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Research Highlights

- Develops a new lot-sizing problem with learning in production.
- The learning curve is bivariate dependent on the group size and the repetitions.
- The bivariate model fitted the data better than the univariate model.
- The model is compared to a classical one that assumes a univariate learning curve.
- Using the bivariate model has little effect on the production-inventory.

Short Communication Ms. No. APM-D-19-01286R1

The effects of learning in production and group size on the lot-sizing problem

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Short Communication:**The effects of learning in production and group size on the lot-sizing problem**

Abstract: One of the lot-sizing problem extensions that received noticeable attention in the literature is the one that investigated the effects of learning in production. The studies along this line of research assumed learning to improve with the number of repetitions following a power form. There is evidence also that the group size, i.e., number of workers learning in a group, affect performance (time per unit). This note revisits the problem and modifies it by incorporating the group size, along with cumulative production, as a proxy for measuring performance. Numerical examples are provided to illustrate the behavior of the modified model. The results of the two models are also compared to draw some meaningful insights and conclusions. Although the results favored using a simple univariate learning curve, considering group size when modeling lot-sizing problems can significantly affect unit production cost.

Keywords: Lot-sizing, learning in production, group size

1. Introduction

The economic order/production quantity (EOQ/EPQ) model, or the lot-sizing problem, has been with us for more than a century and will continue to be for years to come. It has been critiqued, modified, extended. One of the extensions is relaxing the assumption of a constant production rate. In labor-intensive industries, e.g., an assembly operation, the production rate improves with worker's experience or learning-by-doing. As learning becomes faster with every

repetition, inventory builds in a concave form. Calculating the average inventory level requires integrating the production, as a function of time, over the proper limits, resulting in a mathematical expression with no closed-form solution. The literature shows that with learning, it is recommended to produce in smaller lots more frequently. Readers may refer to [1,2] for a review of the papers that studied the effects of learning on inventory decisions, and to [3] for the applications of learning curves in production and operations management.

The literature also shows that most studies assumed learning to follow the relationship proposed by Wright [4] who observed it in an aircraft manufacturing facility that the time to produce an airplane reduced by a constant percentage with doubling output. One drawback of the Wright's learning curve, WLC henceforth, is that its unit production time is zero as cumulative production becomes very large. In reality, learning plateaus. Few researchers rectified this by assuming the learning curve to follow that of de Jong [5] who added a factor of compressibility (a percentage of labor time of total production time) to the WLC forcing it to plateau at some standard time [1-3]. This relaxation, however, of the WLC assumption added complexity to determining the average inventory level where an approximation was necessary [6].

There have been other approaches to modelling the learning curve. Some suggested bi-variate and multivariate models (e.g., [3, 7-8]). In such models, cumulative production, or the number of repetitions, remains as the primary measure of knowledge but added to it other independent variables. Adding more independent variables improves the predictability of the learning curve, but also increases its complexity. Of the learning curve models of relevance to this paper is that of Camm and Womer [9] who had the output per time period as the dependent variable and

crew size (number of workers) and number of repetitions (cumulative production) as independent variables. Camm and Womer [9] used a bivariate model for worker allocation and not for the lot-sizing problem, as this study does. This learning curve is referred to in the paper by the CWLC.

The total cost of an EPQ model is the sum of unit-time setup and holding costs. The first is continuously decreasing in the lot size quantity, while the second is increasing. The unit labor or production cost is usually ignored, as it is constant over time and does not affect the lot sizing quantity when solving for it. The optimal lot size is determined by setting the first derivative of the total cost function to zero and solving to find its optimal value in closed-form. The labor cost is not factored into the closed-form expression, and therefore has no effect on the lot size quantity. The reason for this, mainly, is that the EPQ assumes a constant production rate over time and larger than that of demand. When the time to produce (assemble) a unit reduces because of learning with cumulative output, then the unit-time production cost cannot be ignored. Accounting for it adds complexity to the total cost function and a closed-form solution for the lot size is not possible, but only through a numerical search (e.g., [10]). The papers that investigated learning in production assumed the labour (production) cost per unit of time is given as a lump sum and not a function of the number of workers (group size). This paper, therefore, revisits the work of Jaber and Bonney [10], which assumed learning to follow the WLC and the production cost a lump sum, by assuming the learning curve to follow that of Camm and Womer [9]. A modified version of the model of Jaber and Bonney [10] is numerically compared to the original one to draw some insights and conclusions on the importance of accounting for group size along with cumulative production when investigating such problems.

The numerical example uses learning data from Peltokorpi and Niemi [11], who studied the effects of group size and learning on assembly performance. It worth noting that the literature on lot sizing and learning in production [3] has assumed the learning curve to follow that of Wight [4], with a few papers assuming a plateau learning curve [5], which are univariate with the independent variable being cumulative production (number of repetitions). Glock et al. [3] showed that none assumed a bivariate model with the group size and cumulative production as independent variables. This short communication addresses this limitation by modifying the Jaber and Bonney [10] model and contributing a new lot-sizing model with learning in production.

