Zyuzin, Alexander A.; Simon, Pascal

Disorder-induced exceptional points and nodal lines in Dirac superconductors

Published in:
Physical Review B

DOI:
10.1103/PhysRevB.99.165145

Published: 29/04/2019

Document Version
Publisher's PDF, also known as Version of record

Please cite the original version:

This material is protected by copyright and other intellectual property rights, and duplication or sale of all or part of any of the repository collections is not permitted, except that material may be duplicated by you for your research use or educational purposes in electronic or print form. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone who is not an authorised user.
Disorder-induced exceptional points and nodal lines in Dirac superconductors

Alexander A. Zyuzin\textsuperscript{1,2} and Pascal Simon\textsuperscript{3}

\textsuperscript{1}Department of Applied Physics, Aalto University, P. O. Box 15100, FI-00076 AALTO, Finland
\textsuperscript{2}Ioffe Physical-Technical Institute, 194021 St. Petersburg, Russia
\textsuperscript{3}Laboratoire de Physique des Solides, CNRS, Univ. Paris-Sud, University Paris-Saclay, 91405 Orsay Cedex, France

(Received 23 January 2019; revised manuscript received 1 April 2019; published 29 April 2019)

We consider the effect of disorder on the spectrum of quasiparticles in the point-node and nodal-line superconductors. Due to the anisotropic dispersion of quasiparticles disorder scattering may render the Hamiltonian describing these excitations non-Hermitian. Depending on the dimensionality of the system, we show that the nodes in the spectrum are replaced by Fermi arcs or Fermi areas bounded by exceptional points or exceptional lines, respectively. These features are illustrated by first considering a model of a proximity-induced superconductor in an anisotropic two-dimensional (2D) Dirac semimetal, where a Fermi arc in the gap bounded by exceptional points can be realized. We next show that the interplay between disorder and supercurrents can give rise to a 2D Fermi surface bounded by exceptional lines in three-dimensional (3D) nodal superconductors.

DOI: 10.1103/PhysRevB.99.165145

I. INTRODUCTION

The physics of non-Hermitian systems and of exceptional points typically arises in open quantum systems with energy gain and loss. Their complex spectra can present exceptional points \cite{1} which occur when the linewidths of two neighboring resonance frequencies coalesce under some fine-tuning of the system parameters \cite{2--4}. Such physics has been realized experimentally in various open quantum systems where gain and loss can be introduced in a controllable manner, for example in circuits of resonators \cite{5,6}, in photonic systems \cite{7--13}, or in cold atomic gas \cite{14}. Signatures of exceptional points and loops were recently observed in optical waveguides \cite{15--17}.

In contrast, non-Hermiticity can also naturally emerge in a disordered and/or interacting closed system if we are interested in single-particle excitations. In the presence of disorder, a single-particle excitation of a given momentum acquires a finite lifetime. One can thus associate a non-Hermitian Hamiltonian in order to describe these single-particle excitations \cite{18--23}.

Because topology has completely reshaped our understanding of electronic band structure in solids, there has been in the past years a growing interest in analyzing the topological properties of non-Hermitian Hamiltonians \cite{14,19,21,24--33}. The topologically nontrivial electronic bands are often characterized by the presence of Dirac touching points or line nodes. A natural question is how these band structures are modified for non-Hermitian Hamiltonians. In the simplest non-Hermitian extension of a 2D Dirac semimetal, the quasiparticle conduction and valence bands are connected by a bulk Fermi arc that ends at two topological exceptional points instead of a touching Dirac point \cite{19}. At these singularities, the non-Hermitian Hamiltonian becomes defective.

This extends to a 3D point node semimetal where the two bands, by adding a non-Hermitian term to the Hamiltonian, can stick on a surface bounded by a one-dimensional (1D) loop of exceptional points \cite{14,20,34}. In 3D nodal-line semimetals, the nodal lines can be split into two exceptional lines, connected by a Fermi ribbon \cite{20,23,35--40}. In these cases, the real part of the spectrum with momenta lying within the 1D arc (the 2D area) bounded by the exceptional points (or loops) is zero. More generally, it has been shown that $D$-dimensional non-Hermitian systems can support up to $(D-1)$-dimensional exceptional surfaces thanks to parity-time or parity-particle-hole symmetries \cite{38,39}.

Recently, the interplay of topology and non-Hermiticity was extended on one-dimensional superconductors with a particular emphasis on the difference between Andreev and Majorana bound sates \cite{41--45}. Here, we focus on the band dispersion of the quasiparticles excitations in 2D and 3D nodal superconductors in the presence of weak disorder. As for the semimetallic case, we find that the complex self-energy correction to the Green function of quasiparticles due to disorder gives rise to non-Hermitian Bogoliubov–de Gennes (BdG) Hamiltonians describing quasiparticle bands with exceptional points and lines.

To be more specific, we first consider the proximity induced superconductivity in a 2D Dirac semimetal with a tilted Dirac cone. We find a phase transition as a function of the Dirac cone anisotropy, between the gapped state and the state where the superconducting gap closes in momentum space at the Fermi arc bounded by two exceptional points. We then analyze the effect of weak disorder on 3D nodal-ring and point-node superconductors in the presence of the supercurrent flow. We find that the nodal structure in the spectrum of quasiparticles extends into the flat-band regions bounded by the exceptional lines.

