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Published in:
Composites Part B: Engineering

DOI:
[10.1016/j.compositesb.2019.107224](https://doi.org/10.1016/j.compositesb.2019.107224)

Published: 15/11/2019

Document Version
Peer reviewed version

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Please cite the original version:
Khakalo, S., & Niiranen, J. (2019). Lattice structures as thermoelastic strain gradient metamaterials: Evidence from full-field simulations and applications to functionally step-wise-graded beams. *Composites Part B: Engineering*, 177, [107224]. <https://doi.org/10.1016/j.compositesb.2019.107224>

Lattice structures as thermoelastic strain gradient metamaterials: evidence from full-field simulations and applications to functionally step-wise-graded beams

Sergei Khakalo^{a,1}, Jarkko Niiranen^b

^a*VTT Technical Research Centre of Finland, P.O.Box 1000, Espoo, Finland.*

^b*Aalto University, School of Engineering, Department of Civil Engineering, P.O. Box 12100, 00076 AALTO, Finland.*

Abstract

The present work investigates the mechanical and thermomechanical bending response of beam structures possessing a triangular lattice microarchitecture. The validity of generalized continuum models, in general, and the associated dimensionally reduced models for functionally step-wise-graded microarchitectural beams, in particular, is approved by full-field finite element simulations. Most importantly, the necessity of the temperature gradient in the Helmholtz free energy is substantiated. The corresponding strong and weak forms for the associated Bernoulli–Euler and Timoshenko models of functionally graded beams are derived. The effective classical thermoelastic properties of a metamaterial with a triangular lattice microarchitecture are defined by means of computational homogenization. The additional length scale parameter involved in the generalized beam models, and associated to the particular triangular microarchitecture, is calibrated by fitting the mechanical bending responses of a series of lattice beams to the analytical solutions of the corresponding theoretical models. Strongly size-dependent mechanical and size-independent thermal bending responses are observed for both thin and thick beams with triangular lattice microarchitectures. Finally, different lattice beams with varying microarchitectures are introduced and shown to behave as generalized functionally step-wise-graded beams with respect to the higher-order elastic modulus, i.e., the length scale parameter varying in the direction of the beam axis.

¹Corresponding author, sergei.khakalo@vtt.fi

Keywords: Second grade thermoelasticity, Temperature gradient, Triangular lattice metamaterial, Bernoulli–Euler beam, Timoshenko beam, Microarchitecture, Size effect

1. Introduction

In the framework of Cauchy continuum mechanics – widely and efficiently used for predicting the mechanical behaviour of most engineering materials – the basic constitutive relation for the linearly elastic material model is given in the form of the generalized Hooke’s law which implies a linear dependency of the Cauchy stresses on strains. For the corresponding linearly thermoelastic material model, in turn, the Cauchy stresses depend linearly on both strains and temperature, as formulated in the Duhamel–Neumann constitutive law. In many engineering applications, materials/metamaterials with highly noticeable heterogeneity or microarchitecture, such as lattice-based structures and materials, are widely utilized due to their light weight, high stiffness and/or strength, or tunable dispersion properties [1, 2, 3]. Lattice structures, in particular, can be realized today in different scales via various fastly developing manufacturing technologies for different parent materials (see [4, 5, 6, 7, 8, 9, 10] and the reference therein). However, in many occasions the models of classical mechanics are not able to account for the underlying microstructure at the continuum level [11, 12, 13, 14, 15, 16]. Therefore, more advanced theoretical models and numerical methods have become necessary for reaching higher efficiency levels of design and engineering (e.g., for finding suitable weight and stiffness properties of structural parts to be realized via additive manufacturing).

Generalized continuum theories have been shown to be capable of accounting for the microstructural, or microarchitectural, effects by incorporating additional kinematical variables or higher-order gradients of the basic kinematical variables into continuum models, together with the corresponding length scale parameters. Within strain gradient continuum mechanics [17, 18, 19, 20], in particular, being a representative of generalized continuum mechanics (see [21, 22], for an overview), along with classical strains the continuum model is augmented by including an additional state variable, the gradient of strains. This augmentation implies the appearance of higher-order (or double) stresses being work conjugates of the strain gradients. Regarding the closest relatives of the present work,

bending of beam- and plate-like structures within generalized continua have been studied in [23, 24, 25, 15, 26, 27], whereas studies on other types of configurations, typically involving in-plane deformations, can be found in [28, 29, 29, 30, 15, 14, 31, 32, 33], for instance. The beam-like structures analyzed in [23, 25, 27] are formed by a repetition of a chosen unit cell in one coordinate direction (eventually forming the middle axis of the homogenized beam) – leading to a one-layer beam- or plate-like structure. In [15] and herein, instead, this type of beam-like structures are further stacked (N times) on top of each other in the (thickness) direction perpendicular to the middle axis and, simultaneously, multiplied (N times) in the axial direction – leading to a multi-layer beam-like structure. For this structure, the size dependence related to the transversal-axial multiplication process is studied with respect to uniaxial bending.

For the identification procedures for the elastic constants of enhanced continuum models, literature is quite limited. Regarding beams and plates, [26, 27, 34] rely on assuming strain energy equivalence between the (microscale) unit cell and the (macroscale) micropolar continuum beam or plate (cf. [35]). In [28], studying granular materials in plane, the identification procedure is based on exiting different deformation fields in unit cells (of both regular and irregular grain assemblies) and on comparing the corresponding discrete and continuous material descriptions. In [29] and [36], concerning isotropic and unisotropic strain gradient plane elasticity, respectively, a series of test problems having analytical solutions of classical elasticity is proposed and, accordingly, a set of equations for constitutive parameters is written by utilizing expressions for non-classical reaction forces (associated to non-classical boundary conditions). In the present contribution, we follow the procedure proposed in [15] relying on matching the global responses of full-field simulations with analytical solutions of the generalized beam model for a family of simple test problems.

In the framework of strain gradient continuum thermomechanics, in particular, there is no unanimity of opinion on constitutive relations. On one hand, by following the standard imposed by Coleman–Noll rational thermodynamics [37], on a par with strains and the gradient of strains the Helmholtz free energy includes temperature alone (cf. [38, 39], for instance). Accordingly, the temperature gradient is disallowed in the constitutive laws – here referred as an (thermally) incomplete, or inconsistent, material model. On the other hand, by performing certain treatments it becomes possible to include the temperature gradient into the Helmholtz free energy, and accord-

ingly, into the constitutive laws. This approach leads to a (thermally) complete, or consistent, material model. (This approach relies on the principle of virtual power [40], includes the temperature rate in the set of generalized virtual motions [41, 42], introduces the so-called residual [43] or extra [44] entropy flux in the Clausius–Duhem inequality and considers the gradient of entropy in the free energy density [45].) The lack of experimental data and observations regarding the thermomechanical behaviour of materials classified as strain gradient materials impedes resolving the problem of the correct setting of consistent constitutive laws: to include or not to include the temperature gradient in the Helmholtz free energy. Thus, in literature, there are contributions relying on both incomplete [46, 47, 48] and complete [49, 50] thermomechanical strain gradient material models.

The aim of the present work is to address this problem by providing rather convincing computational results which are based on full-field finite element modeling – relying on classical elasticity and classical thermodynamics together with standard finite element discretization, and therefore considered as a verified and validated reference. The main novelties of the present contribution are listed as follows:

- (1) For both Bernoulli–Euler and Timoshenko beam models, it is recognized and illustrated with analytical solutions that the complete form of strain gradient thermoelasticity exhibits size-independence in thermal bending, whereas the incomplete counterpart produces a size-dependent response for thermal bending, as both of the models do for mechanical bending.
- (2) Lattice trusses with a triangular microarchitecture are revealed to be size-independent in thermal bending but size-dependent in mechanical bending. This combination is, according to item (1), shown to be captured only by beam models within the complete form of strain gradient thermoelasticity.
- (3) Lattice structures with varying triangular microarchitectures are designed and shown to correspond at the homogenized continuum level to functionally step-wise-graded beams with non-homogeneous higher-order elastic properties described by step-wise constant values of the length scale parameter.

This article is organized as follows: In Section 2, the constitutive relations of strain gradient thermoelastic materials accounting for the temperature gradient are addressed. Section 3, based on the principle of virtual work (PVW), is devoted to the derivation of the strain gradient thermoelastic Bernoulli–Euler and Timoshenko models for functionally (FG) graded beams. In Section 4, first the computational homogenization methods are applied for defining the classical thermoelastic effective material moduli of a

2D triangular lattice metamaterial. Next, the size-dependent mechanical and size-independent thermal bending responses for long and thin structures with the triangular microarchitecture are analyzed. The generalized Bernoulli–Euler beam model is then utilized for capturing the simulation results. Mechanical size-dependent and thermal size-independent bending phenomena of short and thick lattice structures are considered in Section 5 and accurately captured by the generalized thermoelastic Timoshenko beam model. In Section 6, the validity of the generalized thermoelastic Bernoulli–Euler and Timoshenko models for functionally step-wise-graded beams is approved by analyzing lattice structures with varying microarchitectures. Finally, Section 7 contains concluding remarks.

