Statistics of Ratios of Random Variables Arising in Analysis of Wireless Poisson Networks

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Abstract
In this paper, the author analyses statistics of some typical ratios of random variables (RVs) occurring in analysis of large-scale wireless networks with node locations modeled by Poisson point processes. In such networks, fading and path-loss effects are the most important factors affecting the signal-to-interference ratio (SIR) under co-channel interference (CCI) or when considering information-theoretic security issues related to the assessment of the secrecy outage probability (SOP) [1]. Under a short-range communication, fading is the main factor affecting the signal strength, and in many previous works, such as [2]-[4], ratios of RVs were analyzed by taking into account fading effects only.

In large-scale wireless networks (WNs) where the transmitter (Tx)-receiver (Rx) distance cannot be neglected, path-loss effects become significant factors affecting the signal strength. The problem of SIR statistical characterization becomes more intriguing if the Tx-Rx distances are random. Nevertheless, analytical results on the outage probability (OP) under CCI (on the coverage probability in cellular networks), and SOP were obtained for Poisson WNs where node locations were modeled by Poisson point processes (PPPs). But to the best of our knowledge, all reported results on ratio statistics had some restrictions summarized below.

- Most of the published works, such as [5]- [8], analyzed cellular-like networks where the closest or strongest Tx communicated with the probe Rx, and the other Txs were sources of CCI.
- The absolute majority of the reported results in cellular-like WNs considered Rayleigh fading only. It is well known, however, that real multipath fading distributions based on practical measurements very often cannot be represented by the Rayleigh fading models, and for obtaining reliable estimates, more sophisticated fading distributions must be applied in design and analysis considerations [9], [10]. Taking into account shadowed fading also results in non-Rayleigh statistics of channel gains [9]. Additionally, various signal-processing techniques applied in multi-antenna transmission may change the statistical distribution of the effective SIR [9], [11].

Reported results on other than Rayleigh fading distributions (see, e.g. [6]-[7]) used substitutions reducing real fading models to Rayleigh ones. Such approaches result in approximate estimates, and additionally they require extra efforts related to the substitutions. Obviously, general techniques applicable to arbitrary fading models rather than to some specific ones, are of special interest.

- 5G WNs are characterized by new technological trends such as spectrum sharing and cooperation [12]. To the best of our knowledge, SIR statistics in Poisson spectrum-sharing networks (where the primary network operates simultaneously with independent interferers (secondary network) within the same frequency band) were analyzed only in [13], [14]. In [13], one operational scenario in spectrum-sharing WNs was analyzed while in [14], SIR statistics were reported for two possible scenarios with application to the average rate evaluation.

In this paper, we derive statistics of ratios of random variables (RVs) occurring under a large variety of typical operational scenarios in cellular-like, spectrum-sharing and cooperative Poisson wireless networks. In contrast to previously reported results on cellular-like WNs, the results of this work are applicable to arbitrary fading distributions of propagation paths. Results of this work concerning spectrum-sharing WNs are more general than those reported in [13], [14] and include them as special cases. In our analysis, we take into account two main factors affecting the signal strength in large-scale wireless networks, which are path loss and fading. All results derived in this work are valid for arbitrary fading distributions of propagation paths. We present application examples showing practical importance of obtained analytical estimates for OP and SOP evaluation under various operational scenarios.

The remainder of this paper is organized as follows. In section 2, we introduce the system model, concepts, and statistical distributions used in our derivations. In section 3, we present analytical results on ratio statistics occurring in Poisson WNs. In section 4, numerical results are given. Finally, the conclusions are drawn in the last section.
II. Preliminaries

A. System model

We consider a random wireless network operating in the two-dimensional Euclidean space $\mathbb{R}^2$ and communicating with the probe Rx located at the origin based on an operational protocol. For instance, the probe Rx may communicate with the strongest or closest Tx. Under this operational scenario, in cellular-like WNs, the other Txs are sources of interference. In spectrum-sharing networks, the interference comes from the secondary Txs, which typically operate independently of the primary Txs. The primary Txs may cooperate.