The paper has four more sections. The next section, Section 2, briefly introduces the reader to the two forms of the learning curves considered here. Section 3 provides the mathematics of the model of Jaber and Bonney [10] that is followed by the modified version. Section 4 is for numerical results and sensitivity analysis. The paper concludes in Section 5.

2. Problem description, assumptions and notations

This paper revisits the economic manufacture/production quantity (EMQ/EPQ) model for learning in production. It takes a model from the literature, which is that of Jaber and Bonney (1998); henceforth, referred to as the JBM98. As workers learn with repetition, their performance improves; i.e., the time per unit reduces with each unit produced following a learning curve. Subsequently, inventory builds in a convex form, thus, affecting the calculation of the average inventory level per cycle. The JBM98 considered aggregated learning curves of many workers and excluded the number of workers. The JBM98 is modified by using a learning

curve that has two dependent variables, the number of workers and cumulative production.

The new model and the JBM98 are analyzed to draw some insights.

The JBM98 and the modified model have the following assumptions: (1) learning occurs only in production, (2) there is no forgetting as a result of intermittent production, (3) demand is deterministic and less than the initial production rate without learning, (4) no shortages are allowed, and (5) the planning horizon is infinite.

The notations used in this paper are listed below:

i	Subscript indicating the cycle number
K	Setup cost (\$)
d	Demand rate (units/day)
w	Worker's daily wage (\$/day)
n	Number of workers or group size, and a parameter of the CWLC
h	Holding cost (\$/unit/day)
T_1	Time to produce the first unit on the learning curve, a parameter of the WLC
b	Learning exponent, a parameter of the WLC
T_x	Time to produce unit x on the WLC
$y_{1,n}$	Time to produce the first unit on the learning curve, a parameter of the CWLC
α	Group size exponent, a parameter of the CWLC
β	Learning exponent, a parameter of the CWLC
$y_{x,n}$	Time to produce unit x on the CWLC
q_i	Lot size (units), a decision variable

3. Background to univariate and bivariate learning curves

The earliest reported and empirically validated learning curve is that of Wright [4]. He showed that the cost (time) to produce a unit (airplane) reduces by a constant percentage with doubling output. The learning curve, WLC, is of the form:

$$T_x = T_1 x^{-b} \quad (1)$$

,where T_x is the time to produce the x th unit, T_1 is the time to produce the first unit, x is cumulative production, and b is a learning exponent ($0 \leq b < 1$). The bivariate learning curve is of the form:

$$y_{x,n} = y_{1,n} n^{-\alpha} x^{-\beta} \quad (2)$$

where $y_{1,n}$ is the time to produce the first unit by group size n , α is a parameter describing the contribution of group size, and β is a learning parameter. Eq. (2) is similar to the model of Camm and Womer [9] except for, in their model, the dependent variable represents the production rate instead of the unit time, and n and x are raised to nonnegative parameters. Although Eq. (1) does account for the group size, n , it worth noting that the parameters of the learning curve in Eq. (1), T_1 and b vary with group size, so, they should be represented as T_1^n and b_n instead, where $T_1^{n-1} \neq T_1^n \neq T_1^{n+1}$ and $b_{n-1} \neq b_n \neq b_{n+1}$. When $n = 1$, then Eq. (2) should be equal to Eq. (1); however, when $n \geq 2$ it is not necessary, as one may surmise, that Eq.(1) is equal to Eq. (2); i.e., $T_1^n \neq y_{1,n} n^{-\alpha}$ and $b_n \neq \beta$ as the result of fitting empirical data show later in this paper. One needs empirical or experimental data to compare the performance of the fits of the two models, Eq. (1) and Eq. (2). In particular, the data should describe the performance of a group of different sizes performing the same number of product repetitions. Group members should be comparable in experience and skills. No such data have been available in the literature, until recently. Peltokorpi and Niemi [11] studied in a laboratory setting the

effects of group size, varying from one to four members, and learning (four repetitions per group) on product assembly time. The product was built from mechanical components found in real products. Sixty-eight undergraduate students, surrogates to industrial workers, participated in the experiment. They found that the mean assembly time decreases as the group size and the number of repetitions increase. The data from this study are used to estimate the parameters of Eqs. (1) and (2), which are then used in the numerical example of this paper.

From Eqs. (1) and (2), the total time to produce q_i units in cycle i are determined, respectively, by integrating these equations over the proper limits as:

$$t(q_i) = T_1 \sum_{x=1}^{q_i} (x + u_i)^{-b} = \int_{u_i}^{q_i+u_i} T_1 x^{-b} dx = \frac{T_1}{1-b} [(q_i + u_i)^{1-b} - (u_i)^{1-b}] \quad (3)$$

$$\begin{aligned} t(q_i, n) &= y_1 n^{-\alpha} \sum_{x=1}^{q_i} (x + u_i)^{-b} = \int_{u_i}^{q_i+u_i} y_1 n^{-\alpha} x^{-\beta} dx \\ &= \frac{y_1 n^{-\alpha}}{1-\beta} [(q_i + u_i)^{1-\beta} - (u_i)^{1-\beta}] \end{aligned} \quad (4)$$

where $0 \leq u_i \leq \sum_{j=1}^i q_j$, is cumulative experience measured in number of units at beginning of cycle i . When forgetting is accounted for, then $0 < u_i < \sum_{j=1}^{i-1} q_j$ when there is partial transfer of learning from cycle to cycle. Total forgetting occurs when $u_i = 0$. This technical note will not consider forgetting as there are no available models that describe how forgetting occurs in a group of workers. Note that some references integrate Eqs. (1) and (2) over the limits $\left[u_i + \frac{1}{2}, q_i + u_i + \frac{1}{2}\right]$ for slightly better accuracy; e.g., $\int_{u_i+1/2}^{q_i+u_i+1/2} T_1 x^{-b} dx - T_1 \sum_{x=1}^{q_i} (x +$