Our paper is structured as follows. Section II contains a short reminder of disorder averaging within the Born approximation. In Sec. III, we consider the effects of disorder on 2D Dirac semimetals in proximity of a superconductor. In Sec. IV, we deal with the effects of disorder scattering in 3D nodal superconductors. Finally, in Sec. V we summarize and discuss our results.
II. DISORDER SCATTERING

Let us first recall the effect of the electron elastic scattering on the random scalar potential disorder on the spectrum of nodal superconductors. One of the prominent effects of the disorder on the unusual superconductivity is to smear the nodes in the superconducting gap in momentum space, which can be described by the imaginary part of the self-energy correction to the disorder averaged Green function, Refs. [46,47]. We consider the short-range impurity scattering potential of the form $V(r) = iG_0 \delta(r - r_i)$, where the sum is over all impurities coordinates $r_i$. The energy and the decay-time of the quasiparticles are defined by the real and imaginary parts of the poles of the disorder-averaged Green function in the superconducting state, respectively.

To find these poles, we will be utilizing the self-consistent equation for the quasiparticle’s retarded self-energy $\Sigma(E)$ (a $4 \times 4$ matrix written in Nambu representation to include spin and particle-hole degrees of freedom), which within Born approximation is given by

$$\Sigma(E) = \gamma \int \frac{d^d k}{(2\pi)^d} \tau_z [E - H(k) - \Sigma(E)]^{-1} \tau_z, \quad (1)$$

where $\gamma = u^2 recalling the effect of the electron elastic scattering on the random scalar potential disorder on the spectrum of nodal superconductors. One of the prominent effects of the disorder on the unusual superconductivity is to smear the nodes in the superconducting gap in momentum space, which can be described by the imaginary part of the self-energy correction to the disorder averaged Green function, Refs. [46,47].

FIG. 1. The lines of zeros in the spectrum of quasiparticles of 2D Dirac semimetal with tilted Dirac cone and proximity-induced superconductivity. The plots show the zero solutions for the spectrum of quasiparticles, given explicitly by Eq. (4) in the text. Panels (a) and (b) show the solutions at two tilt parameters $\kappa = 0.4$ and $\kappa = 1.5$, respectively, for a fixed $\Delta$ and the chemical potential $\mu = 4\Delta$. The nodal lines exist at small tilt parameter $\Delta/\mu < |\kappa| < 1$. Panels (c) and (d) show the nodal lines in the spectrum at $\mu = 0.5\Delta$ and $\mu = 0$, respectively, at fixed $\Delta$ and $\kappa = 1.5$.

around the band-touching points [48,49]. The proximity effect can lead to a gap in the density of states. The linearized Hamiltonian around the single Dirac point (we will neglect disorder scattering between the Dirac valleys) in the presence of the proximity induced superconducting gap is given by

$$H(k) = \kappa v k_x + [v(\sigma_x k_y - \sigma_y k_x) - \mu] \tau_z + \Delta \tau_y, \quad (3)$$

where $\sigma_{x,y}$ are the Pauli matrices acting in the spin space, $\mu$ is the chemical potential, $\kappa$ is a dimensionless parameter related to the inclination of the Dirac cone ($\kappa = 0$ means no tilt), $v$ is the Fermi velocity in absence of tilt, and $\Delta$ is the proximity induced superconducting gap which is considered to be a positive real constant here. We set $\hbar = 1$ throughout our calculations.

The spectrum of quasiparticles is given by

$$E_{\pm,s}(k) = \kappa v k_s \pm \sqrt{(vk - s\mu)^2 + \Delta^2}, \quad (4)$$

where $s = \pm$. The solution of $E_{\pm,s}(k) = 0$ as a function of model parameters is shown in Fig. 1. At $\mu = 0$, the spectrum is gapped provided $|\kappa| < 1$ and hosts nodal lines in the opposite case, $|\kappa| > 1$ [50]. The nodal lines are determined by the equation

$$(\kappa^2 - 1)v^2 k_s^2 = v^2 k_x^2 + \Delta^2. \quad (5)$$

It is worth noting that the linearized Hamiltonian, for which the nodal lines are hyperbolas, is applicable only for momenta below some tilt dependent cutoff $k_0 \gg \Delta/v \sqrt{\kappa^2 - 1}$.
determining the width of the electron and hole pockets. When the chemical potential is placed in the conduction band (here, it is enough to consider $s = 1$), the transition between the gapped and nodal lines state takes place at much smaller values of the tilt parameter $|\kappa| = \frac{s}{2} \ll 1$; see Fig. 1(a). We will focus below on the case of low doping $\mu \to 0$, shown in Fig. 1(d), where the exceptional nodal lines can be realized.

### A. Normal case $\Delta = 0$

Before dealing with the superconducting case, let us first recall results about the normal case $\Delta = 0$ [19,21,51]. At $\kappa = 0$, the Dirac point is smeared due to weak scalar disorder [52]. To proceed, we consider two limiting cases of weak $|\kappa| < 1$ and strong $|\kappa| > 1$ inclinations of the Dirac cone. We search for the self-energy correction to the disorder averaged Green function of the quasiparticles in the presence of the proximity induced superconducting gap by substituting the expression for the Hamiltonian Eq. (3) into the Eq. (1).

We recover, as it was shown in Refs. [21,51], that in the presence of a finite but small tilt such that $|\kappa| < 1$, the self-energy in the normal case at zero frequency acquires a nontrivial matrix structure $\Sigma(0) = -i(1 + \kappa \sigma_y \tau_z) \Gamma_1$, (note that the self-energy in particle-hole space is described by a $\tau_z$ matrix), where

$$\Gamma_1 = 2\Lambda \frac{\sqrt{1 - \kappa^2}}{1 + \kappa^2} \exp \left[ -\frac{2\pi \sqrt{1 - \kappa^2}}{\gamma} \right].$$

Here, $\Lambda$ is the energy cutoff corresponding to the separation between the Dirac point and the closest bulk band (see Appendix A1 for more details). The quasiparticles spectrum is found by solving Eq. (2) which now becomes complex valued, $E_{\pm}(\mathbf{k}) = \kappa v k_x - i \Gamma_1 \pm \sqrt{v^2 k_x^2 + (v k_x + i\kappa \Gamma_1)^2}$. Hence, the dispersion of quasiparticles contains a line segment $k_x \in [-|\kappa| \Gamma_1, |\kappa| \Gamma_1]$ at $k_y = 0$ bounded by two exceptional points. Such a segment in the quasiparticles spectrum is characterized by a vanishing real part, the so-called bulk Fermi arc [19]. This means that the decay rate of a quasiparticle has a strong spatial anisotropy.