In our analysis, we take into account path-loss and fading effects, which are two main factors affecting the signal strength in large-scale wireless networks. Typically, they are modeled by independent RVs. We assume arbitrary fading models of propagation paths and apply a standard distance-dependent path-loss model specified as [9]

$$l(x) = L_0||x||^{-\eta}$$

where $||.||$ means the Euclidean distance between the Tx and Rx, $L_0$ represents the path loss at $||x|| = 1$, and $\eta$ is the path-loss exponent. Typically, $2 < \eta \leq 6$ [9].

We assume that the locations of transmitting nodes are modeled by a PPP $\Phi_T$ with the density $\lambda_T$. Then under communication with the $k$th Tx located at $x_k \in \Phi_T$, the power gain $\gamma_k$ can be expressed as

$$\gamma_k = C_k g_{Tk} L_0 ||x_k||^{-\eta}$$

where $g_{Tk}$ is the fading power gain of the Tx-Rx link, and $C_k$ is a deterministic constant. For instance, $C_k$ may represent the transmit power. Typically, $C_k$ and statistics of $g_{Tk}$ are independent of $k$, and we omit this index below.

In spectrum-sharing networks, we assume that the fading coefficients of interfering links $g_i$ are arbitrary but identical for all nodes denoted as those of $g_1$.

An important application of results derived in this work includes analyzing the SOP in Poisson WNs. Broadcast nature of wireless propagation medium makes information leakage to unintended users an important problem in any wireless network. The SOP may be used as a measure of the physical layer security in wireless networks. It characterizes dominance to unintended users an important problem in any wireless network. Typically, they are modeled by independent RVs. We assume arbitrary fading models of propagation paths and apply a standard distance-dependent path-loss model specified as [9]

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where $E_r(\cdot)$ means the expectation with respect to $x$, and
$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}$ is a Mittag-Leffler’s function [16].

**Proof:** See [13, Proposition 2].

Then the following corollary immediately follows.

**Corollary 1:** The expectation $E\{X_{st}^{-\alpha}\}$, see (7), can be specified as

$$E\{X_{st}^{-\alpha}\} = \frac{1}{K T (\alpha + 1)}.$$  \hspace{1cm} (11)

The following propositions and lemmas specify statistics of ratios of RVs occurring in Poisson wireless networks under different operational scenarios.

**Proposition 1:** Let $y_k$ be independent and identically distributed (i.i.d.) RVs, and $E\{y_k^{-\frac{\pi}{\gamma}}\} = E\{y^{-\frac{\pi}{\gamma}}\}$ exists. Then for arbitrary statistics of power gain $g_T$, the cumulative distribution function (CDF) $F_{\max}(z)$ of an RV $r_{\max} = \max_{x_k \in \Phi_T} P_T g_T |x_k|^{-\alpha}$ can be expressed as

$$F_{\max}(z) = \exp\left[ -\lambda T \pi P_T^2 \frac{\pi}{2} y_k \frac{g_T^{-\frac{\pi}{\gamma}}}{E\{g_T^{-\frac{\pi}{\gamma}}\}} \right].$$  \hspace{1cm} (12)

**Proof:** See Appendix A.

It can be seen from (12) that $r_{\max}$ can be viewed as a negative power of an exponential RV $x_{\exp}$, i.e., $r_{\max} = x_{\exp}^{-\frac{\pi}{\gamma}}$.