$u_i)^{-b} < \int_{u_i}^{q_i+u_i} T_1 x^{-b} dx - T_1 \sum_{x=1}^{q_i} (x + u_i)^{-b}$, with the difference becoming infinitesimal as experience is gained; i.e., as u_i becomes larger. However, to stay in line with [10], we have excluded the half from the limits.

4. EPQ models with learning

Eqs. (3) and (4) are used to determine the average inventory or the holding cost per cycle. In the classical EPQ model, inventory during production builds at a rate $p - d$, where p and d are, respectively, the production and demand rates, $p > d$. The average inventory level of the production time segment, $0 \leq t \leq q/p$, is determined as $\int_0^{q/p} I(t) dt$, where $I(t) = pt - dt$. The maximum inventory level in a cycle is $z = I(t = q/p) = q(1 - d/p)$, which is depleted to zero at a rate d . The average inventory level of the depletion time segment, $q/p \leq t \leq q/d$, is determined as $\int_{q/p}^{q/d} I(t) dt = \int_{q/p}^{q/d} (z - dt) dt$. The average inventory level per cycle is $\int_0^{q/p} I(t) dt + \int_{q/p}^{q/d} I(t) dt = q^2(1 - d/p)/2d$. The total cost per cycle is $TC(q) = K + hq^2(1 - d/p)/2d$ and the total cost per unit time, $TCU(q) = TC(q)/L = Kd/q + hq(1 - d/p)/2$, where $L = q/d$ is the cycle time and K and h are the setup cost per cycle and unit holding cost per unit time. The optimal quantity that minimizes $TCU(q)$ is determined by setting the first derivative of $TC(q)$ equals to zero and solving for q to get $q^* = \sqrt{2Kd/h(1 - d/p)}$. When modelling the holding costs per cycle, the same approach as the one described here is followed, where $I(t) = q(t) - dt$, as $q(t)$ becomes a function of time in Eqs. (3) and (4). In the EPQ cost function, the production cost per unit of time, $c_p d$,

where c_p is the unit production cost (mainly labor, processing and material costs), could be ignored as it does not have any effect. It is not so as seen in the next. The first describes the mathematics of the Jaber and Bonney [10] model, JBM98, which is based on the WLC, while the second modifies JBM98 by assuming learning to follow the form in Eq. (2).

The unit time total cost function of JBM98 for cycle i is the sum of the unit time setup (Term 1; $SCU(q_i)$), production/labor (Term 2; $PCU(q_i)$), and holding (Term 3; $HCU(q_i)$) costs, and written as:

$$TCU_{JB}(q_i) = \frac{Kd}{q_i} + c_p \frac{T_1 d}{(1-b)q_i} [(q_i + u_i)^{1-b} - u_i^{1-b}] + h \left\{ \frac{q_i}{2} - \frac{T_1 d [(q_i + u_i)^{2-b} - u_i^{2-b}]}{(1-b)(2-b)q_i} + \frac{T_1 d u_i^{1-b}}{(1-b)} \right\} \quad (5a)$$

The unit time material cost have been excluded from Eq. (5) as it will not affect the lot size in cycle i , q_i . Eq. (5a) and subsequently (5b) below are convex in q_i (Jaber and Bonney, 1998). The optimal lot size was found using numerical search, Newton-Raphson method, as a closed form was difficult to find. Note that Jaber and Bonney [10] assumed c_p to be given as a lump sum and did not relate it to the number of workers. This is modified and the production/labor cost is wage, w , multiplied by the number of workers, n , or $c_p = wn$. Following Eqs. (1) and (2), we have mentioned that our numerical analysis showed that T_1 and b vary with group size and subsequently should be replaced with T_1^n and b_n , respectively. So, Eq. (5a) could be rewritten as:

$$TCU_{JB}(q_i) = \frac{Kd}{q_i} + wn \frac{T_1^n d}{(1-b_n)q_i} [(q_i + u_i)^{1-b_n} - u_i^{1-b_n}] + h \left\{ \frac{q_i}{2} - \frac{T_1^n d [(q_i + u_i)^{2-b_n} - u_i^{2-b_n}]}{(1-b_n)(2-b_n)q_i} + \frac{T_1^n d u_i^{1-b_n}}{(1-b_n)} \right\} \quad (5b)$$