2. Constitutive laws of strain gradient thermoelastic solids

For the sake of simplicity and clearness, we limit ourselves to small deformations and linear strains. The reference and actual configurations are indistinguishable. Strain tensor in this case is represented by the symmetric part of the displacement gradient, i.e., $\varepsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2$.

For strain gradient (or second gradient) linearly thermoelastic materials, the relevant state variables which are allowed in the specific free energy (details in [42]) are strains, strain gradients, temperature changes and temperature gradients. The Helmholtz free energy can then be constructed as a quadratic form in all these variables (cf. [42]) as

$$\begin{aligned} \rho\psi = & \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2}A_{mijnkl}\partial_m\varepsilon_{ij}\partial_n\varepsilon_{kl} + B_{mijkl}\partial_m\varepsilon_{ij}\varepsilon_{kl} \\ & - \varepsilon_{ij}C_{ijkl}\varepsilon_{kl}^\theta - \partial_m\varepsilon_{ij}A_{mijnkl}\varepsilon_{nkl}^\theta \\ & - \partial_m\varepsilon_{ij}B_{mijkl}\varepsilon_{kl}^\theta - \varepsilon_{mij}^\theta B_{mijkl}\varepsilon_{kl} + F(\theta, \partial_i\theta), \end{aligned} \quad (2.1)$$

where $F(\theta, \partial_i\theta)$ appears as a quadratic form with respect to θ and $\partial_i\theta$

$$F(\theta, \partial_i\theta) = -\frac{1}{2}\eta\theta^2 - \frac{1}{2}M_{ij}\partial_i\theta\partial_j\theta - \theta N_i\partial_i\theta \quad (2.2)$$

and ε_{ij}^θ and ε_{ijk}^θ stand, respectively, for the components of classical thermal eigenstrain and a higher-order counterpart which will be specified later. The components of the fourth, fifth and sixth rank stiffness tensors are denoted, respectively, as C_{ijkl} , B_{mijkl} and A_{mijnkl} . For the symmetry conditions concerning the higher-order elasticity tensors, see [51], for instance. Here and

in what follows, θ stands for the deviation from the reference temperature. In expression (2.2), parameter η corresponds to a classical specific heat, M_{ij} and N_i represent the generalized heat parameters (cf. [42]) associated to thermal length scales. We further assume that the equations of the linear thermoelastic problem are uncoupled and that the temperature fields in Sections 4.4 and 5 correspond to a classical heat problem meaning that not all of the possible thermal length scale effects are arising. This assumption is in agreement with the full-field simulation results (elaborated in Sections 4–6).

The components of the Cauchy (or Cauchy-like) stress tensor and the double stress tensor, the work conjugates of the ordinary strain tensor and the strain gradient tensor, respectively, are defined as (cf. [42] and [52] for the consistent and inconsistent models, accordingly)

$$\tau_{ij} = \frac{\partial \rho \psi}{\partial \varepsilon_{ij}} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^\theta) + B_{klmij}(\partial_k \varepsilon_{lm} - \varkappa_{klm}^\theta), \quad (2.3)$$

$$\tau_{ijk} = \frac{\partial \rho \psi}{\partial (\partial_i \varepsilon_{jk})} = A_{ijklmn}(\partial_l \varepsilon_{mn} - \varkappa_{lmn}^\theta) + B_{ijklm}(\varepsilon_{lm} - \varepsilon_{lm}^\theta). \quad (2.4)$$

Next, we consider centrosymmetric materials for which the odd rank stiffness tensor (see [17]) vanishes. By recalling the components of classical thermal eigenstrain in its standard form the higher-order thermal counterpart can be defined as

$$\varepsilon_{ij}^\theta = \theta \alpha_{ij}, \quad \varkappa_{kij}^\theta = \partial_k \varepsilon_{ij}^\theta = \partial_k (\theta \alpha_{ij}), \quad (2.5)$$

where α_{ij} denote the coefficients of the thermal expansion tensor. The constitutive equations (2.3) and (2.4), respectively, take the forms

$$\tau_{ij} = C_{ijkl}(\varepsilon_{kl} - \theta \alpha_{kl}), \quad \tau_{ijk} = A_{ijklmn} \partial_l (\varepsilon_{mn} - \theta \alpha_{mn}). \quad (2.6)$$

It can be immediately seen that the higher-order thermal strain expressed as in (2.5) lead to constitutive laws compatible with a situation where constant or linear temperature changes produce zero stresses (both Cauchy-like and double) in a body made of a homogeneous material and experiencing zero loadings.

For isotropic materials, the coefficients of thermal expansion and elastic moduli can be written, respectively, as $\alpha_{ij} = \alpha \delta_{ij}$ and $C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$, where α denotes the isotropic modulus of thermal expansion, μ and λ stand for the Lamé parameters and δ_{ij} denotes the Kronecker delta.

Next, we consider the case of weakly non-local isotropy (see [53]) allowing to decompose the six rank elasticity tensor as $A_{ijklmn} = g^2 \delta_{il} C_{jkmn}$, which coincides with the so-called simplified strain gradient model [54] introducing only one length scale parameter g with unit of length. The explicit form of the Helmholtz free energy is finally expressed as

$$\begin{aligned} \rho\psi = & \frac{1}{2}(\lambda\varepsilon_{ii}\varepsilon_{jj} + 2\mu\varepsilon_{ij}\varepsilon_{ij}) - \alpha\theta(2\mu + 3\lambda)\varepsilon_{ii} + F(\theta, \partial_i\theta) \\ & + \frac{1}{2}g^2(\lambda\partial_k\varepsilon_{ii}\partial_k\varepsilon_{jj} + 2\mu\partial_k\varepsilon_{ij}\partial_k\varepsilon_{ij}) - g^2(2\mu + 3\lambda)\partial_i(\alpha\theta)\partial_i\varepsilon_{jj}, \end{aligned} \quad (2.7)$$

leading to the stress expressions of the form

$$\tau_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - \alpha\theta(2\mu + 3\lambda)\delta_{ij}, \quad (2.8)$$

$$\tau_{ijk} = g^2(2\mu\partial_i\varepsilon_{jk} + \lambda\partial_i\varepsilon_{kk}\delta_{jk}) - g^2(2\mu + 3\lambda)\partial_i(\alpha\theta)\delta_{jk}. \quad (2.9)$$

3. Thermoelastic strain gradient FG beam models

In this section, we concentrate on the derivation of the Bernoulli–Euler beam model valid for thin beam structures and the Timoshenko beam model valid for thick beam structures (see Fig. 4.2 (right)) in the framework of strain gradient thermoelasticity. It is assumed that the material of the continuum is linearly isotropic with the Helmholtz free energy defined in (2.7). Material parameters μ , λ , α and g are considered to be functions of the x -coordinate. The derivation of the strong form is based on the PVW reading as $\delta W_{int} = \delta W_{ext}$, where the variation of the stored strain energy can be written as

$$\delta W_{int} = \int_{\Omega} (\tau_{ij}\delta\varepsilon_{ij} + \tau_{ijk}\delta\partial_i\varepsilon_{jk})d\Omega. \quad (3.1)$$

The corresponding variation of the work done by external forces will be specified later.

3.1. Generalized Bernoulli–Euler beam model

It is further assumed that the applied loadings as well as the geometry are selected such that the problem reduces to a planar bending problem (in the xy -plane, see Fig. 4.2 (right)). The classical kinematical Bernoulli–Euler assumptions imply the components of the displacement field in the form

$$u_x = -yw'(x), \quad u_y = w(x), \quad u_z = 0, \quad (3.2)$$

where w represents the transverse deflection of the central axis. By following the basic assumptions regarding the beam model [55], the constitutive relations (2.8) and (2.9) take a reduced form (see the details in Appendix A)

$$\tau_{xx} = E(\varepsilon_{xx} - \alpha\theta), \quad \tau_{xxx} = g^2 E \partial_x(\varepsilon_{xx} - \alpha\theta), \quad \tau_{yxx} = g^2 E \partial_y(\varepsilon_{xx} - \alpha\theta), \quad (3.3)$$

where E denotes Young's modulus.

Next, by substituting assumptions (3.2) and (3.3) into (3.1) and by adopting the kinematical relation $\varepsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2$, one can rewrite the variation of internal energy in the form

$$\begin{aligned} \delta W_{int} = \int_0^L \int_A & (E(y^2 + g^2)w''\delta w'' + E g^2 y^2 w'''\delta w''') \\ & + E\alpha(\theta y + g^2 \partial_y \theta)\delta w'' + E g^2 \partial_x(\alpha\theta)y\delta w''') dA dx, \end{aligned} \quad (3.4)$$

where A stands for the cross-sectional area which, in general, is x -dependent. It is assumed that the temperature field is antisymmetric with respect to the y -argument, i.e., $\theta(x, -y) = -\theta(x, y)$.