**Corollary 2:** If $y_k$ in Proposition 1 represent interference coming from a PPP $\Phi_t$ transmit by the density $\lambda_t$, transmit power $P_t$, and fading statistics for each interfering node are arbitrary and identical (represented by those of $g_t$), then (12) reduces to

$$F_{\max}(z) = \exp\left[ -\lambda_t^2 \pi P_t^2 \frac{\pi}{2} \frac{g_t^{-\frac{\pi}{\gamma}}}{E\{g_t^{-\frac{\pi}{\gamma}}\}} \right].$$  \hspace{1cm} (13)

**Proof:** Eq. (13) follows from (12) where $y$ is a symmetrical $\alpha$-stable RV with $E\{y^{-\frac{\pi}{\gamma}}\}$ specified by (11), and where $K = K_t \equiv \pi \Gamma\left(1 - \frac{\pi}{\gamma}\right) \frac{1}{2} \lambda_t \Gamma\left\{\frac{\pi}{\gamma}\right\}$, see (8), and from an identity for the gamma function, $\Gamma(1+z)\Gamma(1-z) = \frac{\pi}{\sin{\pi z}}$ [17, vol. 3, II.3].

Under conditions of corollary 2 and Eaves’ locations forming the PPP $\Phi_e$, based on the PPP mapping theorem [18], one can find that the CDF expression of the strongest Eaves $F_{r_{\max}}(z)$ has the form of (13), and it can be expressed as

$$F_{r_{\max}}(z) = \exp\left[ -z^{-\frac{\pi}{\gamma}} \frac{\pi}{2} \frac{g_e^{-\frac{\pi}{\gamma}}}{E\{g_e^{-\frac{\pi}{\gamma}}\}} \right].$$  \hspace{1cm} (14)

It can be seen that the fading effects in (13)/(14) are canceled if the statistics of $g_T(g_t)$ and $g_T$ are identical.

In cellular-like networks, Slivnyak’s theorem can be used [18], and with $P_T = P_t$, and $\lambda_T = \lambda_t$, (13) further reduces to

$$F_{r_{\max}}(z) = \exp\left[ -z^{-\frac{\pi}{\gamma}} \frac{\pi}{2} \frac{g_T^{-\frac{\pi}{\gamma}}}{E\{g_T^{-\frac{\pi}{\gamma}}\}} \right].$$  \hspace{1cm} (15)

Then we consider the ratio of two RVs with the CDFs having the form (12).

**Corollary 3:** Let Rvs $a$ and $b$ be characterized by the respective CDFs $F_a(z) = \exp\left[-K_a z^{-\alpha}\right]$ and $F_b(z) = \exp\left[-K_b z^{-\alpha}\right]$. Then the CDF of the ratio $F_{\frac{a}{b}}(z)$ can be represented as

$$F_{\frac{a}{b}}(z) = \frac{1}{1 + \frac{K_a}{K_b} z^{-\alpha}}.$$  \hspace{1cm} (16)

**Proof:** Eq. (16) can be obtained by inserting $F_a(t)$ and $f_b(t)$ into (9).

**Proposition 2:** If $x$ is a symmetrical $\alpha$-stable RV, see (7), and $y$ is an arbitrary RV such that the MGF $M_{\frac{a}{b}}(-s)$ exists, then

$$M_{\frac{a}{b}}(-s) = M_{y^{-\alpha}}(-K s^\alpha).$$  \hspace{1cm} (17)

**Proof:**

$$M_{\frac{a}{b}}(-s) = \exp\left\{ -s \frac{\pi}{\gamma} \right\} = E_y \left\{ M_x \left( -s \frac{\pi}{\gamma} \right) \right\}$$

\[ \xrightarrow{\text{(a)}} E_y \left\{ \exp\left( -K_x \frac{s^\alpha}{y^{-\alpha}} \right) \right\} \xrightarrow{\text{(b)}} M_{y^{-\alpha}}\left( -K_x s^\alpha \right) \]  \hspace{1cm} (18)

where (a) is due (7), and (b) is due to the MGF definition.