In the limit of strong tilt $|\kappa| > 1$, introducing a cutoff $\propto k_0$ for the width of the electron-hole pockets in momentum space, we obtain in the limit $|\kappa| k_0 \gg |E|$, where the density of states is determined by $k_0(\kappa)$. Importantly, the self-energy contains a frequency independent imaginary contribution, which also results in an unusual bulk Fermi arc in the spectrum of quasiparticles [19].

It is worth noting that the linearized model introduced in Eq. (3) cannot be applied in the limit $|\kappa| \to 1$ due to unavoidable higher-order corrections in momentum, which are not taken into account here.

### B. Superconducting case

We now consider the superconducting case where the superconducting gap is induced by proximity and analyze how the spectrum of 2D quasiparticles might be affected by disorder scattering.

**Weak tilt.** In the superconducting case the spectrum is gapped at $\kappa = 0$. At a weak tilt $0 < |\kappa| < 1$, in the limit $\Gamma_1 \ll \Delta$ where the first-order Born approximation should apply, we obtain (details are presented in Appendix A2)

$$\Sigma(E) = -\frac{\gamma}{2\pi \sqrt{1 - \kappa^2}} \ln \frac{2\Lambda}{\sqrt{\Delta^2 - E^2}} \times \left[ \frac{E}{1 - \kappa^2} \left(1 + \kappa \sigma_y \tau_z\right) - \Delta \tau_z \right].$$

We can notice in this expression that the increase of disorder $\gamma$ decreases the proximity induced superconducting gap and increases the anisotropy of the dispersion.

In the strong disorder limit, $\Gamma_1 \gg \Delta/|\kappa|^3$, we can self-consistently obtain the Fermi arc in the spectrum bounded by the exceptional points due to the imaginary part of the self-energy

$$\Sigma(0) = \frac{1 - \kappa^2}{2\kappa^2} \Delta \tau_z - i(1 + \kappa \sigma_y \tau_z) \Gamma_1,$$

which renders the disorder averaged Green function of quasiparticles non-Hermitian (the derivation of this result is detailed in Appendix A2). This means, in particular, that disorder drives the system through a Lifshitz transition between a gapped state and a state where the superconducting gap closes at the bulk Fermi arc.

**Strong tilt.** Such a phase transition can also take place in the strong tilt case, as we show below. Indeed, in the limit $|\kappa| > 1$ and considering that $|\kappa| k_0 \gg |E|$ and $|\kappa| k_0 \gg \Delta$, one recovers Eq. (7),

$$\Sigma(E) = -i \Gamma_2 \left(1 + \frac{\sigma_y}{\kappa} \tau_z\right) + O(E/|\kappa| k_0; \Delta/|\kappa| k_0).$$

The self-energy contains a frequency independent imaginary contribution that results in an unusual complex quasiparticles spectrum:

$$E_{\pm}(\mathbf{k}) = \kappa v k_x - i \Gamma_2 \pm \sqrt{\Delta^2 + v^2 k_y^2 + \left( v k_x + i \frac{\Gamma_2}{\kappa} \right)^2}.$$
the material (here the mean free path), the measured spectral
of a width typically larger than characteristic length scale of
a system with a given disorder realization, for a light beam
probe the spectral function at the surface of a compound. In
toemission spectroscopy seems the most natural technique to
proposed phase transition at which exceptional points emerge
depicted in Fig. 3, let us address the question of how the
of the Andreev modes predicted for the 2D Josephson junction
smaller than $\frac{\gamma}{\Delta}$.

FIG. 2. Panel (a) shows the gapped real part of the spectrum
$E(k)$ given by Eq. (10) at $k_y = 0$ as a function of momentum $k_x$
at $\Gamma_2/|k| < \Delta$. Panel (b) shows the spectrum at the phase
transition point $\Gamma_2/|k| = \Delta$, at which the superconducting gap closes
and then transforms into two exceptional points. Panels (c) and
(d) show respectively the real and imaginary parts of the spectrum at
$\Gamma_2/|k| > \Delta$.

assuming the width of the Josephson junction to be much
smaller than $v(k)|k|^{-1}/\Delta$; namely we set $L_x \to 0$.

Consider the interface along the line $y = 0$, at which
the proximity induced superconducting gap is given by $\Delta(y) = \Delta e^{-\gamma y/2}$ for $y > 0$ and $\Delta(y) = \Delta e^{\gamma y/2}$ for $y < 0$, where gap $\Delta$
is the same in both proximitized regions. Note that two bulk
superconductors are coupled through the 2D Dirac semimetal
and there is no direct coupling between them. At $\phi \neq 0$
there are two chiral modes, which propagate with the
momentum $q$ along $x$ in the same direction, with the wave
function $\Psi(q, y) \propto \exp \left( i q x - y \sin \frac{\phi}{2} \right) \Delta/\nu$ localized at the
interface [48]. The spectrum of these bound modes is similar to
the spectrum of bulk modes given by Eq. (10), namely

$$E_{\pm}(q) = \kappa v q - i \Gamma_2 \pm \sqrt{\Delta^2 \cos^2 \frac{\phi}{2} + \left( v q + i \frac{\Gamma_2}{\kappa} \right)^2}.$$  \hspace{1cm} (11)

In the special case, at $\phi = \pi$, the two chiral modes are
gapless and propagate in the same direction with dispersions
$E_{\pm}(q) = (\kappa \pm 1)v q - i \Gamma_2 (1 \mp 1/\kappa)$. The edge states have a
momentum independent imaginary part. Due to the strong tilt,
the edge states coexist with the pockets of bulk states, but
can however be separated in energy at $|k| < 1$. The evolution
of the spectrum as a function of the tilt is shown in Fig. 3.
We note that there are no localized modes at the interface
$x = 0$ between two superconductors for $|k| > 1$. This result
is consistent with Refs. [20,53].