The variation of the work done by external forces can be expressed now as (see [56])

$$\delta W_{ext} = \int_0^L f \delta w dx + Q \delta w|_0^L + M \delta w'|_0^L + R \delta w''|_0^L. \quad (3.5)$$

By substituting (3.4) and (3.5) into the PVW expression, one can derive the governing differential equation in the form

$$(E(I + g^2 A)w'')'' - (g^2 E I w''')''' = f - (\alpha E(\gamma_1 + g^2 \gamma_2))'' + (g^2 E \gamma_3)''', \quad (3.6)$$

valid for all $x \in (0, L)$, with the essential – Dirichlet type – boundary conditions (BCs) (left column) and the natural – Neumann type – BCs (right column) defined as

$$\begin{aligned} w = w_0 \quad \text{or} \quad Q = & - (E(I + g^2 A)w'')' + (g^2 E I w''')'' \\ & - (\alpha E(\gamma_1 + g^2 \gamma_2))' + (g^2 E \gamma_3)'', \end{aligned} \quad (3.7)$$

$$\begin{aligned} w' = w_1 \quad \text{or} \quad M = & E(I + g^2 A)w'' - (g^2 E I w''')' \\ & + \alpha E(\gamma_1 + g^2 \gamma_2) - (g^2 E \gamma_3)', \end{aligned} \quad (3.8)$$

$$w'' = w_2 \quad \text{or} \quad R = g^2 E I w''' + g^2 E \gamma_3 \quad \text{at } x = 0, L. \quad (3.9)$$

Here I denotes the second moment of area, f acts as a distributed bending force, Q and M stand, respectively, for the ordinary external transverse bending force and bending moment, R denotes the higher-order external load. Temperature-dependent resultant coefficients γ_1 , γ_2 and γ_3 are defined, respectively, as

$$\gamma_1(x) = \int_A \theta y dA, \quad \gamma_2(x) = \int_A \partial_y \theta dA, \quad \gamma_3(x) = \int_A \partial_x(\alpha \theta) y dA. \quad (3.10)$$

It is worth noting that for the incomplete thermoelastic material model (disallowing the temperature gradient in the Helmholtz free energy) temperature coefficients γ_2 and γ_3 are excluded since they contain the gradients of the temperature field.

3.2. Generalized Timoshenko beam model

For the Timoshenko beam model, the kinematical assumptions read as

$$u_x = -y\beta(x), \quad u_y = w(x), \quad u_z = 0, \quad (3.11)$$

where, along with deflection $w(x)$, an independent field variable, rotation $\beta(x)$, is introduced for measuring the rotation of a beam cross section about the neutral axis. The non-zero components of the ordinary and double stress tensors are defined in the form

$$\begin{aligned} \tau_{xx} &= E(\varepsilon_{xx} - \alpha\theta), \quad \tau_{xy} = \tau_{yx} = 2\kappa G \varepsilon_{xy}, \\ \tau_{xxx} &= g^2 E \partial_x(\varepsilon_{xx} - \alpha\theta), \quad \tau_{yxx} = g^2 E \partial_y(\varepsilon_{xx} - \alpha\theta), \\ \tau_{xxy} &= \tau_{xyx} = 2g^2 \kappa G \partial_x \varepsilon_{xy}, \end{aligned} \quad (3.12)$$

where $G = E/(2(1 + \nu))$ denotes the shear modulus and κ stands for the shear correction factor (cf. [57]).

By substituting expressions (3.11) and (3.12) into (3.1) and by adopting linear strain assumptions $\varepsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2$, the variation of the internal energy takes the form

$$\begin{aligned} \delta W_{int} &= \int_0^L \int_A (E(y^2 + g^2)\beta' \delta\beta' + E g^2 y^2 \beta'' \delta\beta'' \\ &\quad + \kappa G(w' - \beta)(\delta w' - \delta\beta) + g^2 \kappa G(w'' - \beta')(\delta w'' - \delta\beta') \\ &\quad + E\alpha(\theta y + g^2 \partial_y \theta) \delta\beta' + E g^2 \partial_x(\alpha\theta) y \delta\beta'') dA dx. \end{aligned} \quad (3.13)$$

As in Section 3.1, it is assumed that the temperature field is antisymmetric with respect to the y -argument (see Fig. 4.2 (right)). The variation of the work done by external forces is written as

$$\delta W_{ext} = \int_0^L (f\delta w + m\delta\beta)dx + Q_1\delta w|_0^L + Q_2\delta w'|_0^L + M_1\delta\beta|_0^L + M_2\delta\beta'|_0^L, \quad (3.14)$$

where m acts as a distributed bending moment, Q_1 and M_1 stand, respectively, for the ordinary external transverse bending force and bending moment, whereas Q_2 and M_2 denote, respectively, the higher-order external load resultants.

By substituting equations (3.13) and (3.14) into the PVW expression, one can derive the governing differential equations of the isotropic FG Timoshenko beam model in the form

$$(\kappa GA(w' - \beta))' - (\kappa GA g^2(w'' - \beta'))'' = -f, \quad (3.15)$$

$$(E(I + g^2 A)\beta')' - (EI g^2 \beta'')'' + \kappa GA(w' - \beta) - (\kappa GA g^2(w'' - \beta'))' = -\tilde{m}, \quad (3.16)$$

valid for all $x \in (0, L)$, where $\tilde{m} = m + (\alpha E(\gamma_1 + g^2 \gamma_2))' - (g^2 E \gamma_3)''$ with temperature resultants γ_1 , γ_2 and γ_3 defined already in (3.10). The corresponding boundary conditions at $x = 0, L$ are given as

$$w = w_0 \quad \text{or} \quad Q_1 = \kappa GA(w' - \beta) - (\kappa GA g^2(w'' - \beta'))', \quad (3.17)$$

$$w' = w_1 \quad \text{or} \quad Q_2 = \kappa GA g^2(w'' - \beta'), \quad (3.18)$$

$$\begin{aligned} \beta = \beta_0 \quad \text{or} \quad M_1 = E(I + g^2 A)\beta' - (EI g^2 \beta'')' - \kappa GA g^2(w'' - \beta') \\ + \alpha E(\gamma_1 + g^2 \gamma_2) - (g^2 E \gamma_3)', \end{aligned} \quad (3.19)$$

$$\beta' = \beta_1 \quad \text{or} \quad M_2 = EI g^2 \beta'' + g^2 E \gamma_3. \quad (3.20)$$

In case of homogeneous elastic beams with $\gamma_i = 0$, the model reduces to the corresponding one in [58].

4. Triangular lattice strips in bending

In this section, we consider 2D triangular lattice structures (see Fig. 4.1 (left)) being one of the representatives of gradient metamaterials (addressed

in [15] regarding mechanical bending; cf. [9] considering extension and compression). First, thin long strips made of the triangular lattice metamaterial (see Fig. 4.2 (left)) are subjected to mechanical and thermal bending. The structures are analyzed by accomplishing finite element (FE) modeling and analysis within the Abaqus FE-software. Second, the effective classical thermoelastic moduli, e.g., Young’s modulus and the coefficient of thermal expansion, are determined via computational homogenization applied to the corresponding representative volume element (RVE). Finally, the bending response of the lattice truss is captured by the generalized thermoelastic Bernoulli–Euler beam model of Section 3.1 derived for the homogenized continuum. The length scale parameter g is fitted to the full-field simulation results by following the procedure introduced in [15]. It should be noted that the generalized elastic moduli could be estimated via other computational homogenization techniques as well (cf. [59], for instance).

Appendix B demonstrates that the basic kinematical assumption on straight normal fibers of the central axis, in particular, is valid for the triangular lattice microarchitecture beams. This observation supports staying in the regime of the lowest order beam models in the hierarchy (cf. [60]).

4.1. Lattice structure description

Let us consider a 2D triangular lattice metamaterial illustrated in Fig. 4.1 (left) with the RVE depicted in Fig. 4.1 (right) [15]. The characteristic dimensions and effective material parameters are given in Table 1 (cf. the parallelogrammic RVE in [9]). The base material is taken to be isotropic with Young’s modulus $E = 2$ GPa, Poisson’s ratio $\nu = 0.25$, thermal expansion coefficient $\alpha = 0.00074$ $^{\circ}C^{-1}$ and thermal conductivity $k = 0.13$ W/(m $^{\circ}C$), corresponding to polylactic acid (PLA) thermoplastic. We note that for the slenderness ratio of the struts we have chosen value $t/h_1 = 1/10$ (cf. [25]) but other physically appropriate values can be used as well, meaning that the relative density of the RVE can be varied (see [15]) from close to 0% (extremely slender members) to 100% (full solid).