**Corollary 4:** If $x$ and $y$ are two symmetrical $\alpha$-stable RVs with the respective parameters $K$ in (7) $K = K_x$ and $K = K_y$, then the MGF of the ratio $\frac{x}{y}$, $M_{\frac{x}{y}}(-s)$, can be expressed as

$$M_{\frac{x}{y}}(-s) = E_x \left\{ -K_x \frac{s^\alpha}{K_y} \right\}.$$  \hspace{1cm} (19)

**Proof:** Eq. (19) follows from (18) and (10).

**Corollary 5:** The PDF $f_{\frac{a}{b}}(z)$ and CDF $F_{\frac{a}{b}}(z)$ of the ratio $\frac{a}{b}$ of two $\alpha$-stable RVs specified in Corollary 3 can be expressed as

$$f_{\frac{a}{b}}(z) = \frac{1}{\alpha} \left( \frac{K_y}{K_x} \right)^{\frac{\pi}{gamma}} \times H_{2,2}^{1,1} \left[ \left( \frac{K_y}{K_x} \right)^{\frac{\pi}{gamma}} \left( 1 - \frac{1}{\alpha} \right), (0, 1) \right]$$

\[ \times E \left\{ \left( \frac{\pi}{gamma} \right)^{\frac{\pi}{gamma}} \right\} \]

$$F_{\frac{a}{b}}(z) = \frac{1}{\alpha} H_{3,3}^{1,2} \left[ z \left( \frac{K_y}{K_x} \right)^{\frac{\pi}{gamma}} \left( 1 - \frac{1}{\alpha} \right), (1, 1) \right]$$

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where $H[.]$ is the Fox-H function [19].

**Proof:** See Appendix B.

**Lemma 2:** If points $x_k \in \Phi_T$, and $y$ is an arbitrary RV such that $E\{y^{-\frac{\pi}{\gamma}}\}$ exists, then the CDF $F_{r_{\max}}(x)$ of the ratio

$$r_{\max} = \max_{x_k \in \Phi_T} C g_T |x_k|^{-\alpha} \left( \frac{y^{-\alpha}}{y^{-\alpha}} \right)$$

can be expressed as

$$F_{r_{\max}}(z) = E_y \left\{ \exp\left[ -\pi \lambda T (C L_0)^{\frac{\pi}{gamma}} \right] \times E \left\{ \left( \frac{\pi}{gamma} \right)^{\frac{\pi}{gamma}} \right\} \right\}$$

$$\times E \left\{ \left( \frac{\pi}{gamma} \right)^{\frac{\pi}{gamma}} \right\} \right\}$$

$$= M_{y^{-\alpha}} \left\{ -\pi \lambda T (C L_0)^{\frac{\pi}{gamma}} E \left\{ \left( \frac{\pi}{gamma} \right)^{\frac{\pi}{gamma}} \right\} \right\}$$

\[ \times E \left\{ \left( \frac{\pi}{gamma} \right)^{\frac{\pi}{gamma}} \right\} \right\} \]

where $M_{y^{-\alpha}}(-s)$ is the MGF of $y^{-\frac{\pi}{gamma}}$. 


Proof: See [13, eqs. (14)-(16)].

Corollary 5: If \( y \) in (22) is an \( \alpha \)-stable RV with the MGF (7) where \( \alpha = \frac{2}{7} \), then \( F_{t_m}(z) \) can be expressed as

\[
F_{t_m}(z) = E_{\gamma} \left( -Gz^{-\frac{2}{\gamma}} \right)
\]

where

\[
G = \frac{\pi \lambda_T (CL_0)^{\frac{2}{\gamma}} E \left\{ (g_T)^{\frac{2}{\gamma}} \right\}}{K}.
\]

It can be seen that an RV with the CDF of the form (23) is inverse to the Pillai RV [20]. Eq. (23) coincides with a result obtained in [13, eq. (8)] in a different way.

Lemma 3: If RVs \( p \) and \( b \) are characterized by CDFs of forms (23) and (12), respectively, i.e., \( F_p(z) = E_\alpha \left( -G_p z^{-\alpha} \right) \), and \( F_b(z) = \exp \left( -K_b z^{-\alpha} \right) \), then the ratio \( \frac{z}{b} \) follows an H-function distribution [21], and the CDF \( F_{\frac{z}{b}}(z) \) can be expressed as

\[
F_{\frac{z}{b}}(z) = H_{2,2}^1 \left[ \frac{G_p}{K_b} z^{-\alpha} \right] \left\{ (0,1), (0,1) \right\} \left\{ (0,1), (1,\alpha) \right\}.
\]