Experimental signatures. Besides the specific dependence
of the Andreev modes predicted for the 2D Josephson junction
depicted in Fig. 3, let us address the question of how the
proposed phase transition at which exceptional points emerge
can be directly probed experimentally. Angle-resolved photoemission spectroscopy seems the most natural technique to
probe the spectral function at the surface of a compound. In
a system with a given disorder realization, for a light beam
of a width typically larger than characteristic length scale of
the material (here the mean free path), the measured spectral
function will be disorder averaged because different parts of
the surface are probed. Therefore, the spectral function probed
experimentally, $A(k, E) = -\frac{i}{\pi} \text{Im} \text{Tr} [E - H(k) - \Sigma(E)]^{-1}$
is defined as the disorder averaged retarded Green function of
quasi-particles. We plot the spectral function at zero frequency
in Figs. 4(a) and 4(b), where one directly observes the transi-
tion to a state hosting bulk Fermi arc as the proximity
induced gap is varied. It is also instructive to comment on the
frequency dependence of $A(k, E)$ in the limit of zero wave
vector which is related to the average density of states which
can be probed by scanning tunneling microscopy (STM). As shown in Figs. 4(c) and 4(d) the spectral function has
a single peak in the case of the state with the bulk Fermi arc.
At strong proximity effect, $\Delta \gg \Gamma_2$, the height of two
peaks located at $E_{\pm} = \pm |\Delta^2 - \Gamma_2^2 (1 + \kappa^{-2})|^{1/2}$ is given by
given by superfluid $^3$He) in the presence of the supercurrent flow with perconductor (our results can be mapped to a polar phase of the superfluid $^3$He) [54,55]. Notice that our discussion equally applies to the superfluid $^3$He [54,55].

IV. 3D NODAL SUPERCONDUCTORS UNDER WEAK DISORDER

Let us now consider several representative realizations of 3D topological superconductors with nodal lines or nodal points in their quasiparticles spectrum. Notice that our discussion equally applies to the superfluid $^3$He [54,55].

A. Nodal-line superconductors

The BCS Hamiltonian of a characteristic nodal line superconductor (our results can be mapped to a polar phase of superfluid $^3$He) in the presence of the supercurrent flow with velocity $v_z$, is given by [54]

$$H(k) = \frac{k^2 - k_F^2}{2m} \tau_z + v_z \cdot k + \Delta n_z \tau_x,$$  \hspace{1cm} (12)

where $n = k/k$ is the unit vector in the direction of momentum, $k_F$ is the Fermi momentum, $m$ is the effective mass, and $\Delta(v_z)$ is the gap in the spectrum (the gap is suppressed with the increase of supercurrent). The spectrum of quasiparticles at $|v_z| = 0$ has a nodal ring defined by the equation $k_x^2 + k_y^2 = k_F^2$ in the plane $k_z = 0$. We first consider a case of the supercurrent flowing along the $z$ axis. In this situation the Fermi surface appears above the Lifshitz transition occurring at $|v_z|k_F = \Delta$.

The self-energy due to scattering on the scalar disorder can be found self-consistently. Consider first the situation where the tilt is along the $z$ axis; namely we set for concreteness $v_z = v_z(0, 0, 1)$. The retarded self-energy taken at zero frequency within the first order in powers of $|v_z|k_F/\Delta < 1$ is given by

$$\Sigma(0, k_F) = -i\tau_0 \left[ 1 + \frac{v_z k_F}{\Delta} \tau_z \right],$$  \hspace{1cm} (13)

where $\Gamma = \Delta \cosh(2\tau \Delta)$ and $1/2\tau = \pi v \gamma$ is the scattering rate (details of this result are shown in Appendix B). The first term in Eq. (13) is well known [46,47] and implies that an infinitesimally weak scattering smears the nodal ring in three dimensions.

We show that this smearing becomes anisotropic due to the tilt, which gives rise to the off-diagonal terms in the self-energy matrix in Eq. (13), with the sign being defined by $v_z$, i.e., by the direction of the supercurrent. Indeed, taking into account the self-energy in Eq. (13), we can write the spectrum of quasiparticles in the plane $k_z = 0$ in the following form:

$$E_{\pm}(k) = -i\Gamma \pm \sqrt{\left( k_x^2 + k_y^2 - k_F^2 \right) / 2m} - \frac{\left( v_z k_F / \Delta \right)}{2} \left( k_x^2 + k_y^2 - k_F^2 \right),$$  \hspace{1cm} (14)

The nodal ring in the disordered superconductor extends into a ring of finite width (called a Fermi ribbon) with the inner and outer radiuses defined by $k_{\pm} = k_F \sqrt{1 \pm \frac{v_k v_z}{\Delta}},$ where the matrix of the disorder averaged Green function is defective. This result is analogous to the theoretically predicted exceptional rings in the nodal-line semimetals [20,23,35–39]. The spectral properties of the Fermi ribbon is similar to that of the Fermi arc discussed in the previous section. The dispersion is shown in Fig. 5. The nodal-loop superconductor has so-called “drum-head” states localized at any surfaces not lying parallel to the $z$ axis [56]. Due to the smearing of the nodal line the spectral region of these edge states vanishes.