By replicating the RVE along the x_1 -axis (a few times depending on the desired thickness-to-length ratio), we first build a long thin truss structure with length L_0 and thickness h_0 (an example is shown in Fig. 4.2 (left)). The thickness of the truss is equal to the h_2 -dimension of the RVE. Next, by duplicating this lattice truss in the x_1 - and x_2 -directions we produce subsequent lattice trusses with lengths $L = NL_0$ and thicknesses $h = Nh_0$ (see Fig. 4.2 (left)), where $N = 1, 2, 3, \dots$ denotes the number of RVEs along

Figure 4.1: Left: 2D triangular lattice metamaterial. Right: Representative volume element [15].

the thickness direction. It should be noted that the trusses are composed in such a way that the length-to-thickness ratio remains constant, i.e., $L/h = L_0/h_0$.

4.2. Lattice metamaterial homogenization

The effective classical thermo-mechanical properties are defined according to [61]. It is assumed that the effective homogenized continuum is governed by the Duhamel–Neumann law for orthotropic materials. For defining the effective thermal expansion coefficients, we solve a coupled temperature-displacement problem for an RVE with the following boundary conditions: $u_1 = 0$ at $x_1 = \pm h_1/2$ and $u_2 = 0$ at $x_2 = \pm h_2/2$, see Fig. 4.1 (right). A constant temperature field θ is given to the RVE.

The effective thermal expansion coefficients are defined by resolving the equations of the generalized Duhamel–Neumann law with respect to the two moduli α_1^* and α_2^* :

$$\alpha_1^* = -\frac{1}{\theta E_1^*} (\langle \tau_{11} \rangle - \nu_{12}^* \langle \tau_{22} \rangle), \quad (4.1)$$

$$\alpha_2^* = -\frac{1}{\theta E_2^*} (\langle \tau_{22} \rangle - \nu_{21}^* \langle \tau_{11} \rangle), \quad (4.2)$$

Figure 4.2: Left: Lattice structure strips with $N = 1, 2, 4$, respectively. Right: A schematic representation of a long prismatic body in the xy - and yz -planes and a plot of the temperature distribution across the thickness.

where E_1^* , E_2^* and ν_{12}^* , ν_{21}^* have been defined in [15]. The averaging procedure is defined as

$$\langle \tau \rangle = \frac{1}{V} \int_V \tau dV, \quad (4.3)$$

where $V = h_1 h_2$ denotes the RVE volume and τ_{ij} , $i, j = 1, 2$, stands for the stress tensor components in 2D. For more details, one can consult [61].

The values of the effective thermo-mechanical material constants are listed in Table 1. As shown in [15], the mechanical properties in the two coordinate directions coincide, i.e., $E_1^* = E_2^* = E^*$ and $\nu_{12}^* = \nu_{21}^* = \nu^*$ indicating isotropy of the triangular lattice metamaterial, at least in the classical sense. The same holds true for the thermal expansion coefficients: $\alpha_1^* = \alpha_2^* = \alpha^*$. It can be seen that the value of the effective modulus of thermal expansion coincides with the one of the base material.

Table 1: Dimensions and effective properties of RVE.

h_1 , [mm]	h_2 , [mm]	t , [mm]	E^* , [MPa]	ν^*	α^* , [$^{\circ}C^{-1}$]
5	4.33	0.5	246.7	0.335	0.00074

4.3. Mechanically induced bending

In the present section, our focus lies on the global response of the truss structure which, with reliable accuracy, can be modelled by a reduced strain

gradient model [15] (almost identical to the modified couple stress model according to [56]). An analogous reduction (concerning the higher-order terms) is applied here to the temperature-related resultants meaning that γ_3 vanishes. We note that when dropping the higher-order terms the identification procedures for the non-classical constitutive parameters relying on the utilization of the higher-order reaction forces (the key quantities in [29, 36]) are not applicable.

The strong formulation of the strain gradient thermoelastic Bernoulli–Euler beam model of (3.6)–(3.9) is reduced to

$$(E(I + g^2 A)w'')'' = f - (\alpha E(\gamma_1 + g^2 \gamma_2))'', \quad \forall x \in (0, L) \quad (4.4)$$

$$w = w_0 \quad \text{or} \quad Q = -(E(I + g^2 A)w'')' - (\alpha E(\gamma_1 + g^2 \gamma_2))', \quad (4.5)$$

$$w' = w_1 \quad \text{or} \quad M = E(I + g^2 A)w'' + \alpha E(\gamma_1 + g^2 \gamma_2), \quad x = 0, L. \quad (4.6)$$

In what follows, the material parameters used in the beam model are the ones obtained via the computational homogenization of the preceding subsection, i.e., $E = E^*$ (from Table 1).

For a simply supported beam bended by moments applied at both ends (see Fig. 4.3 (top)), we exclude body forces ($f = 0$) and eliminate the temperature-dependent parameters ($\gamma_1 = 0$, $\gamma_2 = 0$). The boundary conditions read as $w(0) = w(L) = 0$ and $M(0) = M(L) = -M_0$ resulting in the following expression for the deflection of the homogeneous beam:

$$w(x) = \frac{M_0}{2E(I + g^2 A)}(Lx - x^2). \quad (4.7)$$

Let us define the rotation angle at $x = L$ by φ and the measure for the rigidity of the beam in pure bending by $D = M_0/(h^2 \varphi)$. For rectangular cross-sections with width b , cross-sectional area $A = bh$ and second moment of area $I = bh^3/12$, the bending rigidity can be expressed as

$$D = \frac{Eb}{6} \frac{h}{L} \left(1 + 12 \frac{g^2}{h^2}\right), \quad (4.8)$$

where $\varphi = -w'(L)$, according to the Bernoulli–Euler beam model. For the classical Bernoulli–Euler beam model, the rigidity measure is calculated from (4.8) by setting $g = 0$. Here and in what follows, we assume a unit width, i.e., $b = 1$ mm.

For the numerical simulations of the lattice trusses in bending, the problem setting is the following: The vertical edges at $x = 0$ and $x = L$ are

fixed such that they can freely rotate around the corresponding neutral axes and they are loaded by bending moments M_0 acting around the z -direction as shown in Fig. 4.3 (top). The length-to-thickness ratio is fixed to $L_0/h_0 = 20.8$ for all samples (see Fig. 4.2 (left)). The lattice trusses are modelled as 2D domains discretized by quadrilateral elements (type CPS4 in Abaqus). The bending rigidity is calculated as the ratio of the applied moment M_0 to the rotation angle φ_L at $y = 0$, $x = L$ (represented in Fig. 4.3 (top)) and scaled by h^2 as $D = M_0/(h^2\varphi_L)$.

Figure 4.3: Problem settings for the computational analysis of simply supported truss structures. Top: Mechanical bending by applied external moments M_0 : a lattice structure strip (red) with $h = 4.33$ mm and $L = 90$ mm, the corresponding strain gradient (blue) and classical (green) Bernoulli-Euler beams. Bottom: Thermal bending by applied external temperatures $\pm T_0$: a lattice structure strip (red) with $h = 4.33$ mm and $L = 90$ mm, the corresponding strain gradient thermoelastic Bernoulli-Euler beam by including only temperature (grey) and both temperature and temperature gradient (blue) into the Helmholtz free energy.

The deformation response is depicted as the dependence of the bending rigidity on the truss thickness in Fig. 4.4 (left), where the blue dots correspond to the lattice trusses with $N = 1, 2, 3, 4, 8$ and 16 , respectively. The horizontal dashed black line represents the bending rigidity corresponding to the classical Bernoulli-Euler beam model (eq. (4.8) with $g = 0$) with Young's

modulus $E = E^* = 246.7$ MPa taken from Table 1 for the homogenized continuum. The red solid curve corresponds to the generalized Bernoulli–Euler beam model (eq. (4.8)) which is fitted to the simulation results (blue dots) by setting $g = 1.57$ mm. Comparing the calibrated length scale parameter to the unit cell thickness ($h_2 = 4.33$ mm) it can be seen that the ratio is roughly $1/3$. It should be noted that the same value for the g -parameter has been estimated from the bending of cantilever lattice trusses in [15]. It can be seen that for the thin lattice trusses with very few RVEs in the thickness direction ($N = 1, 2, 3, 4$) the value of the bending rigidity is much higher (approximately 2.5 times for $N = 1$ ($h = 4.33$ mm)) than the value for the densest lattice truss ($N = 16$ or $h = 69.28$ mm) practically equal to the one of the classical beam model.