Proof: In view of the forms of \( F_p(z) \) and \( F_b(z) \), the integral in (9) specifying \( F_{\frac{z}{b}}(z) \) can be evaluated via a Laplace transform formula [16, eq. (11.10)], and we can obtain that

\[
F_{\frac{z}{b}}(z) = \psi_1 \left( -\frac{G_p}{K_b} z^{-\alpha} \right) \left\{ (1,1), (1,1) \right\} \left\{ (1,\alpha) \right\},
\]

where \( \psi_1(.) \) is the Wright generalized hypergeometric function [19]. It can be expressed via the Fox H-function as in [19, eq. (1.140)], and (25) follows.

Lemma 4: If \( y_k \), \( k = 1,2,... \) are arbitrary RVs such that \( E \left\{ y_k^{\frac{1}{2}} \right\} = E \left\{ y^{\frac{1}{2}} \right\} \) exists, an RV \( r_\Sigma \) for the primary Tx-probe Rx links, is a symmetrical \( \alpha \)-stable RV, see (7), with \( K = K_\Sigma \) specified as

\[
K_\Sigma = -\lambda E \pi (L_0 P_T)^{\frac{2}{\gamma}} E \left\{ y^{\frac{1}{2}} \right\} \times \Gamma \left( 1 - \frac{2}{\gamma} \right) E \left\{ g_\Sigma^{\frac{2}{\gamma}} \right\}.
\]

Proof: See Appendix C.

Corollary 6: If \( y_k \) in (26) are symmetrical \( \alpha \)-stable RVs characterized by the MGF (7), then in view of (11), (26) reduces to

\[
K_\Sigma = -\lambda E \frac{E \pi (L_0 P_T)^{\frac{2}{\gamma}}}{K T} \times \frac{1}{\Gamma \left( 1 - \frac{2}{\gamma} \right)} E \left\{ g_\Sigma^{\frac{2}{\gamma}} \right\}.
\]

The derived results are summarized in Table I.

IV. APPLICATIONS AND NUMERICAL RESULTS

In this section, we give a few examples of application in different types of wireless networks. In all figures, single points report simulation results.

A. Outage Probability in Dense Cellular-like and Spectrum-Sharing Networks for Arbitrary Statistics of Propagation Paths

Network densification was specified as the crucial factor in 5G evolution [22], [12]. In dense wireless networks, CCI dominates over background noise, and formulas obtained in section III for interference-limited scenarios can be used for OP and SOP specification under different operational scenarios.

First, we considered cellular-like networks where the probe Rx communicated with the strongest Tx, and the other Txs created interference. For such scenarios, \( \lambda_T = \lambda_1, P_T = P_1 \), and based on Slivynak’s theorem, we applied (15) for the OP evaluation. We assumed that each Tx was equipped with \( N_T \) antennas. OP curves for a few operational scenarios are shown in Fig. 1. For single-antenna transmission (\( N_T = 1 \)), we tested a few fading models, and our simulation results confirmed that the OP was invariant to the fading distribution. For \( N_T > 1 \), we assumed Tx maximal ratio combining through a unitary precoding vector. Then for Rayleigh fading models of transmission links, \( g_T \) followed a chi-squared distribution with \( 2N_T \) degrees of freedom, and \( g_t \) was an exponentially distributed RV [11, Appendix A]. The curves in Fig. 1 are given for a few path-loss exponent values, \( \eta = \{ 2; 6; 3.8 \} \), corresponding to typical propagation conditions [23]. Although both the useful and interfering signals boost as the path-loss exponent decreases, it can be seen that under considered conditions, path-loss exponent increasing resulted in OP decreasing.