Let us also contrast these results with the situation in which the supercurrent is applied within the $k_z = 0$ plane. The supercurrent tilts the nodal ring and gives rise to Fermi-surface pockets connected by two pseudo-Weyl points, which are located at $\pm k_F v_z \times \hat{z}/|v_z|$ [57]. Here the density of states is finite even at infinitesimally small $|v_z|$ and is given by $v |v_z| k_F / \Delta$, in which $v = m k_F / 2\pi^2$ is the density of states at the Fermi level for one spin projection in the normal state of the system. At $|v_z| k_F / \Delta \gg \tau \Gamma$, the self-energy at

FIG. 4. Top: momentum resolved spectral function $A(k)$ at zero energy for the cases of (a) strong proximity effect, in which $\Delta/\Gamma^2 \sqrt{1 + k^2} \approx 4 > 1$, and (b) weak proximity effect ($\Delta/\Gamma^2 \sqrt{1 + k^2} \approx 0.4 < 1$) for fixed $|k|$ = 1.3. The thick light green lines indicate a large spectral weight. One sees the emergence of the bulk Fermi arc as the value of the proximity induced gap $\Delta$ decreases. Bottom: spectral function $A(0, E)$ at zero momentum for the cases of strong (c) and weak (d) proximity effect, which can be distinguished by the number of peaks in the spectral function.

$$A(0, E_{\pm}) = 2/\pi \Gamma^2_2.$$  \hspace{1cm} (14)

In the limit of weak proximity effect, $\Delta \ll \Gamma_2$, the height of the peak at zero frequency is instead given by $A(0, 0) = \frac{4\pi}{\pi} \left[ \Delta^2 + \Gamma^2_2 (1 - k^2) \right]^{-1}$.

FIG. 5. Panels (a) and (b) show respectively the real and imaginary parts of the spectrum $E(k)$, given by Eq. (14), as a function of momentum $k$, at $k_z = k_y = 0$ for the disordered nodal-ring superconductor (or polar phase of the superfluid $^3$He) with the supercurrent applied along the $z$ axis. The interplay of the supercurrent and disorder splits the nodal-ring into a region where the real part of the spectrum is zero, bounded with the outer and inner exceptional circumferences. The spectrum has a square root singularity at the exceptional rings.
zero frequency is linear, \( \Sigma = -i|v_s|k_F/2\tau \Delta \) in \( v_s \), while in the limit \( |v_s|k_F/\Delta \to 0 \) it reaches the value \( \Sigma = -i\Gamma[1 + O(|v_s|^2k_F^2/\Delta^2)] \). We did not find any exceptional points in this case because \( v_s \cdot n_z = 0 \).

B. Point-node superconductors

Let us finally comment on the effect of the interplay between the supercurrent and disorder scattering in a 3D Weyl superconductor with nodal points in their quasiparticles spectrum. The BCS Hamiltonian describing a Weyl superconductor (or equivalently the \( \Lambda \) phase of \(^3\)He) is given by

\[
H(k) = \frac{k^2 - k_F^2}{2m} v_z + v_s \cdot k + \Delta(n_z \tau_x - n_y \tau_y).
\]

At \( |v_s| = 0 \), the spectrum of quasiparticles has two Weyl nodes at \( k = (0, 0, \pm k_F) \). The surface of the superconductor might host Andreev-Majorana localized chiral modes. The dispersion of surface modes has the form of the Fermi arc, which connects the projections of the bulk Weyl points to the surface; for a review see Ref. [55].

Consider qualitatively the situation where the supercurrent flows parallel to the \( x \) axis, \( v_x = v_x(1, 0, 0) \). At \( |v_s|k_F < \Delta \), similarly to 3D Dirac-Weyl semimetals, weak disorder satisfying \( \tau \Delta < \pi/4 \) does not lead to a finite imaginary part of the self-energy at zero frequency and hence does not form any exceptional lines.

V. DISCUSSION AND CONCLUSIONS

In this paper we have studied the effect of weak scalar disorder on the band structure of nodal superconductors. We have argued that the nodes in the anisotropic superconducting gap in the presence of weak disorder may be replaced by Fermi arcs or 2D Fermi areas bounded by exceptional points or exceptional lines, respectively. At these exceptional points or lines the quasiparticles Green function becomes defective. Here we have analyzed the smearing of the nodes in the superconducting gap in the presence of scalar disorder within the self-consistent Born approximation. Going beyond this approximation and taking into account a more general form for the disorder [58] shall be addressed in the future.

We believe that the proposed non-Hermitian superconducting phase might be probed in several materials, such as the quasi-2D organic conductor \( \sigma-(BEDT-TTF)_2\text{I}_3 \) salt [50,59] and (001) surface states of the crystalline insulator SnTe [60], which host 2D massless Dirac fermions with anisotropic dispersion. The proximity induced superconducting gap in these structures might be established experimentally. Nodal phases with the zeros in the energy spectrum are known to exist in superfluid \(^3\)He-A (for a review see [55]). Signatures of nontrivial topological nodal superconductivity were observed in \( \text{Cu}_2\text{Bi}_2\text{Se}_3 \), in noncentrosymmetric heavy fermion systems, and in cuprate-based superconductors [61].

Finally, it would be interesting to extend our work on the interplay between proximity induced superconductivity and disorder in systems with triple Dirac points [62]. Therein, higher-order non-Hermitian degeneracies with cubic-root singularities at the nontrivial exceptional curves are expected [63].