The comparison of the lattice truss response with the corresponding prediction based on the classical and generalized Bernoulli–Euler beam models is shown in Fig. 4.3 (top) for deflection distributions. Fig. B.1 (left) of Appendix B represents the cross-sectional displacements showing a good accordance with the basic kinematical assumption on straight normal fibers of the central axis. The lattice truss is shown in red colour. It can be seen that the deflection curve of the generalized Bernoulli–Euler beam model (blue solid line) coincides with the deflection of the lattice truss middle line, whereas the classical beam model (green line) obviously fails to describe the lattice truss response, with a difference in the maximal deflection higher than 2.5 times. The lattice trusses have been modelled by standard 2D quadrilateral solid finite elements (CPS4 element type in Abaqus), while the classical Bernoulli–Euler beam has been modelled by standard 2D cubic beam finite elements (B23 element type in Abaqus). For the numerical analysis of the generalized Bernoulli–Euler beam problem of (3.6)–(3.9), isogeometric NURBS-based beam elements have been implemented in the Abaqus user element framework [62]. The corresponding weak and discrete formulations are given in Appendix C.

Figure 4.4: Bending rigidity (left) and rotation angle (right) against the strip thickness.

4.4. Thermally induced bending

For thermally induced bending of a simply supported beam without body forces ($f = 0$), constant thermal loading conditions are given, i.e., constant temperatures $-T_0$ and T_0 are prescribed at the upper ($y = -h/2$) and lower ($y = h/2$) edges, respectively, as shown in Fig. 4.3 (bottom). The temperature field is assumed to be linear across the beam thickness, i.e., $\theta(x, y) = 2T_0y/h$ (see Fig. 4.2 (right)), which implies that the isothermal lines are parallel to the beam middle line. With the boundary conditions $w(0) = w(L) = 0$ and $M(0) = M(L) = 0$ and temperature coefficients calculated as $\gamma_1 = bT_0h^2/6$ and $\gamma_2 = 2bT_0$, one can derive the following expression for the deflection of the homogeneous beam with a rectangular cross-section:

$$w(x) = \frac{\alpha T_0}{h}(Lx - x^2), \quad (4.9)$$

where the thermal expansion coefficient $\alpha = \alpha^*$ is taken from Table 1. It should be noted that within the incomplete thermoelastic material model the beam deformation is expressed in the form

$$w(x) = \frac{\alpha T_0}{h(1 + 12g^2/h^2)}(Lx - x^2), \quad (4.10)$$

where an extra multiplier $(1 + 12g^2/h^2)$ in the denominator acts as a source of size dependency as shown in Fig. 4.4 (right).

For the numerical simulations of the thermally induced bending of the lattice trusses, the coupled temperature-displacement problem is utilized in Abaqus. The problem setting is similar to the one of Subsection 4.3 except the applied bending moment M_0 . Instead, the upper ($y = -h/2$) and lower ($y = h/2$) boundaries are subjected to thermal heating, i.e., constant temperatures $-T_0$ and T_0 are prescribed at the upper and lower edges, respectively, as shown in Fig. 4.3 (bottom). The length-to-thickness ratio is kept the same ($L_0/h_0 = 20.8$). In order to be consistent with the assumption of a linear temperature distribution in the corresponding homogenized continuum, we apply Robin, or mixed, boundary conditions (known as Newton cooling) to the rest of the lattice edges. This disallows the insulation condition (zero heat flux) at the free boundaries being the default natural boundary condition in heat transfer problems.

For the problem of thermal bending, the lattice truss response is presented in Fig. 4.4 (right) as the dependence of the rotation angle φ_L (shown in Fig. 4.3 (bottom)) on the truss thickness. The blue dots correspond to the computational results. As can be seen, the value of the rotation angle remains constant as the truss thickness increases demonstrating the lattice truss size-independency in thermal bending, which is perfectly captured by the complete thermoelastic material model. The horizontal red solid line is defined as $\varphi_L = -w'(L) = \alpha T_0(L/h)$ according to equation (4.9) with the thermal expansion coefficient taken from Table 1 for the homogenized continuum. For the incomplete thermoelastic material model, the rotation angle is prescribed as $\varphi_L = -w'(L) = \alpha T_0(L/h)/(1 + 12g^2/h^2)$ according to equation (4.10) and represented by the dashed black line in Fig. 4.4 (right). As noted earlier, the lack of the temperature gradient in the Helmholtz free energy leads to the incomplete representation of the strain gradient thermoelastic material model, resulting in a size-dependent behaviour which does not correspond to the obtained simulation results.

As in Subsection 4.3, the lattice truss response in thermal bending is compared in Fig. 4.3 (bottom) to the corresponding predictions based on the complete and incomplete thermoelastic material models. The beam bending curve within the incomplete model is presented by the solid grey line which does not match to the computational results. The solid blue line corresponds to the complete thermoelastic material model. As can be seen, the deflection curve coincides with the deformation of the lattice truss middle line.

5. Thick triangular lattice trusses in bending

By neglecting the higher-order terms and eliminating the temperature resultant γ_3 , the strong formulation of the strain gradient thermoelastic Timoshenko beam model (3.15)–(3.20) is reduced to the system

$$(\kappa GA(w' - \beta))' - (\kappa GA g^2(w'' - \beta'))'' = -f, \quad (5.1)$$

$$(E(I + g^2 A)\beta')' + \kappa GA(w' - \beta) - (\kappa GA g^2(w'' - \beta'))' = -\tilde{m}, \quad (5.2)$$

valid for all $x \in (0, L)$, where $A = bh$ and $\tilde{m}$ is defined as $\tilde{m} = m + (\alpha E(\gamma_1 + g^2 \gamma_2))'$. The corresponding boundary conditions at $x = 0, L$ are rewritten in the form

$$w = w_0 \quad \text{or} \quad Q_1 = \kappa GA(w' - \beta) - (\kappa GA g^2(w'' - \beta'))', \quad (5.3)$$

$$w' = w_1 \quad \text{or} \quad Q_2 = \kappa GA g^2(w'' - \beta'), \quad (5.4)$$

$$\begin{aligned} \beta = \beta_0 \quad \text{or} \quad M_1 = E(I + g^2 A)\beta' - \kappa GA g^2(w'' - \beta') \\ + \alpha E(\gamma_1 + g^2 \gamma_2). \end{aligned} \quad (5.5)$$

In what follows, the material parameters used in the beam model are the ones of Table 1 obtained via computational homogenization, i.e., $E = E^*$, $G = E^*/(2(1 + \nu^*))$, $\alpha = \alpha^*$.

For a cantilever beam bended by a transversal force applied at the free end (see Fig. 5.1 (left)), we exclude body forces ($f = 0$ and $m = 0$) and eliminate the temperature coefficients ($\gamma_1 = 0$ and $\gamma_2 = 0$). The boundary conditions are defined as $w(0) = 0$, $\beta(0) = 0$, $Q_2(0) = 0$, $Q_1(L) = F$, $M_1(L) = 0$, $Q_2(L) = 0$, resulting in the following expression for the deflection of the homogeneous beam:

$$w(x) = \frac{F}{6E(I + g^2 A)}(3Lx^2 - x^3) + \frac{F}{\kappa GA}x. \quad (5.6)$$

The bending rigidity defined as $D = F/w(L)$ is expressed in the form (cf. [15])

$$D = \frac{\kappa GAE(I + g^2 A)}{L(L^2 \kappa GA/3 + EI + EA g^2)}, \quad (5.7)$$

where, for a rectangular cross section, $I = bh^3/12$ and $\kappa = 0.85$ [57]. For the classical Timoshenko beam model, the bending rigidity is calculated from (5.7) by setting $g = 0$.

For the numerical simulations of a thick lattice truss, the problem setting is the following: the left edge ($x = 0$) is fixed, the right edge ($x = L$) is affected by a concentrated transversal force F as shown in Fig. 5.1 (left). The length-to-thickness ratio is fixed to $L_0/h_0 = 2.31$ for all samples (see Fig. 4.2 (left)). The bending rigidity is calculated as the ratio of the applied force to the deflection w_L at $y = 0$, $x = L$ (represented in Fig. 5.1 (left)) as $D = F/w_L$.

The lattice truss response is depicted in Fig. 5.2 (left) as the dependence of the bending rigidity on the truss thickness. As can be seen, the strain gradient Timoshenko model (Gradient T, red line) captures the size-dependent bending of thick beams very accurately, unlike the Bernoulli–Euler beam model (Gradient BE, green line). It is worth noting that the red line is defined according to equation (5.7) with the values for effective classical moduli E and G (or E and ν) taken from Table 1 and with the g -parameter calibrated in Section 4.3.