The unit time total cost function for the modified model for cycle i is determined as described above and using Eq. (4), and is written as:

$$TCU_M(q_i) = \frac{Kd}{q_i} + wn \frac{y_1 n^{-\alpha} d}{(1-\beta)q_i} \left[(q_i + u_i)^{1-\beta} - u_i^{1-\beta} \right] + h \left\{ \frac{q_i}{2} - \frac{y_1 n^{-\alpha} d \left[(q_i + u_i)^{2-\beta} - u_i^{2-\beta} \right]}{(1-\beta)(2-\beta)q_i} + \frac{y_1 n^{-\alpha} d u_i^{1-\beta}}{(1-\beta)} \right\} \quad (6)$$

For a fixed value of n , Eq. (6) is convex in q_i as shown in Jaber and Bonney (1998). Note that n , in Eq. (6) is not a decision variable since Eq. (6) is, for a given value of q_i , a continuously increasing function in n , as shown in Appendix A.

Note that when there is no learning, $b = 0$ in Eqs(5a) and (5b) and $\alpha = 0$ and $\beta = 0$ in Eq. (6), the models reduce to the economic manufacture/production quantity (EMQ/EPQ) model, which is of the form $TCU(q) = \frac{Kd}{q} + wnd/p + hq(1-d/p)/2$ and whose solutions is $q^* = \sqrt{2Kd/h(1-d/p)}$ and $TCU_{EMQ} = \sqrt{2Kdh(1-d/p)}$, where $p = 1/T_1$ in Eqs(5a) and (5b) and $p = 1/y_1$ in Eq. (6). On the other hand, when experience is gained; i.e., $u_i \rightarrow \infty$, the same equations reduce to the economic order quantity (EOQ) model which is of the form $TCU(q) = \frac{Kd}{q} + wnd/p + hq/2$, and whose solutions is $q^* = \sqrt{2Kd/h}$ and $TCU_{EOQ} = \sqrt{2Kdh}$.

Eqs. (5b) and (6) could be optimized using Excel Solver. However, we have also developed the following search solution procedure for Eqs. (5b) and (6).

STEP 1: Input the parameters T_1^n and b_n (or y_1, α , and β), n , K , d , w , and h .

STEP 2: For cycle i , starting with $i = 1$, let $j=1$, $u_i = 0$, $q = q_j = \sqrt{2Kdh}$, determine $TCU(q_j)$. Set Value1 = $TCU(q_j)$.

STEP 3: Increment q_j from Step 1 by 1, determine $TCU(q_j + 1)$. Set Value2 = $TCU(q_j + 1)$ and $q = q_j + 1$.

STEP 4: If Value1 < Value2, then set lot size to $q = q_j = q_i$, ELSE GO TO STEP 3, repeat until $TCU(q^* + k - 1) > TCU(q^* + k) < TCU(q^* + k + 1)$, where $j = 1, \dots, k$ and k is the number of increments each time STEP 3 is visited from STEP 4. The optimal solution for cycle i occurs when $q_i^* = q^* + k$.

STEP 5: Let $u_{i+1} = u_i + q_i^*$, and $i = i + 1$. GO TO STEP 2 and calculate the optimal lot size for the next cycle I and repeat STEPS 3-5.

STEP 6: REPEAT STEPS 2-5 as needed, that is for $i = 1, \dots, I$

The next section provides a numerical example to illustrate the behavior of both models and demonstrate how accounting for group size affect the inventory policy.

5. Numerical example

To have a meaningful comparison between the two models, Eqs. (1) and (2), and subsequently, Eqs. (5) and (6), empirical or experimental data have to be available to provide some guidance of estimating the parameters of the learning curves, which are T_1^n , b_n , y_1 , α , and β . As indicated earlier, the data from Peltokorpi and Niemi [11] were used to estimate them. Table 1 shows the learning curve parameters for the WLC for the four groups, those of the learning curve in Eq. (2) or Camm and Womer [9], CWLC, and the SSE and R^2 values. Unfortunately, only four repetitions were collected per group. The results would be more meaningful if more repetitions were performed and more group sizes were considered.

Table 1. Learning curve parameters

Model	n	T_1^n	b_n	MSE	R^2
WLC	1	40.947	-0.600	7.489	0.976
	2	24.759	-0.672	2.450	0.981
	3	20.478	-0.798	1.637	0.985
	4	16.830	-0.815	2.961	0.961
	y_1	α	β	MSE	R^2
CWLC	41.730	-0.717	-0.656	24.685	0.981

