration, and buckling analysis of laminated composite Reissner–Mindlin plates using NURBS-based isogeometric approach, *Int. J. Numer. Meth. Eng.* 91 (6) (2012) 571–603.
- [23] N. Valizadeh, S. Natarajan, O. A. Gonzalez-Estrada, T. Rabczuk, T. Q. Bui, S. P. A. Bordas, NURBS-based finite element analysis of functionally graded plates: static bending, vibration, buckling and flutter, *Compos. Struct.* 99 (2013) 309–326.
- [24] C. Adam, S. Bouabdallah, M. Zarroug, H. Maitournam, Improved numerical integration for locking treatment in isogeometric structural elements. Part II: Plates and shells, *Comput. Methods Appl. Mech. Engrg.* 284 (2015) 106–137.
- [25] B. Oesterle, E. Ramm, M. Bischoff, A shear deformable, rotation-free isogeometric shell formulation, *Comput. Methods Appl. Mech. Engrg.* 307 (2016) 235–255.
- [26] W. Dornisch, R. Müller, S. Klinkel, An efficient and robust rotational formulation for isogeometric Reissner–Mindlin shell elements, *Comput. Methods Appl. Mech. Engrg.* 303 (2016) 1–34.
- [27] A. T. Karttunen, R. von Hertzen, J. N. Reddy, J. Romanoff, Bridging plate theories and elasticity solutions, *Int. J. Solids Struct.* 106 (2017) 251–263.

- [28] A. T. Karttunen, R. von Hertzen, J. N. Reddy, J. Romanoff, Exact elasticity-based finite element for circular plates, *Comput. Struct.* 182 (2017) 219–226.
- [29] R. D. Gregory, The general form of the three-dimensional elastic field inside an isotropic plate with free faces, *J. Elast.* 28 (1) (1992) 1–28.
- [30] M. Z. Wang, B. S. Zhao, The decomposed form of the three-dimensional elastic plate, *Acta Mech.* 166 (1-4) (2003) 207–216.
- [31] B. Zhao, D. Wu, M. Wang, The refined theory and the decomposed theorem of a transversely isotropic elastic plate, *Eur. J. Mech. A-Solid.* 39 (2013) 243–250.
- [32] S. Cheng, Elasticity theory of plates and a refined theory, *J. App. Mech.* 46 (3) (1979) 644–650.
- [33] F. Y. Wang, Two-dimensional theories deduced from three-dimensional theory for a transversely isotropic body. I: Plate problems, *Int. J. Solids Struct.* 26 (4) (1990) 455–470.
- [34] F. Y. Wang, Two-dimensional theories deduced from three-dimensional theory for a transversely isotropic body. II: Plane problems, *Int. J. Solids Struct.* 28 (2) (1991) 161–177.
- [35] R. Piltner, Three-dimensional stress and displacement representations for plate problems, *Mech. Res. Commun.* 18 (1) (1991) 41–49.
- [36] R. Piltner, The derivation of a thick and thin plate formulation without ad hoc assumptions, *J. Elast.* 29 (2) (1992) 133–173.
- [37] M. Batista, An exact theory of the bending of transversely inextensible elastic plates, *Acta Mech.* 226 (9) (2015) 2899–2924.
- [38] M. Levinson, An accurate, simple theory of the statics and dynamics of elastic plates, *Mech. Res. Commun.* 7 (6) (1980) 343–350.
- [39] A. T. Karttunen, R. von Hertzen, Exact theory for a linearly elastic interior beam, *Int. J. Solids Struct.* 78 (2016) 125–130.
- [40] A. T. Karttunen, R. von Hertzen, On the foundations of anisotropic interior beam theories, *Compos. B-Eng.* 87 (2016) 299–310.
- [41] J. N. Reddy, *An Introduction to Nonlinear Finite Element Analysis*, 2nd Edition, Oxford University Press, 2015.
- [42] W. H. Wittrick, Analytical, three-dimensional elasticity solutions to some plate problems, and some observations on Mindlin’s plate theory, *Int. J. Solids Struct.* 23 (4) (1987) 441–464.