Next, we analyzed the OP in cooperative spectrum-sharing networks where the primary nodes jointly communicated with the probe Rx while the secondary nodes jointly communicated with a secondary Rx. In this case, both the useful and interfering signal powers were symmetrical \( \alpha \)-stable RVs, and (21) with \( \alpha = \frac{2}{7} \) was applied to OP evaluation. We assumed line-of-sight (LOS) fading conditions (Rician fading [9]) with the Rician K-factor \( K_R \) for the primary Tx-probe Rx links, and non-LOS fading conditions (Rayleigh fading [9]) for the secondary Txs-probe Rx links. The curves in Fig. 2 are shown for \( \lambda_T / \lambda_1 = 10 \), \( P_T / P_1 = \{ 1; 2 \} \), and for path-loss exponent values \( \eta = \{ 2.6; 3.8 \} \). It can be seen that effects of the path-loss exponent on \( P_{out}(z) \) depend on values of \( z \), and effects of the LOS component (Rician K-factor \( K_R \)) on the OP is more evident for larger values of path-loss exponent.

B. Secrecy Outage Probability in Cellular-like and Cooperative Poisson Networks

In this subsection, we present application examples showing practical importance of the presented theoretical results for SOP evaluation. As earlier, we considered dense wireless networks where the interference dominates over the background noise. In all considered cases, we applied (5) for the SOP evaluation.

First we analyze the SOP in cellular-like networks.

For NCE scenarios with \( \Gamma_E = \min_{\Gamma_E \in \Phi_E} \Gamma_E \), [5], and for arbitrary fading models of propagation paths, the CDFs of \( \Gamma_R \) and \( \Gamma_E \) can be expressed with the help of (13) and (14),
respectively. Thus the SOP can be evaluated with the help of (16) and (5) as

\[ P_{\text{out-cel,CE}}(R_s) = \frac{1}{1 + K_{\text{NCE}} \exp \left( -\frac{2\ln 2}{\eta} R_s \right)} \]  

(28)

where \( K_{\text{NCE}} = \frac{\lambda_T E \left( (g_T)^{\frac{2}{\alpha}} \right)}{\lambda_E E \left( (g_E)^{\frac{2}{\alpha}} \right)} \).

The SOP estimates versus the secrecy rate \( R_s \) are shown in Fig. 3 for a few \( \lambda_T/\lambda_E \) and path-loss exponent values. We tested a few identical fading models of \( g_T \) and \( g_E \) and confirmed our theoretical result stating that the SOP in this case is independent of a concrete fading distribution.

For CE scenarios, a SOP formula is not directly given in section III, but it can easily be obtained based on the presented results. \( \Gamma_E \) is an \( \alpha \)-stable RV, see lemma 4, and the CDF of \( \Gamma_R \) can be represented with the help of (15). Then via proposition 2, one can find that the MGF of the inverse variable \( \Gamma_R \), \( \mathcal{M}_{\Gamma_R}(-s) \), can be expressed as

\[ \mathcal{M}_{\Gamma_R}(-s) = \frac{1}{1 + s^{\frac{2}{\alpha}} \frac{2\pi K_\Sigma}{\eta \sin \pi s}} \]

(29)

where (29) with an MGF expression [16, eq. (19.4)], we conclude that the RV \( \Gamma_R \) is a Pillai RV with the CDF

\[ F_{\Gamma_R}(z) = 1 - \exp \left( -\frac{\eta \sin \pi \frac{z}{\eta} \Gamma}{2\pi K_\Sigma \frac{z}{\eta}} \right) \]

We tested a few identical fading models of transmitting and eavesdropping links, and again observed that for these conditions, the SOP is independent of concrete fading models.

For CE scenarios, one can note that Eave’s collusion significantly deteriorates the SOP. It can also be seen that under both scenarios, the SOP decreases as the path-loss exponent becomes larger.