Figure 5.1: Problem settings for the computational analysis of cantilever structures. Left: Mechanical bending by applied transversal force F : a lattice structure strip (red) with $h = 4.33$ mm and $L = 10$ mm, the corresponding strain gradient Timoshenko beam (blue) and classical Timoshenko beam (green) models. Right: Thermal bending by applied external temperatures $\pm T_0$: a lattice structure strip (red) with $h = 4.33$ mm and $L = 10$ mm, the corresponding strain gradient thermoelastic Timoshenko beam by including only temperature (grey) and both temperature and temperature gradient (blue) into the Helmholtz free energy.

As in Section 4.3, the comparison of the lattice truss response with the corresponding prediction based on the classical and generalized Timoshenko beam models is depicted in Fig. 5.1 (left), and in Fig. B.1 (right) of Appendix B. The lattice truss is shown in red colour. It can be seen that the deflection curve of the generalized Timoshenko beam model (blue solid line) follows the deformation of the lattice truss middle line almost exactly,

Figure 5.2: Bending rigidity (left) and rotation angle (right) against the strip thickness.

whereas the classical Timoshenko beam model (green line) overestimates the behaviour roughly by factor 2. For the isogeometric analysis of the generalized Timoshenko beam problem (3.15)–(3.20), the corresponding weak and discrete formulations are given in Appendix D.

For thermally induced bending of a cantilever beam, the constant temperatures T_0 and $-T_0$ are prescribed at the upper ($y = -h/2$) and lower ($y = h/2$) edges, respectively, as represented in Fig. 5.1 (right). The temperature field is assumed to be linear across the beam thickness, i.e., $\theta(x, y) = -2T_0y/h$. With boundary conditions $w(0) = 0$, $\beta(0) = 0$, $Q_2(0) = 0$, $Q_1(L) = 0$, $M_1(L) = 0$, $Q_2(L) = 0$ and temperature coefficients defined as $\gamma_1 = -bT_0h^2/6$ and $\gamma_2 = -2bT_0$, one can derive the following expression for the deflection of the Timoshenko beam with a rectangular cross-section:

$$w(x) = \frac{\alpha T_0}{h} x^2, \quad (5.8)$$

corresponding to the complete thermoelastic material model. For the incomplete one (γ_2 is omitted), the beam deformation is expressed in the form

$$w(x) = \frac{\alpha T_0}{h(1 + 12g^2/h^2)} x^2 \quad (5.9)$$

containing the extra multiplier $(1 + 12g^2/h^2)$, as for the incomplete Bernoulli–Euler beam model in (4.10), and leading to the underestimation and thickness-dependent predictions depicted in Fig. 5.2 (right).

The dependence of the normalized deflection of the free end $x = L$, $y = 0$ (w_L/L , see Fig. 5.1 (right)) on the truss thickness is shown in Fig. 5.2 (right) and represented by the blue dots. The size-independent lattice truss bending response is precisely captured by the complete thermoelastic Timoshenko beam model represented by the horizontal red solid line (based on equation (5.8)). The size-dependent prediction based on the incomplete thermoelastic material model (based on equation (5.9)) is represented by the dashed black line. The deformation response in thermal bending is compared to the fine scale truss model in Fig. 5.1 (right).

6. Application to functionally step-wise-graded beams

In this section, we consider thin and thick trusses with a triangular lattice microarchitecture varying in the length direction as shown in Figs. 6.1 and 6.2. The left half of each truss is made of a triangular lattice with the RVE dimensions given in Table 1. The right half is composed of a microarchitecture with RVE scaled either by factor 0.5 as in Fig. 6.1 (middle) and Fig. 6.2 (left), referred as trusses of type 1, or by factor 0.25 as in Fig. 6.1 (bottom) and Fig. 6.2 (right), referred as trusses of type 2. Upon homogenization, the effective classical moduli coincide with each other, i.e., $E_1 = E_2 = E^* = 246.7$ MPa and $\alpha_1 = \alpha_2 = \alpha^* = 0.00074$ $^{\circ}C^{-1}$ (cf. Table 1), while the length scale parameters are changed proportionally to the change of the RVE size: $g_2 = g_1/2$ and $g_2 = g_1/4$ for scaling factors 0.5 and 0.25, respectively, with $g_1 = 1.57$ mm (equal to g of the preceding subsection; see [15], for details). Such trusses are modelled as homogenized functionally step-wise-graded beams depicted in Fig. 6.1 (top) with models described in Section 3. Contact conditions at the interphase at the homogenized level correspond to the continuous deflection, first derivative of deflection, ordinary bending force and bending moment in Bernoulli–Euler beams (see Appendix E) and to the continuous deflection, first derivative of deflection, rotation, higher-order bending force, ordinary bending force and bending moment in Timoshenko beams (see Appendix F).

Figure 6.1: Representation of functionally step-wise-graded beam with piecewise constant thermoelastic properties (top). Long, thin lattice trusses ($L = 92.5$ mm and $h = 4.33$) with the right half containing a microarchitecture scaled by factor 0.5 (middle) and 0.25 (bottom) with respect to the basic microarchitecture of the left half. Upon homogenization, the effective classical moduli coincide ($E_1 = E_2$ and $\alpha_1 = \alpha_2$), while the length scale parameters g_1 and g_2 are different.

Figure 6.2: Short, thick lattice trusses ($L = 12.5$ mm and $h = 4.33$) with the right half (shaded with dark grey) containing a microarchitecture scaled by factor 0.5 (left) and 0.25 (right) with respect to the basic microarchitecture of the left half (shaded with light grey).

Figure 6.3: Mechanical bending of long, thin lattice structure strips (red) of type 1 (top) and type 2 (bottom), the corresponding strain gradient Bernoulli–Euler beam (blue) and classical Bernoulli–Euler beam (green).

For the long and thin trusses, the deformed geometries are shown in Fig. 6.3 for mechanical bending and in Fig. 6.6 for thermal bending. They are compared to the corresponding predictions of the Bernoulli–Euler beam models of Section 3 (both complete and incomplete thermoelastic models). It is remarkable that the classical beam model fails to describe the truss behaviour not only quantitatively, but qualitatively as well, as shown in Fig. 6.3. For the trusses of type 1 and type 2, the location of the maximal deflection is shifted by 5.3 mm and by 6.6 mm, respectively, with respect to the middle point ($x = L/2$) being the point with maximal deflection within the classical model.

Figure 6.4: Deflection (left) and rotation (right) against the normalized x -coordinate with $g_1 = 1.57$ mm and $g_2 = g_1/2$.

Figure 6.5: Deflection (left) and rotation (right) against the normalized x -coordinate with $g_1 = 1.57$ mm and $g_2 = g_1/4$.

The deflection and rotation of the lattice truss cross sections are compared in Figs. 6.4 and 6.5 with the corresponding analytical expressions derived in Appendix E. In Figs. 6.4 (right) and 6.5 (right), it can be seen that due to the variation of the truss microarchitecture in the longitudinal direction the rotation angle of the truss cross sections is a piecewise linear function of the x -coordinate, perfectly captured by the generalized Bernoulli–Euler beam model.

Figure 6.6: Thermal bending of long, thin lattice structure strips (red) of type 1 (top) and type 2 (bottom), the corresponding strain gradient thermoelastic Bernoulli-Euler beam within the incomplete (grey) and complete (blue) models.

For the short and thick beams, the mechanical and thermal bending responses of the lattice trusses are presented in Figs. 6.7 and 6.10, respectively, and compared to the corresponding predictions of the Timoshenko beam models of Section 3 (both complete and incomplete thermoelastic models). Figs. 6.8 and 6.9 compare the deflection and rotation of the lattice truss cross sections with the corresponding analytical solutions of Appendix F and demonstrate the validity of the generalized Timoshenko beam model. In particular, it is shown that the rotation angle of the truss cross sections, a piecewise quadratic function of the x -coordinate, is accurately described by the generalized model.

Figure 6.7: Mechanical bending of short, thick lattice structure strips (red) of type 1 (left) and type 2 (right), the corresponding strain gradient Timoshenko beam (blue) and classical Timoshenko beam (green).

Figure 6.8: Deflection (left) and rotation (right) against the normalized x -coordinate with $g_1 = 1.57$ mm and $g_2 = g_1/2$.

Figure 6.9: Deflection (left) and rotation (right) against the normalized x -coordinate with $g_1 = 1.57$ mm and $g_2 = g_1/4$.

Figure 6.10: Thermal bending of short, thick lattice structure strips (red) of type 1 (left) and type 2 (right), corresponding strain gradient thermoelastic Timoshenko beam within the incomplete (grey) and complete (blue) models.