ACKNOWLEDGMENTS

We thank G. Volovik for important comments. A.A.Z. acknowledges the hospitality of the Université Paris-Sud and Pirinen School of Theoretical Physics as well as the support by the Academy of Finland.

APPENDIX A: CALCULATION OF THE SELF-ENERGY IN TWO DIMENSIONS

1. Normal case \( \Delta = 0 \)

Let us calculate the disorder-induced self-energy correction for the case of a two-dimensional semimetal. The Hamiltonian describing such a system reads

\[
H(k) = \kappa v_k + v(k_\sigma + k_\sigma). \tag{A1}
\]

The self-consistent equation for the retarded self-energy within the first Born approximation is given by

\[
\Sigma(E) = \gamma \int \frac{d^2k}{(2\pi)^2} [E - H(k) - \Sigma(E)]^{-1}. \tag{A2}
\]

We search for the solution of this equation in the form

\[
\Sigma = t + d\sigma_x + \epsilon\sigma_z, \tag{A3}
\]

where \( t = t_1 - it_2, d = d_1 - id_2 \), and \( \epsilon \) are complex functions of the energy \( E \) such that \( t_{1,2} \) and \( d_{1,2} \) are real and satisfy \( t_2 > |d_2| \geq 0 \) because we are dealing with a retarded self-energy. Since the tilt is along the \( k \) direction, the \( \epsilon \sigma_z \) contribution in the self-energy vanishes to guarantee its momentum independence. This point can also be checked explicitly. Hence

\[
\Sigma = \gamma \int \frac{d^2k}{(2\pi)^2} \frac{E - \kappa v_k + \sigma_x(v_k + \epsilon) - \sigma_y(v_k - d)}{(2\pi)^2 (E + \kappa v_k)^2 - (v_k + \epsilon)^2 - (v_k - d)^2}. \tag{A4}
\]

From hereon we consider \( E = 0 \) to make progress with the algebra. The poles in (A4) can be found from the equation

\[

(\nu_k - \frac{d + \kappa t}{1 - \kappa^2})^2 = -\frac{(v_k + \epsilon)^2}{1 - \kappa^2} + \left(\frac{t + \kappa d}{1 - \kappa^2}\right)^2. \tag{A5}
\]

When \( |\kappa| < 1 \), we note that provided

\[
|d_2 + \kappa t_2| < |\text{Re}(\nu_k + \epsilon)(1 - \kappa^2) - (t + \kappa d)^2| \tag{A6}
\]

the integration over momentum \( k_y \) yields

\[
\Sigma = \frac{\gamma}{1 - \kappa^2} \left[ t - d\sigma_y + (\kappa + \sigma_x)\frac{d + \kappa t}{1 - \kappa^2} - \sigma_y(w + \epsilon) \right] \\
\times \int_{-\Lambda}^{\Lambda} \frac{d\epsilon}{4\pi v^2} \left[ (w + \epsilon)^2 - \left(\frac{t + \kappa d}{1 - \kappa^2}\right)^2\right]^{-1/2}, \tag{A7}
\]

where \( \Lambda \) is the energy cutoff and \( w \equiv v_k \). It can be now seen that the \( \sigma_x \) term is small, \( \propto O(\epsilon/\Lambda) \), and will be neglected in what follows.
The integration over \( w \) in Eq. (A22) after some simplifications results in two equations for \( t \) and \( d \):
\[
\begin{align*}
t &= \gamma \frac{t + \kappa d}{2\pi v^2} \ln \frac{2\Lambda \sqrt{1 - \kappa^2}}{i(t + \kappa d)}, \\
d &= \gamma \frac{t + \kappa d}{2\pi v^2} \ln \frac{2\Lambda \sqrt{1 - \kappa^2}}{i(t + \kappa d)}.
\end{align*}
\]
Noting that \( d = \kappa t \), we obtain for \( t \neq 0 \)
\[
1 = \gamma \frac{1 + \kappa^2}{2\pi v^2} \ln \frac{2\Lambda \sqrt{1 - \kappa^2}}{i(t + \kappa)}.
\]
which gives \( \Sigma = -i(1 + \kappa \sigma_y) \Gamma_1 \), where
\[
\Gamma_1 = 2\Lambda \sqrt{1 - \kappa^2} \exp \left[ -\frac{2\pi v^2}{\gamma} \frac{(1 - \kappa^2)^{3/2}}{1 + \kappa^2} \right].
\]  
This is the expression given in Eq. (6) of the main text. 

When \(|\kappa| > 1\) it is enough to consider the self-energy within the first Born approximation similarly as it was shown for the nodal-line semimetal in [20]. Integrating first over \( k_y \),
\[
\Sigma = -i \frac{\gamma}{8\pi} \sum_{x=\pm} \int_{-k_0}^{k_0} dk_x \int dk_y \left[ 1 + \frac{k_y}{k} \right] \delta(\kappa v k_x - sv k_y)
\]
\[
= -i \frac{\gamma}{2\pi v} \left( 1 + \frac{\sigma_y}{\kappa} \right) \frac{|k_0(\kappa)|}{\kappa^3 - 1},
\]  
where \( k_0(\kappa) \) is the momentum cutoff on the \( k_x \) axis describing the width of the electron and hole pockets, which might depend on the tilt parameter \( \kappa \). This is the expression given in Eq. (7) of the main text. 

### 2. Superconducting case

The BdG Hamiltonian describing the low-energy states in the system in the presence of the proximity-induced superconducting gap reads
\[
H(\mathbf{k}) = \kappa v k_x + v(k_y \sigma_x - k_x \sigma_y) t_x + \Delta t_x,
\]  
where \( \Delta \) can be considered to be real and positive without loss of generality. In order to obtain an analytical solution for \( \Sigma \), we again proceed by considering two limiting cases of weak and strong tilts, \(|\kappa| < 1\) and \(|\kappa| > 1\).