The next analyzed scenario is that of cooperative spectrum-sharing networks with NCE. We consider scenarios where the probe Rx communicates with the strongest primary Tx under the interference coming from the secondary WN. Since in this case, \( \Gamma_R \) and \( \Gamma_E \) follow distributions specified by (23) and (14), respectively, the SOP can be evaluated with the help of lemma 3 as

\[ P_{\text{out-cel,CE}}(R_s) = \]

\[ H_{2,2}^{1,2} \left[ K_{\Sigma} \exp \left( -\frac{2\ln 2}{\eta} R_s \right) \right] (0, 1), (0, 1) \]  

(31)
where \( K_{SS} = \frac{\lambda_T}{\lambda_E} \left\{ \frac{4}{\eta^2} \right\} \Gamma \left( 1 + \frac{z}{\eta} \right) \frac{\lambda_E}{\lambda_T} \left\{ \frac{2}{\eta^2} \right\} \Gamma \left( 1 + \frac{z}{\eta} \right) \). SOP estimates evaluated with the help of (31) for identical fading models of transmitting and eavesdropping links are shown in Fig. 5 where we presented SOP estimates versus the ratio \( \lambda_T/\lambda_E \) for \( R_s = \{1;2\} \). As in the previous examples, we observed SOP independence of concrete fading model. It can be seen that the path-loss exponent effects on the SOP depend on networks’ parameters. For example, for all tested values of \( \lambda_T/\lambda_E \) and for the targeted secrecy rate \( R_s = 2 \), smaller SOP values are observed for larger path-loss exponent values, and more sophisticated effects are observed for the targeted secrecy rate \( R_s = 1 \). In the latter case, these effects are specified by the ratio \( \lambda_T/\lambda_E \).

V. Conclusion

In this paper, we derived statistics of ratios of random variables occurring in analysis of large-scale wireless Poisson networks where path-loss and fading effects are basic factors affecting signal strength. Main applications of derived theoretical results include evaluation of outage probability and secrecy outage probability in cellular-like and spectrum-sharing networks under different operational scenarios. Since the SIR is the main factor affecting various performance metrics of WNs, the presented results can successfully be applied to their evaluation. All results obtained in this work are applicable to arbitrary fading statistics of transmission links.

References

Fig. 4: Secrecy outage probability $P_{\text{out}}(R_s)$ in cellular-like WNs for CE scenarios and identical fading models of transmitting and eavesdropping links.

Fig. 5: Secrecy outage probability $P_{\text{out}}(R_s)$ versus $\lambda_T/\lambda_E$ in spectrum-sharing WNs for NCE scenarios and identical fading models of transmitting and eavesdropping links.

APPENDIX A
PROOF OF PROPOSITION 1

The CDF $F_{r_{\text{max}}}(z)$ can be specified as

$$F_{r_{\text{max}}}(z) = \Pr \left\{ \max_{x_k \in \Phi_T} \frac{P_T g_{T_k} ||x_k||^{-\eta}}{y_k} \leq z \right\}$$

$$\overset{(a)}{=} E \left\{ \prod_{x_k \in \Phi_T} E_{y_k} \left\{ F_{g_T} \left( z ||x_k||^{-\eta} P_T^{-1} y_k \right) \right\} \right\}$$

$$\overset{(b)}{=} \exp \left[ -\lambda_T \int_{\mathbb{R}^2} E_{y_k} \left\{ \left( 1 - F_{g_T} \left( \frac{z}{||y|| P_T^{-1} y_k} \right) \right) \right\} dx \right]$$

$$\overset{(c)}{=} 2 \eta \int_0^{\infty} t^{\frac{\eta-1}{\eta}} \left( 1 - F_{g_T}(t) \right) dt$$

where $F_{g_T}(t)$ is the CDF of power gain $g_T$. (a) is due to the CDF of max of RVs, (b) results from the probability generating functional (PGFL) of a PPP [], (c) follows from conversion from Cartesian to polar coordinates and employing a change of variable $t = z r^{\eta} P_T^{-1} y$. Then (12) follows since the integral in (32) represents $E \left\{ \hat{g}_T^2 \right\}$.