It should be noted that within strain gradient theories there are several admissible contact conditions at interphases as addressed in [63]. The first type, given as a *higher order contact with complete transmissibility of energy*, implies that both displacement vector and strain tensor are continuous across interphases. According to (3.2) and (3.11), this leads, respectively, to a continuous second derivative of deflection for strain gradient Bernoulli–Euler beams and to a continuous first derivative of rotation for strain gradient Timoshenko beams. The second type, given as a *kinematical contact as in classical elasticity with partial transmissibility of energy*, implies a continuous first derivative of deflection for strain gradient Bernoulli–Euler beams and a continuous rotation for strain gradient Timoshenko beams. It should be further emphasized that, first, the second type contact can be realized in the reduced strain gradient beams introduced in Sections 4.3 and 5, and second, metamaterials with triangular microarchitectures demonstrate at interphases the kinematical contact as in classical elasticity at the homogenized level.

7. Conclusions

This work is a part of the development and applicability investigation of generalized continuum theories in the context of thermomechanics of solids, structures, materials and meta-materials. In this respect, the main concluding remarks are summarized as follows:

1. A clear necessity of a reconsideration of the classical thermodynamics towards generalized continuum mechanics is manifested.
2. Constitutive relations which include both temperature and temperature gradient are shown to be a consistent framework of strain gradient thermoelasticity.
3. Thermomechanical bending of lattice beams having homogeneously and functionally step-wise distributed triangular microarchitectures is accurately represented, respectively, by gradient-thermoelastic homogeneous and functionally graded beam models in the frameworks of both Bernoulli–Euler and Timoshenko assumptions.
4. From the point of view of functional grading, the gradient parameter of the generalized beam model can be effectively used for designing a desired microarchitecture with respect to the deflection distribution of a lattice beam by tuning the corresponding microarchitectural density.
5. A general remark concerning the bending theories of elastic gradient beams recently revisited in [64] can be stated as follows: The existing experimental

results on the size dependency of beams in bending, obtained for different length scales in both real [65, 66] and virtual [15] experiments (as in the present work), support the "full gradient" approach for the model derivation originally proposed in [67] rather than the "axial gradient" approach originally proposed in [68]. It should be mentioned that both models have been revisited and extended to plates in a bunch of later works and compared to each other and to other generalized beam models in [56, 64], in particular. Finally, we note that another very recent comparison of these two approaches in the framework of Kirchhoff–Love shells [69] supports the "full gradient" approach by providing a comparison to the corresponding 3D strain gradient solid.

6. The demonstrated size-dependent mechanical and size-independent thermal beam bending phenomena are assumed to be observed in bending of the analogous plate structures. This investigation can be considered as a natural step for future research.

Acknowledgements

Authors have been supported by Academy of Finland through the project *Adaptive isogeometric methods for thin-walled structures* (decision numbers 270007, 304122). The second author gratefully acknowledges the support of the August-Wilhelm Scheer Visiting Professors Program established by TUM International Center and funded by the German Excellence Initiative. Access and licenses for the commercial FE software Abaqus have been provided by CSC – IT Center for Science (www.csc.fi). The authors are grateful to DSc Viacheslav Balobanov for useful discussions.

Appendices

A. On stress and kinematical assumptions

For convenience, constitutive relations (2.8)–(2.9) are duplicated below

$$\tau_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - \alpha\theta(2\mu + 3\lambda)\delta_{ij}, \quad (\text{A.1})$$

$$\tau_{ijk} = g^2(2\mu\partial_i\varepsilon_{jk} + \lambda\partial_i\varepsilon_{ll}\delta_{jk}) - g^2(2\mu + 3\lambda)\partial_i(\alpha\theta)\delta_{jk}. \quad (\text{A.2})$$

Along with the kinematical relations (3.2), it is assumed that

$$\tau_{\alpha\beta} = 0, \quad \alpha, \beta = 2, 3, \quad (\text{A.3})$$

which is used in the constitutive equation (A.1) in order to eliminate $\varepsilon_{\alpha\beta}$ leading to

$$\varepsilon_{yy} = \varepsilon_{zz} = \frac{\alpha\theta(2\mu + 3\lambda) - \lambda\varepsilon_{xx}}{2(\mu + \lambda)}, \quad \varepsilon_{yz} = 0. \quad (\text{A.4})$$

By redefining the Lamé parameters as $\mu = E/2/(1 + \nu)$ and $\lambda = \nu E/(1 + \nu)/(1 - 2\nu)$, the classical expression for τ_{xx} is retrieved

$$\tau_{xx} = E(\varepsilon_{xx} - \alpha\theta). \quad (\text{A.5})$$

The rest of the stress components are equal to zero.

Regarding double stresses, we adopt the following simplifications (see [15]):

$$\tau_{i\alpha\beta} = 0, \quad \alpha, \beta = 2, 3, \quad i = 1, 2, 3, \quad (\text{A.6})$$

which eliminates $\partial_i\varepsilon_{\alpha\beta}$ in the constitutive relation (A.2) giving

$$\partial_i\varepsilon_{yy} = \partial_i\varepsilon_{zz} = \frac{\partial_i(\alpha\theta)(2\mu + 3\lambda) - \lambda\partial_i\varepsilon_{xx}}{2(\mu + \lambda)}, \quad \partial_i\varepsilon_{yz} = 0. \quad (\text{A.7})$$

The active double stress components are redefined in the form

$$\tau_{xxx} = g^2 E \partial_x(\varepsilon_{xx} - \alpha\theta), \quad \tau_{yxx} = g^2 E \partial_y(\varepsilon_{xx} - \alpha\theta). \quad (\text{A.8})$$

B. On the cross section kinematics

Deformations in the x -direction of the lattice truss cross sections are compared in Fig. B.1 with the deformations following the kinematical assumptions (3.2) and (3.11). The left figure corresponds to the simply supported long trusses of Section 4.3 with length to thickness ratio $L/h = 20.8$. Cross sections are considered at $x = L/4$. The normalization is performed with respect to the displacement at point $x = L/4$, $y = -h/2$ corresponding to the classical Bernoulli–Euler beam model, i.e., $u_x = 3M_0L/(2Ebh^2)$. The right figure corresponds to the clamped short trusses of Section 5 with length to thickness ratio $L/h = 2.3$. Cross sections are considered at $x = L/2$. The normalization is performed with respect to displacement at point $x = L/2$, $y = -h/2$ corresponding to the classical Timoshenko beam model, i.e., $u_x = 9FL^2/(4Ebh^2)$.

Black dots, green circles, blue triangles and red diamonds correspond, respectively, to the grid point x -displacements of the lattice trusses with $N = 1$,

2, 3 and 8 (see the description in Section 4.1). The black solid lines represent the analytical solutions describing the beam cross section deformations (3.2) and (3.11) which are assumed to behave as linear functions with respect to the y -argument. As it can be seen, the lattice truss cross sections remain straight and the corresponding grid points lie on the black solid lines, which approve the validity of kinematical assumptions (3.2) and (3.11).

Figure B.1: Displacement profiles of lattice truss cross sections in case of simply supported long (left) and clamped short (right) beam bending problems.

C. Weak formulation of gradient-thermoelastic Bernoulli–Euler beam problem

The variational, or weak, formulation of the strong form (3.6)–(3.9) reads as follows:

Find $w \in W \subset H^3(\Omega)$, $\Omega = (0, L)$, such that

$$a(w, \hat{w}) = l(\hat{w}) \quad \forall \hat{w} \in V \subset H^3(\Omega), \quad (\text{C.1})$$

where bilinear form $a: W \times V \rightarrow \mathbb{R}$ and load functional $l: V \rightarrow \mathbb{R}$, respectively, are defined as

$$a(w, \hat{w}) = \int_0^L (E(I + g^2 A)w''\hat{w}'' + g^2 EIw'''\hat{w}''')dx, \quad (\text{C.2})$$

$$\begin{aligned} l(\hat{w}) = & \int_0^L f\hat{w}dx + Q\hat{w}|_0^L + M\hat{w}'|_0^L + R\hat{w}''|_0^L \\ & - \int_0^L (E\alpha(\gamma_1 + g^2\gamma_2)\hat{w}'' + g^2 E\gamma_3\hat{w}''')dx, \end{aligned} \quad (\text{C.3})$$

with coefficients $\gamma_1(x)$, $\gamma_2(x)$ and $\gamma_3(x)$ defined in (3.10).

For the problem formulation given in (3.6)–(3.9), the trial and test function sets consist of $H^3(\Omega)$ functions satisfying suitable boundary conditions. The conforming Galerkin formulation for the problem reads as follows: Find $w_h \in W_h \subset W$ such that

$$a(w_h, \hat{w}) = l(\hat{w}) \quad \forall \hat{w} \in V_h \subset V. \quad (\text{C.4})$$

For further details on implementation utilizing NURBS-based isogeometric Galerkin methods, we refer to [62].