The values of T_1^n and y_1 in Table 1 are in minutes. The results in Table 1 confirm that $T_1^n \neq y_1 n^{-\alpha}$, i.e., $T_1^{n=1} = 40.947 \neq 41.73 \times 1^{-0.717}$, $T_1^{n=2} = 24.759 \neq 41.73 \times 2^{-0.717} = 25.387$, $T_1^{n=3} = 20.478 \neq 41.73 \times 3^{-0.717} = 18.982$, and $T_1^{n=4} = 16.83 \neq 41.73 \times 4^{-0.717} = 15.444$. Further analysis showed that $T_1^{n-1} > T_1^n > T_1^{n+1}$ and $b_{n-1} < b_n < b_{n+1}$, where we found the following relationships $T_1^n = T_1(n) = 40.19n^{-0.633}$, with $R^2 = 0.9933$, and $b_n = b(n) = 0.0771n + 0.5285$, with $R^2 = 0.9345$, respectively. These relationships suggest a learning curve of the following form:

$$T(n, x) = T_1(n)x^{b(n)} = 40.19n^{-0.633}x^{-(0.0771n+0.5285)}$$

Note that, although we found two slightly better fits for $b(n)$, which are $b(n) = 0.1662\ln(n) + 0.5285$, with $R^2 = 0.9411$, and $b(n) = 0.593n^{0.2366}$, with $R^2 = 0.9411$, the linear relationship shown above produced equal average percentage absolute deviation (4.71%) to the logarithmic form and better than that of the power(5.15%). The results are summarized in Table 2 for $n = 1, 2, 3$, and 4 and $x = 1, 2, 3$, and 4, where $PADj = ABS[(y(n, x) - Model j)/y(n, x)]$. The linear is a simple representation and it was, therefore, suggested. However, The authors believe

that $T_1(n)$ would not decrease in n , but rather would have a convex form, i.e., for a given n , say n^* , where $T_1(n^* - 1) > T_1(n^*) < T_1(n^* + 1)$, or $T_1(n)$ may plateau at some value $T_{1,min}$. They also believe that $b(n)$ would not increase (learning become faster) with more workers in the team (increasing n), they think for a given value of n , say n^{**} , $b(n)$ would become slower, suggesting a concave like behaviour for $b(n)$; i.e., $b(n^{**} - 1) < b(n^{**}) > b(n^{**} + 1)$, or plateauing at some value beyond which $b(n)$ will not increase with increasing n . There is some evidence in the project management literature. Brook's heuristics developed a resource-constrained project where activities are scheduled according to a priority (e.g., [12]). The results of the heuristics show that project duration reduces, and resource productivity improves as the number of resources (workers) increases. However, if the number of workers keeps increasing beyond a certain point, the project duration will not reduce further, and workers' productivity reduces as their idle time increases. Unfortunately, there is no data on group learning curves with a sufficient number of workers and repetitions. Viewing group performance as the additive and interactive efforts of individuals in a group [11], there is a lack of group learning curve models derived from measurable practical elements from real industrial tasks performed by a group. Furthermore, no group forgetting curves have been developed in the literature [3]. The literature that treated learning and forgetting considered learning and forgetting curves as aggregated curves of individual learners. Later studies argued that such an approach is inaccurate when modelling individual learners in a group [3]. The mentioned limitations of group learning curves, including forgetting, which limit this study, are on the authors' research agenda, hopefully, to be addressed in future work.

Table 2. Comparison between the group learning curve and three proposed ones.

n	x	$y(n, x)$	Logarithmic	Linear	Power	PAD 1	PAD 2	PAD 3
			Model 1	Model 2	Model 3			
1	1	41.73	40.18	40.18	40.18	3.71%	3.71%	3.71%
1	2	26.48	27.86	26.41	26.64	5.18%	0.29%	0.58%
1	3	20.30	22.48	20.66	20.94	10.76%	1.77%	3.19%
1	4	16.81	19.31	17.35	17.66	14.90%	3.25%	5.07%
2	1	25.39	25.91	25.91	25.91	2.06%	2.06%	2.06%
2	2	16.11	16.58	16.14	15.96	2.93%	0.19%	0.92%
2	3	12.35	12.77	12.24	12.03	3.45%	0.89%	2.62%
2	4	10.22	10.61	10.06	9.84	3.81%	1.65%	3.81%
3	1	18.98	20.04	20.04	20.04	5.59%	5.59%	5.59%
3	2	12.05	12.24	11.84	11.76	1.64%	1.74%	2.36%
3	3	9.23	9.18	8.70	8.61	0.61%	5.79%	6.74%
3	4	7.65	7.48	6.99	6.90	2.17%	8.56%	9.72%
4	1	15.44	16.71	16.71	16.71	8.18%	8.18%	8.18%
4	2	9.80	9.87	9.35	9.44	0.73%	4.57%	3.66%
4	3	7.51	7.26	6.66	6.76	3.39%	11.32%	9.98%
4	4	6.22	5.83	5.24	5.34	6.20%	15.82%	14.20%
						4.71%	4.71%	5.15%