APPENDIX B
PROOF OF COROLLARY 4

The Mittag-Leffler function in (18) can be expressed in terms of the Fox H-function as [16, eq. (8.13)]

$$E_{\alpha}(\frac{-K_x}{K_y} s^\alpha) = H_{1,1}^{1,1} \left[ \frac{K_x}{K_y} s^\alpha \right] \left( 0, 1 \right), \left( 0, 1, \alpha \right).$$

Then the PDF $f_{\hat{g}}(z)$ can be found via the inverse LT of the Fox H-function in (33) with the help of [16, eq. (12.7)] and a transformation formula [21, eq. (2.4)], and we obtain (19).
The CDF \( F_Z(z) \) can be obtained via an integration formula [17, vol. 3, eq. (1.16.4.1)] and transformation formula [21, eq. (2.4)], and (20) follows.

**APPENDIX C**

**PROOF OF LEMMA 4**

The MGF of \( r_\Sigma \) can be specified as

\[
\mathcal{M}_{r_\Sigma}(-s) = E_{E_k \in \Phi} \prod E_{y_k, g_E} \left\{ \exp \left( -s \frac{L_0 P_T g_E \|x_k\|^{-\eta}}{y_k} \right) \right\}
\]

\[
\overset{(b)}{=} \exp \left( -\lambda_E \int_{\mathbb{R}^2} E_y \left\{ 1 - \mathcal{M}_{g_E} \left( -s \frac{L_0 P_T \|x\|^{-\eta}}{y} \right) \right\} dx \right)
\]

\[
\overset{(c)}{=} \exp \left( \left[ -\lambda_E \pi (sL_0 P_T)^{\frac{2}{\eta}} E_y \left\{ y^{-\frac{2}{\eta}} \right\} \right] \right)
\]

\[
\times \int_0^\infty \frac{2}{\eta} t^{-\frac{2}{\eta}-1} \left[ 1 - M_{g_E}(-t) \right] dt
\]

\[
\overset{(d)}{=} \exp \left[ -\lambda_E \pi (sL_0 P_T)^{\frac{2}{\eta}} E_y \left\{ y^{-\frac{2}{\eta}} \right\} \right]
\]

\[
\times \Gamma \left( 1 - \frac{2}{\eta} \right) E \left\{ g_E^{\frac{2}{\eta}} \right\}
\]

(34)

where (a) is due to the expression of the MGF of sum via the product of individual MGFs; (b) is due to the PGFL of PPP; (c) follows from algebraic manipulations after converting from Cartesian to polar coordinates \( r, \theta \), and substituting \( t = s \frac{L_0 P_T r^{-\eta}}{y} \); (d) follows from evaluation of the integral by parts.

Using a series expansion \( \mathcal{M}_{g_E}(-t) = \sum_{l=0}^{\infty} \frac{(-t)^l m_{g_E}^l}{l!} \), it can be shown that \( \lim_{t \to 0} t^{-\frac{2}{\eta}} \left[ 1 - M_{g_E}(-t) \right] = 0 \) since \( \frac{2}{\eta} < 1 \) and the moments \( m_{g_E}^l \) are assumed to be finite. Then taking into account that \( \frac{d \mathcal{M}_{g_E}(-t)}{dt} = -E\{g_E \exp(-t g_E)\} \), we find that

\[
\text{Int} = \int_0^\infty t^{-\frac{2}{\eta}} E\{g_E \exp(-t g_E)\} dt
\]

\[
\overset{(a)}{=} \int_0^\infty g_E f_{g_E}(g_E) \int_0^\infty t^{-\frac{2}{\eta}} \exp(-t g_E) dt dg_E \overset{(b)}{=} \Gamma \left( 1 - \frac{2}{\eta} \right) E \left\{ g_E^{\frac{2}{\eta}} \right\}
\]

\[
\times \int_0^\infty y^{\frac{2}{\eta}} f_{g_E}(y) dy = \Gamma \left( 1 - \frac{2}{\eta} \right) E \left\{ (g_E)^{\frac{2}{\eta}} \right\}
\]

(35)

where (a) follows from the representation of \( E\{g_E \exp(-t g_E)\} \) in an integral form and Fubini’s theorem [25], and (b) results from an expression of the inner integral in terms of gamma function [17, vol. 1, eq. (2.3.18.2)].