- [43] A. Nosier, J. N. Reddy, On boundary layer and interior equations for higher-order theories of plates, *ZAMM-Z. Angew. Math. Me.* 72 (12) (1992) 657–666.
- [44] B. Häggblad, K. J. Bathe, Specifications of boundary conditions for Reissner/Mindlin plate bending finite elements, *Int. J. Numer. Meth. Eng.* 30 (5) (1990) 981–1011.
- [45] D. N. Arnold, R. S. Falk, The boundary layer for the Reissner–Mindlin plate model, *SIAM J. Math. Anal.* 21 (2) (1990) 281–312.
- [46] D. N. Arnold, R. S. Falk, Asymptotic analysis of the boundary layer for the Reissner–Mindlin plate model, *SIAM J. Math. Anal.* 27 (2) (1996) 486–514.
- [47] S. S. Singiresu, *The Finite Element Method in Engineering*, 5th Edition, Elsevier, 2010.
- [48] J. N. Reddy, *Energy Principles and Variational Methods in Applied Mechanics*, 3rd Edition, John Wiley & Sons, New York, NY, 2017.
- [49] M. H. Sadd, *Elasticity – Theory, Applications and Numerics*, 3rd Edition, Academic Press, Oxford, 2014.
- [50] T. Nguyen, C. H. Thai, H. Nguyen-Xuan, On the general framework of high order shear deformation theories for laminated composite plate structures: A novel unified approach, *Int. J. Mech. Sci.* 110 (2016) 242–255.
- [51] A. Adini, R. W. Clough, Analysis of plate bending by the finite element method, NSF Report G7337, 1961.
- [52] P. Lascaux, P. Lesaint, Some nonconforming finite elements for the plate bending problem, *Revue française d’automatique, informatique, recherche opérationnelle. Analyse numérique* 9 (R1) (1975) 9–53.
- [53] T. Q. Bui, T. V. Do, L. H. T. Ton, T. H. Doan, S. Tanaka, D. T. Pham, T. A. Nguyen-Van, T. Yu, S. Hirose, On the high temperature mechanical behaviors analysis of heated functionally graded plates using fem and a new third-order shear deformation plate theory, *Compos. Part B-Eng.* 92 (2016) 218–241.
- [54] S. Liu, T. Yu, T. Q. Bui, S. Yin, D. K. Thai, S. Tanaka, Analysis of functionally graded plates by a simple locking-free quasi-3d hyperbolic plate isogeometric method, *Compos. Part B-Eng.* 120 (2017) 182–196.
- [55] R. Piltner, A quadrilateral hybrid-Trefftz plate bending element for the inclusion of warping based on a three-dimensional plate formulation, *Int. J. Numer. Met. Eng.* 33 (2) (1992) 387–408.
- [56] S. P. Timoshenko, S. Woinowsky-Krieger, *Theory of Plates and Shells*, 2nd Edition, McGraw-Hill, New York, 1959.
- [57] T. M. Teo, K. M. Liew, A differential quadrature procedure for three-dimensional buckling analysis of rectangular plates, *Int. J. Solids Struct.* 36 (8) (1999) 1149–1168.

- [58] Ö. Civalek, Three-dimensional vibration, buckling and bending analyses of thick rectangular plates based on discrete singular convolution method, *Int. J. Mech. Sci.* 49 (6) (2007) 752–765.
- [59] H. Altenbach, J. W. Altenbach, W. Kissing, *Mechanics of Composite Structural Elements*, Springer, 2013.
- [60] S. Srinivas, C. V. J. Rao, A. K. Rao, An exact analysis for vibration of simply-supported homogeneous and laminated thick rectangular plates, *J. Sound Vib.* 12 (2) (1970) 187–199.
- [61] J. Hutchinson, Vibrations of thick free circular plates, exact versus approximate solutions, *J. Appl. Mech.* 51 (1984) 581–585.
- [62] N. G. Stephen, Mindlin plate theory: best shear coefficient and higher spectra validity, *J. Sound Vib.* 202 (4) (1997) 539–553.