Now back to solving Eqs.(5b) and (6), let us assume a 480 minutes working day, then the initial production rates from T_1 and y_1 are respectively, 11.72 and 11.50 units per day. The models in Eqs. (5) and (6) assume no shortages, so the demand rate is assumed to be $d = 10 < \min(11.50, 11.72)$. We assume a daily wage of $w = \$100$, a setup cost of $K = \$200$, and a unit daily holding cost of $h = \$0.2/\text{unit}/\text{day}$. The initial cumulative experience is $u_1 = 0$. The optimal production policies are determined from Eqs. (5) and (6) by using Excel Solver. Table 3 shows the optimal policies for JBM98 and the Modified model for ten consecutive cycles. The cumulative experience values, for example, for the case when $n = 1$, for ten consecutive cycles

is $u_1 = 0, u_2 = 179, u_3 = 325, u_4 = 469, u_5 = 613, u_6 = 756, u_7 = 899, u_8 = 1042, u_9 = 1185$, and $u_{10} = 1327$, respectively, for JBM98. The u_i for the other cases, $n = 2, 3$, and 4 , and for the modified model are shown in Table 2. It also shows, along the $TCU(q_i)$ values, its three cost components, setup, $SCU(q_i)$, labor/production, $PCU(q_i)$, and holding, $HCU(q_i)$, and the lot size, q_i , are listed with their average values ($\bar{q}, \overline{SCU}, \overline{PCU}, \overline{HCU}$, and $\overline{TCU}$ in bold).

<Insert Table 3 about here (provided on pages 15 and 16)>

The results show that JBM98 recommends hiring three workers ($n = 3$) to minimize the unit time cost (in Bold), where $\bar{q} = 146.6$ units and $\overline{TCU} = 29.90$ \$/day. Learning being relatively faster ($b = -0.798$, Table 1) and, further, lower production cost for three-worker group explains its slightly better performance in comparison to other group sizes. The modified model behaved differently. The optimal policy is to have one worker ($n = 1$) where $\bar{q} = 146.2$ units and $\overline{TCU} = 29.94$ \$/day. Higher inefficiency ($\alpha = -0.717$, Table 1) and, further, rising production cost in larger groups mainly explains the decreasing performance (or increasing total cost) with increasing group size. The percentage differences in the average lot sizes and total costs between the Modified and the JBM98 model are $-0.48, 0.14, 0.14$, and 0.07% for $n = 1, 2, 3$, and 4 , respectively, for the first and $-1.56, 0.67, 3.08$, and 3.53% , respectively, for the second. The percentage differences in $\overline{SCU}$ and $\overline{HCU}$ values are $0.45, -0.11, -0.20$, and -0.16% and $0.10, 0.01, -0.16$, and -0.18% , respectively. The main variation has to do with the learning curve in Eq. (2), where the time to assemble a unit (performance) is a decreasing function of both the number of workers and the cumulative production. The percentage difference in the average production cost, $\overline{PCU}$, for $n = 1, 2, 3$, and 4 are $-27.60, 9.18, 37.99$, and 40.38% , respectively. These results show that although the form of the learning curve has a significant effect on the

production cost, it has little (almost insignificant) effect on the behavior of the production-inventory policy. The modified, bivariate, model performs better only for $n = 1$, and due to lower production cost. In general, WLC model performs better. The results suggest using simple WLC model when modelling the lot-sizing problem and not a complex one like the group size learning curve.

Sensitivity analysis

A sensitivity analysis was conducted to find out how changes in group learning profiles (or data) affect the difference in the average unit time cost, $\overline{TCU}$, between WLC and the modified model. For this end, the parameter values of T_1 and b (WLC) in Table 1 were modified, in the same proportion for each group size. The corresponding learning data for four consecutive repetitions was generated for each group size. The modified model was then fitted to the data to obtain parameter values of y_1 , α and β . Both T_1 and b have three different values, i.e., in total $3^2 = 9$ alternative scenarios were studied. The values may remain the same, decrease by 30%, i.e., $0.7T_1$ and $0.7b$, or T_1 increases by 30% and b by 20%, i.e. $1.3T_1$ and $1.2b$, respectively. Note that $b < 1$ for WLC limits the fastest learning speed to $1.2b$. Studying different learning parameter values and group learning profiles, as presented above, is justified as a similar variation between the groups was also found in the experimental study in [13].

Figure 1 is a pictorial of the differences (%) in the unit time cost, $\overline{TCU}$, for each group size and various combinations of T_1 and b when using the Modified model instead of the WLC (or JBM98) model. The reference line is from the data in Table 1.

Figure 1. Difference (%) in $\overline{TCU}$ for each group size (n) and various combinations of T_1 and b when using the Modified model instead of WLC.

Figure 1 shows that, for each scenario of $n = 1$, the modified model predicts lower $\overline{TCU}$ than the WLC does. The modified model also predicts lower $\overline{TCU}$ for $n=3$ and 4, for the scenarios when learning speed increases to $1.2b$. In all other cases, the modified model predicts higher $\overline{TCU}$. When learning speed decreases to $0.7b$ (the three topmost curves) the difference in $\overline{TCU}$ between the models becomes significant for $n=3$ and 4 ($>5\%$). Performance of models is more sensitive to changes in the learning exponent b than the unit time in the first repetition, T_1 . The scenarios relating to $1.2b$ correspond to learning rates of 51 to 61%, depending on the group size. These are extremely fast learning rates and uncommon for industrial tasks (e.g., [12]). The scenarios of $0.7b$ correspond to learning rates of 67 to 75%, which are more realistic. The

sensitivity analysis suggests using the simple WLC model (repetition as a dependent variable) for each group size instead of a bivariate (repetition and group size as independent variables) model when studying the effects of learning on the lot-sizing problem.