D. Weak formulation of gradient-thermoelastic Timoshenko beam problem

The variational, or weak, formulation of the strong form (3.15)–(3.20) reads as follows:

Find $w \in W \subset H^2(\Omega)$ and $\beta \in V \subset H^2(\Omega)$, $\Omega = (0, L)$, such that

$$a(w, \beta; \hat{w}, \hat{\beta}) = l(\hat{w}, \hat{\beta}) \quad \forall \hat{w} \in \hat{W}, \forall \hat{\beta} \in \hat{V}, \quad (\text{D.1})$$

where bilinear form $a: (W \times V) \times (\hat{W} \times \hat{V}) \rightarrow \mathbb{R}$ and load functional $l: \hat{W} \times \hat{V} \rightarrow \mathbb{R}$, respectively, are defined as

$$a(w, \beta; \hat{w}, \hat{\beta}) = \int_0^L (E(I + Ag^2)\beta' \hat{\beta}' + EIg^2\beta'' \hat{\beta}'' + \kappa GA(w' - \beta)(\hat{w}' - \hat{\beta}) + \kappa GA g^2(w'' - \beta'')(\hat{w}'' - \hat{\beta}'')) dx, \quad (\text{D.2})$$

$$l(\hat{w}, \hat{\beta}) = \int_0^L (q\hat{w} + m\hat{\beta}) dx + Q_1 \hat{w}|_0^L + Q_2 \hat{w}'|_0^L + M_1 \hat{\beta}|_0^L + M_2 \hat{\beta}'|_0^L - \int_0^L (E\alpha(\gamma_1 + g^2\gamma_2)\hat{\beta}' + g^2 E\gamma_3 \hat{\beta}'') dx. \quad (\text{D.3})$$

The corresponding trial and test function sets consist of $H^2(\Omega)$ functions satisfying suitable boundary conditions. The conforming Galerkin formulation for the problem reads as follows:

Find $w_h \in W_h \subset W$ and $\beta_h \in V_h \subset V$ such that

$$a(w_h, \beta_h; \hat{w}, \hat{\beta}) = l(\hat{w}, \hat{\beta}) \quad \forall \hat{w} \in \hat{W}_h \subset \hat{W}, \quad \forall \hat{\beta} \in \hat{V}_h \subset \hat{V}. \quad (\text{D.4})$$

It is worth noting that the weak formulation is given in the standard form which is shown to suffer from numerical shear locking phenomena in [58].

E. Analytical solutions for functionally step-wise-graded Bernoulli–Euler beams

For a constant Young's modulus E and a piecewise constant length scale parameter g

$$g(x) = \begin{cases} g_1, & x \in (0, L/2) \\ g_2, & x \in (L/2, L) \end{cases}, \quad (\text{E.1})$$

the closed-form solution of strong form (4.4)–(4.6) (with $f = 0$) takes the general form

$$w(x) = \begin{cases} w_1(x) = A_1 x^3 + A_2 x^2 + A_3 x + A_4, & x \in (0, L/2) \\ w_2(x) = B_1 x^3 + B_2 x^2 + B_3 x + B_4, & x \in (L/2, L) \end{cases}. \quad (\text{E.2})$$

For mechanically induced bending ($\gamma_1 = 0, \gamma_2 = 0$) of a simply supported beam with boundary conditions $w(0) = w(L) = 0, M(0) = M(L) = -M_0$ and compatibility conditions at $x = L/2$: $\llbracket w \rrbracket = 0, \llbracket w' \rrbracket = 0, \llbracket Q \rrbracket = 0, \llbracket M \rrbracket = 0$, integration constants A_i and $B_i, i = 1, 2, 3, 4$, are defined as

$$\begin{aligned} A_2 &= -\frac{M_0}{2E(g_1^2 A + I)}, \quad A_1 = 0, \quad A_4 = 0, \\ A_3 &= \frac{LM_0(g_1^2 A + 3Ag_2^2 + 4I)}{8E(g_1^2 A + I)(Ag_2^2 + I)}, \\ B_2 &= -\frac{M_0}{2E(g_2^2 A + I)}, \quad B_1 = 0, \\ B_3 &= \frac{LM_0(5g_1^2 A - Ag_2^2 + 4I)}{8E(g_1^2 A + I)(Ag_2^2 + I)}, \\ B_4 &= -\frac{M_0 L^2 A(g_1^2 - g_2^2)}{8E(g_1^2 A + I)(Ag_2^2 + I)}. \end{aligned} \quad (\text{E.3})$$

The double square braces denote the jump of a quantity at interphase as $\llbracket w \rrbracket = w_1 - w_2$, for instance.

For thermally induced beam bending within the complete thermoelastic material model, temperature coefficients (3.10) are explicitly defined as $\gamma_1 = bT_0 h^2/6$ and $\gamma_2 = 2bT_0$, which with $M(0) = M(L) = 0$ and $I = bh^3/12$ results in the following expression for the integration constants:

$$\begin{aligned} A_1 &= 0, \quad A_4 = 0, \quad A_2 = -\frac{\alpha T_0}{h}, \quad A_3 = \frac{\alpha T_0 L}{h}, \\ B_1 &= 0, \quad B_4 = 0, \quad B_2 = -\frac{\alpha T_0}{h}, \quad B_3 = \frac{\alpha T_0 L}{h}. \end{aligned} \quad (\text{E.4})$$

F. Analytical solutions for functionally step-wise-graded Timoshenko beams

For constant Young's modulus E and shear modulus G and a piecewise constant length scale parameter g

$$g(x) = \begin{cases} g_1, & x \in (0, L/2) \\ g_2, & x \in (L/2, L) \end{cases}, \quad (\text{F.1})$$

the closed-form solution of strong form (5.1)–(5.5) (with $f = 0$ and $\tilde{m} = 0$) takes the general form

$$w(x) = \begin{cases} w_1(x) = A_4x^3 + A_3x^2 + A_2x + A_1 \\ \quad + A_5 \sinh(x/g_1) + A_6 \cosh(x/g_1), & x \in (0, L/2) \\ w_2(x) = B_4x^3 + B_3x^2 + B_2x + B_1 \\ \quad + B_5 \sinh(x/g_2) + B_6 \cosh(x/g_2), & x \in (L/2, L) \end{cases}, \quad (\text{F.2})$$

$$\beta(x) = \begin{cases} \beta_1(x) = 3A_4x^2 + 6A_4E/(\kappa G)(g_1^2 + I/A) + 2A_3x + A_2, & x \in (0, L/2) \\ \beta_2(x) = 3B_4x^2 + 6B_4E/(\kappa G)(g_2^2 + I/A) + 2B_3x + B_2, & x \in (L/2, L) \end{cases}. \quad (\text{F.3})$$

For mechanically induced bending ($\gamma_1 = 0$, $\gamma_2 = 0$) of a cantilever beam with boundary conditions $w(0) = 0$, $\beta(0) = 0$, $Q_2(0) = 0$, $Q_1(L) = F$, $Q_2(L) = 0$, $M_1(L) = 0$ and compatibility conditions at $x = L/2$: $\llbracket w \rrbracket = 0$, $\llbracket w' \rrbracket = 0$, $\llbracket \beta \rrbracket = 0$, $\llbracket Q_1 \rrbracket = 0$, $\llbracket Q_2 \rrbracket = 0$, $\llbracket M_1 \rrbracket = 0$, integration constants A_i and B_i , $i = 1, \dots, 6$, take the form

$$\begin{aligned} A_2 &= \frac{F}{\kappa GA}, \quad A_1 = 0, \quad A_5 = 0, \quad A_6 = 0, \\ A_3 &= \frac{FL}{2E(I + g_1^2 A)}, \quad A_4 = -\frac{F}{6E(I + g_1^2 A)}, \\ B_1 &= \frac{FAL^3(g_1^2 - g_2^2)}{12E(I + g_1^2 A)(I + g_2^2 A)}, \quad B_5 = 0, \quad B_6 = 0, \\ B_2 &= F \frac{3A^2\kappa GL^2(g_2^2 - g_1^2) + 8E(I + g_1^2 A)(I + g_2^2 A)}{8\kappa GAE(I + g_1^2 A)(I + g_2^2 A)}, \\ B_3 &= \frac{FL}{2E(I + g_2^2 A)}, \quad B_4 = -\frac{F}{6E(I + g_2^2 A)}. \end{aligned} \quad (\text{F.4})$$

For thermally induced beam bending within the complete thermoelastic material model, we set $Q_1(L) = 0$, $I = bh^3/12$, $\gamma_1 = -bT_0h^2/6$ and $\gamma_2 = -2bT_0$, which defines the integration constants in the form

$$\begin{aligned} A_3 &= \frac{\alpha T_0}{h}, \quad A_1 = 0, \quad A_2 = 0, \quad A_4 = 0, \quad A_5 = 0, \quad A_6 = 0, \\ B_3 &= \frac{\alpha T_0}{h}, \quad B_1 = 0, \quad B_2 = 0, \quad B_4 = 0, \quad B_5 = 0, \quad B_6 = 0. \end{aligned} \quad (\text{F.5})$$

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