When \(|\kappa| < 1\) and \( \Delta \gg \Gamma_1 \), we can use the first-order Born approximation. For convenience we transform to Matsubara frequencies \( E + i0^+ \rightarrow i\omega_n \). The self-energy \( \Sigma \rightarrow \Sigma_{\omega_n} \) then reads
\[
\Sigma_{\omega_n} = \gamma \int \frac{d^2 k}{(2\pi)^2} \frac{i\omega_n - \kappa v k_x + v(k_y \sigma_x - k_x \sigma_y) t_x - \Delta t_x}{i\omega_n - \kappa v k_x} = \frac{i\gamma}{2(1 - \kappa^2)} \int \frac{d^2 k}{(2\pi)^2} \frac{1}{\sqrt{\lambda}}
\]
\[
\times \sum_{s=\pm} \frac{i\omega_n - \Delta t_x - v k_x (k + \sigma_y t_x)}{v k_x + i\omega_n \sqrt{\kappa - is\sqrt{\lambda}}},
\]
where
\[
\lambda = \frac{k^2 + \Delta^2}{1 - \kappa^2} + \frac{\omega_n^2}{(1 - \kappa^2)^2}.
\]  
Provided \( \text{Re} \sqrt{\lambda} > \frac{|\omega_n|}{1 + \kappa^2} \) (in the opposite case the integral is zero) after the integration over \( k_x \) one obtains
\[
\Sigma_{\omega_n} = -\gamma \int_0^\Lambda d\omega \frac{i\omega_n - \Delta t_x + (\kappa + \sigma_y t_x) \frac{\text{Im} \omega_n}{\sqrt{(\omega_n^2 + \Delta^2)(1 - \kappa^2)}}}{\sqrt{(\omega_n^2 + \Delta^2)(1 - \kappa^2)} + \omega_n^2},
\]
where \( \Delta \) is the energy cutoff. The integration over \( \omega \) gives
\[
\Sigma_{\omega_n} = -\gamma \frac{\omega_n}{2\pi v^2} \sqrt{1 - \kappa^2} \ln \frac{2\Lambda}{\sqrt{\Delta^2 + \omega_n^2}} \times \left[ \frac{i\omega_n}{\sqrt{1 - \kappa^2} (1 + \kappa \sigma_y t_x - \Delta t_x)} \right].
\]  
Notice that there is no contribution proportional to \( \sigma_y \). After the transformation \( i\omega_n \rightarrow E + i0^+ \) and \( \Sigma_{\omega_n} \rightarrow \Sigma \), one obtains the expression in Eq. (8) of the main text. 

When \(|\kappa| < 1\) and \( |\kappa| \Gamma_1 \gg \Delta \), following the same reasoning as in Appendix A 1, we now search for the self-energy in the form
\[
\Sigma = t + d\sigma_x t_x + S t_x,
\]  
where \( S = S_1 - i S_2 \). The integrand in the self-consistent equation for the self-energy now reads
\[
\Sigma = \gamma \int \frac{d^2 k}{(2\pi)^2} \left[ E - t - \kappa v k_x - (v k_y - d) \sigma_y t_x - (\Delta + S) t_x \right]
\]
\[
\times \left[ (t - \kappa v k_x)^2 - v^2 k_y^2 - (v k_y - d)^2 - (\Delta + S)^2 \right]^{-1}
\]
\[
\]  
and can be linearized in powers of \( \Delta \). Performing the same derivations as were done in Appendix A 1, we obtain an additional equation in addition to Eq. (A8) in order to determine \( S \) at \( E = 0 \):
\[
S = -\frac{\gamma (\Delta + S)}{2\pi v^2} \ln \frac{2\Lambda \sqrt{1 - \kappa^2}}{i(t + \kappa d)}.
\]
Together with Eq. (A8), this gives
\[
S = (\Delta + S) \frac{1 - \kappa^2}{1 + \kappa^2}.
\]  
Therefore, \( S = \Delta (1 - \kappa^2)/2\kappa^2 \) and
\[
\Sigma = \frac{1 - \kappa^2}{2\kappa^2} \Delta t_x - i(1 + \kappa \sigma_y t_x) \Gamma_1.
\]  
In the limit of a strong tilt, when \(|\kappa| > 1\), we have a finite density of states at the Dirac point and we can use the first-order Born approximation to calculate the self-energy,
\[
\Sigma(E) = \gamma \int \frac{d^2 k}{(2\pi)^2} \frac{E + i0^+ - (\kappa + \sigma_y t_x) v k_x - \Delta t_x}{(E + i0^+ - \kappa v k_x)^2 - v^2 k_y^2 - \Delta^2},
\]  
where the integral over \( k_x \) is taken in the region \( k_x \in [-k_0, k_0] \) to qualitatively account for the width of the electron-hole
where
\begin{equation}
M = v^2(k^2 - 1)[(k_x - p)^2 - q],
\end{equation}
in which \(p = \kappa E/(k^2 - 1)v\) and \(q = (E^2/\kappa^2 - \Delta^2)/(k^2 - 1)v^2\), and note that
\begin{equation}
\text{sgn}(E - \kappa v k_x) = -\text{sgn}\left[k_x - p + E/(\kappa v)\right].
\end{equation}
We can integrate in Eq. (A22) over \(k_x\) separately in two regions, \(M < 0\) and \(M > 0\), taking into account that \(\text{sgn}(E - \kappa v k_x)\) does not depend on \(k_x\). For \(M > 0\) the integration over \(k_x\) gives:
\begin{equation}
\Sigma(E) = -i\gamma \int_{-\sqrt{q}}^{\sqrt{q}} \frac{dk_x}{4\pi v} \sqrt{E - \Delta \tau_x - (\kappa + \sigma_x \tau_x) v k_x} x
\end{equation}
\begin{equation}
\text{x} \frac{\text{sgn}(E - \kappa v k_x)}{\sqrt{M}}
\end{equation}
\begin{equation}
= -i\gamma \frac{|\kappa| k_0}{2\pi v} \left(1 + \frac{\sigma_x}{\kappa} \tau_x\right).
\end{equation}
This term is purely imaginary since we neglect small corrections \(\Delta/p/k_0 \ll 1\). At \(M < 0\) we obtain
\begin{equation}
\Sigma(E) = -\gamma \int_{-\sqrt{q}}^{\sqrt{q}} \frac{dk_x}{4\pi v} \sqrt{E - \Delta \tau_x - (\kappa + \sigma_x \tau_x) v k_x} x
\end{equation}
\begin{equation}
= -\frac{i\gamma}{\sqrt{\kappa^2 - 1}} \left[\frac{E}{1 - \kappa^2} \left(1 + \kappa \sigma_x \tau_x\right) - \Delta \tau_x\right]
\end{equation}
\begin{equation}
- \frac{i\gamma}{2\pi v} \left(1 + \frac{\sigma_x}{\kappa} \tau_x\right) \frac{|\kappa| k_0}{\sqrt{\kappa^2 - 1}}.
\end{equation}
We arrive at the expression for the self-energy,
\begin{equation}
\Sigma(E) = -\frac{\gamma}{4\pi v} \left[\frac{E}{1 - \kappa^2} \left(1 + \kappa \sigma_x \tau_x\right) - \Delta \tau_x\right]
\end{equation}
\begin{equation}
- \frac{i\gamma}{2\pi v} \left(1 + \frac{\sigma_x}{\kappa} \tau_x\right) \frac{|\kappa| k_0}{\sqrt{\kappa^2 - 1}}.
\end{equation}
Finally, we note that \(|\kappa| k_0 \gg \Delta, |E|/(\kappa^2 - 1)\) and write
\begin{equation}
\Sigma(E) = -i\gamma \frac{|\kappa| k_0}{2\pi v} \left(1 + \frac{\sigma_x}{\kappa} \tau_x\right) \frac{|\kappa| k_0}{\sqrt{\kappa^2 - 1}},
\end{equation}
which is given in Eq. (10) of the main text.