6. Summary and conclusions

This paper revisited a lot-sizing problem with learning in production [10] by assuming the learning curve to be a bivariate [9] rather than a univariate [4] function. The first has performance improvements by increasing the worker's group size and cumulative production. The latter does not account for the group size. The two learning curves were fitted to data collected from a laboratory study, where students (surrogates for workers) of different group sizes assembled a near-real product. The univariate model for the four group sizes fitted the data better than the bivariate learning curve model. The production-inventory model of Jaber and Bonney [10], which assumes a univariate learning curve was modified assuming a bivariate learning curve. The inventory policies of the two models were compared. It has been shown that using a bivariate learning curve model has an insignificant effect on the behavior of the production-inventory policy. This finding may suggest favoring Wright's learning curve for its simple mathematics. The results, however, showed that considering group size when modeling the lot-sizing problem can have a significant effect on unit production and, consequently, total cost, depending on the learning profiles of groups. It worth noting that the data used in this study have some limitations. The number of repetitions (totaling four) and the group sizes (from one to four students) considered are not enough to generalize the results. The diminishing returns with increasing group size indicate that, at some point, additional workers will reduce group productivity (see also [14, 15]). This limitation in group size is a consequence

of limited work content and workspace. On the other hand, some industrial tasks demand physical or cognitive capability that an individual worker may not have, thus, additional workers will improve productivity up to a threshold point [e.g., 12]. Consequently, the bivariate group learning curve studied in this paper could be further developed by modeling both the group size and learning parameters as a function of group size. Unlike Jaber and Bonney [10], the forgetting effect was not considered as there is neither work that studies forgetting in groups of varying sizes nor data that supports it. The listed limitations could be addressed in future work shall data become available. New group learning data and models would show the production conditions that benefit from modeling the effect of group size alongside learning in the lot-sizing problem. This paper, therefore, could be the start of a new research path.

Appendix A:

Treating n as a real number, and differentiating Eq. (6) with respect to n it we get the first partial derivative as:

$$\begin{aligned} \frac{\partial}{\partial n} TCU_M(q_i, n) &= (1 - \alpha)w \frac{y_1 n^{-\alpha} d}{(1 - \beta)q_i} \left[(q_i + u_i)^{1-\beta} - u_i^{1-\beta} \right] \\ &+ h \left\{ \alpha \frac{y_1 n^{-1-\alpha} d \left[(q_i + u_i)^{2-\beta} - u_i^{2-\beta} \right]}{(1 - \beta)(2 - \beta)q_i} - \alpha \frac{y_1 n^{-1-\alpha} d u_i^{1-\beta}}{(1 - \beta)} \right\} \end{aligned}$$

The second partial derivative is:

$$\begin{aligned}
& \frac{\partial^2}{\partial n^2} TCU_M(q_i, n) \\
&= -\alpha(1-\alpha)w \frac{y_1 n^{-1-\alpha} d}{(1-\beta)q_i} [(q_i + u_i)^{1-\beta} - u_i^{1-\beta}] \\
&+ h \left\{ -(1+\alpha)\alpha \frac{y_1 n^{-2-\alpha} d [(q_i + u_i)^{2-\beta} - u_i^{2-\beta}]}{(1-\beta)(2-\beta)q_i} + (1+\alpha)\alpha \frac{y_1 n^{-2-\alpha} d u_i^{1-\beta}}{(1-\beta)} \right\}
\end{aligned}$$

Since $0 < \alpha, \beta < 1$, $q_i > 0$, $u_i > 0$, $y_1 > 0$, and $d > 0$, then first term in the second partial derivative is negative for all values of n . If we set the second term to less than zero, then it reduces to

$$-\frac{[(q_i + u_i)^{2-\beta} - u_i^{2-\beta}]}{(2-\beta)q_i} + u_i^{1-\beta} < 0 \Rightarrow \frac{[(q_i + u_i)^{2-\beta} - u_i^{2-\beta}]}{(2-\beta)q_i} > u_i^{1-\beta} \Rightarrow \frac{(q_i + u_i)^{2-\beta} - u_i^{2-\beta}}{u_i^{1-\beta}} > 0, \text{ which is true since}$$

$q_i > 0$ and $u_i > 0$. Therefore, the first and second partial derivatives are, respectively, positive and negative for all values of $n > 0$, making Eq.(6) a continuously increasing function in n . This suggests that it is better to keep the group size at a minimum.