APPENDIX B: CALCULATION OF THE SELF-ENERGY IN THREE DIMENSIONS

Let us consider a nodal-line superconductor in the presence of a supercurrent flow with velocity \(v_\parallel = v_\parallel(0, 0, 1)\) parallel to the \(z\) axis. The BdG Hamiltonian is given by
\begin{equation}
H = v_\parallel k_z + \xi \tau_z + \Delta n_z \tau_x,
\end{equation}
where \(\xi = k_x^2/2m - \mu\) and \(n_z = k_z/k\) is the unit vector in the direction of \(k_z\). We focus on the limit \(|v_\parallel| k_F \ll \Delta\). The nodal ring is defined by \(\xi = 0, k_z = 0\). The equation for the self-energy is given by
\begin{equation}
\Sigma = -\gamma \int \frac{d^3k}{(2\pi)^3} \frac{v_\parallel k_z + t - \xi \tau_z + (\Delta n_z + S) \tau_x}{(v_\parallel k_z + t)^2 - \xi^2 - (\Delta n_z + S)^2}.
\end{equation}
We neglect the contributions to \(\Sigma\) with \(\tau_z\) matrix, assuming \(\mu \gg \text{Re} \text{Tr} \tau_x \tau_z\). Integration over \(\xi\),
\begin{equation}
\int \frac{d^3k}{(2\pi)^3} F(k) \rightarrow v_\parallel \int_{-\infty}^{\infty} d\xi \int_{0}^{\pi} \sin \theta d\theta \frac{dF(\xi, \theta)}{2},
\end{equation}
where \(v = mp_F/2\pi^2\) is the density of states in the normal-metal state, we obtain
\begin{equation}
\Sigma = \frac{i}{2\pi} \int_{-2}^{1} \frac{dz}{\sqrt{(v_\parallel k_F z + t + (\Delta z + S) \tau_x)^2 - (\Delta n_z + S)^2}}
\end{equation}
where \(\tau = 1/2\pi v_\parallel \gamma\) is the mean free time. Using the condition \(|v_\parallel| k_F \ll \Delta\), we obtain two equations for \(t\) and \(S\):
\begin{equation}
2\tau \Delta = \arcsin \frac{\Delta}{t}
\end{equation}
\begin{equation}
S = -\frac{i}{2\pi \Delta} \left[\frac{v_\parallel k_F}{\Delta} \arcsin \frac{\Delta}{t} + t \frac{v_\parallel k_F - \Delta S}{\sqrt{t^2 - \Delta^2}}\right],
\end{equation}
which together give
\begin{equation}
t = -\frac{i\Delta}{\text{sh}(2\pi \Delta)} \frac{v_\parallel k_F}{\Delta},
\end{equation}
\begin{equation}
S = \frac{i\Delta}{\text{sh}(2\pi \Delta)} \left(1 + \frac{v_\parallel k_F}{\Delta}\right).
\end{equation}
Hence one recovers the expression in Eq. (13) of the main text:
\begin{equation}
\Sigma = -\frac{i\Delta}{\text{sh}(2\pi \Delta)} \left(1 + \frac{v_\parallel k_F}{\Delta}\right).
\end{equation}