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Table 3. Optimal policies for JBM98 and modified model

JBM98 (WLC)							
n	i	u_i	q_i	$SCU(q_i)$	$PCU(q_i)$	$HCU(q_i)$	$TCU(q_i)$
1	1	0	179	11.17	9.49	15.47	36.14
1	2	179	146	13.70	3.13	14.12	30.95
1	3	325	144	13.89	2.37	14.05	30.30
1	4	469	144	13.89	1.96	14.11	29.96
1	5	613	143	13.99	1.70	14.05	29.74
1	6	756	143	13.99	1.52	14.08	29.58
1	7	899	143	13.99	1.38	14.10	29.46
1	8	1042	143	13.99	1.27	14.12	29.37
1	9	1185	142	14.08	1.18	14.03	29.29
1	10	1327	142	14.08	1.11	14.04	29.23
			146.90	13.68	2.51	14.22	30.40
2	1	0	180	11.11	9.60	16.70	37.41
2	2	180	144	13.89	2.55	14.20	30.64
2	3	324	143	13.99	1.87	14.16	30.01
2	4	467	143	13.99	1.51	14.19	29.69
2	5	610	142	14.08	1.29	14.11	29.48
2	6	752	142	14.08	1.13	14.12	29.34
2	7	894	142	14.08	1.02	14.13	29.23
2	8	1036	142	14.08	0.93	14.13	29.15
2	9	1178	142	14.08	0.86	14.14	29.08
2	10	1320	142	14.08	0.80	14.14	29.02
			146.20	13.75	2.16	14.40	30.30
3	1	0	187	10.70	9.75	17.69	38.13
3	2	187	143	13.99	1.55	14.22	29.76
3	3	330	142	14.08	1.08	14.15	29.31
3	4	472	142	14.08	0.84	14.16	29.09
3	5	614	142	14.08	0.70	14.17	28.95
3	6	756	142	14.08	0.60	14.17	28.86
3	7	898	142	14.08	0.53	14.17	28.79
3	8	1040	142	14.08	0.48	14.18	28.74
3	9	1182	142	14.08	0.43	14.18	28.70
3	10	1324	142	14.08	0.40	14.18	28.66
			146.60	13.74	1.64	14.53	29.90
4	1	0	191	10.47	10.49	18.25	39.21
4	2	191	143	13.99	1.53	14.24	29.75
4	3	334	142	14.08	1.06	14.16	29.30
4	4	476	142	14.08	0.83	14.17	29.08
4	5	618	142	14.08	0.68	14.18	28.94
4	6	760	142	14.08	0.59	14.18	28.85
4	7	902	142	14.08	0.52	14.18	28.78
4	8	1044	142	14.08	0.46	14.18	28.73
4	9	1186	142	14.08	0.42	14.18	28.69
4	10	1328	142	14.08	0.38	14.19	28.65
			147.00	13.71	1.69	14.59	30.00

Table 3. Optimal policies for JBM98 and modified model (Continued)

Modified (CWLC)							
n	i	u_i	q_i	$SCU(q_i)$	$PCU(q_i)$	$HCU(q_i)$	$TCU(q_i)$
1	1	0	177	11.30	8.48	15.47	35.24
1	2	177	145	13.79	2.37	14.13	30.29
1	3	322	144	13.89	1.74	14.14	29.77
1	4	466	143	13.99	1.41	14.09	29.49
1	5	609	143	13.99	1.21	14.12	29.32
1	6	752	142	14.08	1.07	14.05	29.20
1	7	894	142	14.08	0.96	14.06	29.11
1	8	1036	142	14.08	0.88	14.07	29.04
1	9	1178	142	14.08	0.81	14.08	28.98
1	10	1320	142	14.08	0.76	14.09	28.93
			146.20	13.74	1.97	14.23	29.94
2	1	0	181	11.05	10.17	16.73	37.95
2	2	181	145	13.79	2.85	14.28	30.92
2	3	326	143	13.99	2.10	14.14	30.23
2	4	469	143	13.99	1.71	14.17	29.87
2	5	612	142	14.08	1.47	14.09	29.65
2	6	754	142	14.08	1.30	14.11	29.49
2	7	896	142	14.08	1.17	14.12	29.37
2	8	1038	142	14.08	1.07	14.12	29.27
2	9	1180	142	14.08	0.99	14.13	29.20
2	10	1322	142	14.08	0.92	14.13	29.14
			146.40	13.73	2.37	14.40	30.51
3	1	0	185	10.81	11.24	17.47	39.52
3	2	185	145	13.79	3.16	14.34	31.29
3	3	330	143	13.99	2.34	14.18	30.51
3	4	473	143	13.99	1.91	14.21	30.10
3	5	616	142	14.08	1.64	14.12	29.85
3	6	758	142	14.08	1.45	14.13	29.66
3	7	900	142	14.08	1.31	14.14	29.53
3	8	1042	142	14.08	1.19	14.14	29.42
3	9	1184	142	14.08	1.10	14.15	29.33
3	10	1326	142	14.08	1.03	14.15	29.26
			146.80	13.71	2.64	14.50	30.85
4	1	0	188	10.64	12.07	17.96	40.66
4	2	188	145	13.79	3.40	14.37	31.57
4	3	333	143	13.99	2.53	14.21	30.72
4	4	476	143	13.99	2.07	14.22	30.28
4	5	619	142	14.08	1.77	14.14	29.99
4	6	761	142	14.08	1.57	14.14	29.80
4	7	903	142	14.08	1.41	14.15	29.65
4	8	1045	142	14.08	1.29	14.15	29.53
4	9	1187	142	14.08	1.20	14.16	29.44
4	10	1329	142	14.08	1.11	14.16	29.36
			147.10	13.69	2.84	14.57	31.10