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*Published in:*
2018 IEEE International Symposium on Antennas and Propagation and USNC-URSI Radio Science Meeting

*DOI:*
10.1109/APUSNCURSINRSM.2018.8608178

*Published:*
01/01/2018

*Document Version*
Peer reviewed version

*Please cite the original version:*

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General Boundary Conditions in Electromagnetics

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Abstract—The most general form of linear and local boundary
conditions (General Boundary Conditions, GBC) is considered in
this paper. The conditions are defined in terms of four vectors
tangential to the boundary surface and two scalars. The number
of parameters needed to define the conditions is at most 8. Well-
known examples of boundary conditions are found to emerge for
various choices of the vectors and scalars. The reflection dyadic
yielding the reflected wave for a given incident plane wave is
found and its eigenproblem is solved with examples considered
in the paper.

I. INTRODUCTION

The general linear and local boundary conditions (GBC) at
a boundary surface have been defined in [1] by two scalar
conditions
\[
\begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{pmatrix}
\begin{pmatrix}
E \\
H
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
(1)
when the medium above the boundary surface is assumed
isotropic with parameters \( \varepsilon_0, \mu_0 \) and \( \eta_0 = \sqrt{\mu_0/\varepsilon_0} \). For a
given boundary, the representation (1) is not unique. In fact,
the matrix of four vectors can be replaced by another one
defined by
\[
\begin{pmatrix}
a'_1 & b'_1 \\
a'_2 & b'_2
\end{pmatrix}
= \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{pmatrix},
\]
(2)
in terms of any scalars satisfying \( AD - CB \neq 0 \). To obtain
a more unique representation for the GBC conditions, we can
expand the four vectors in normal and tangential components,
denoted by \( \mathbf{n} \cdot \mathbf{a} = a_n \) and \( \mathbf{a} = \mathbf{a} - \mathbf{n}a_n \), where \( \mathbf{n} \)
is the unit vector normal to the boundary surface. Assuming
\[
\Delta = a_{1n}b_{2n} - b_{1n}a_{2n} \neq 0,
\]
(3)
we can successively eliminate \( \mathbf{n} \cdot \mathbf{E} \) and \( \mathbf{n} \cdot \mathbf{H} \) from the two
scalar equations of (1). Changing the definitions of the vectors,
the general form of GBC conditions can be written as
\[
\mathbf{n} \cdot \begin{pmatrix}
\alpha \mathbf{E} \\
\beta \eta_0 \mathbf{H}
\end{pmatrix}
+ \begin{pmatrix}
a_{1t} & b_{1t} \\
a_{2t} & b_{2t}
\end{pmatrix}
\begin{pmatrix}
\mathbf{E}_t \\
\eta_0 \mathbf{H}_t
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
(4)
where \( \alpha \) and \( \beta \) are some scalars. In the general case with
nonzero \( \alpha \) and \( \beta \), we can set \( \alpha = \beta = 1 \) in which case,
the representation (4) for a given GBC boundary is unique.
The number of parameters required for the definition is thus
\( 4 \times 2 = 8 \). For \( \Delta = 0 \), (1) can be reduced to a simpler form
where at least one of \( \alpha \) and \( \beta \) is zero.

When the medium is not isotropic, the condition (4) must
be written in the form
\[
\mathbf{n} \cdot \begin{pmatrix}
\alpha \eta_0 \mathbf{cD} \\
\beta \mathbf{eB}
\end{pmatrix}
+ \begin{pmatrix}
a_{1t} & b_{1t} \\
a_{2t} & b_{2t}
\end{pmatrix}
\begin{pmatrix}
\mathbf{E}_t \\
\eta_0 \mathbf{H}_t
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
(5)
with \( c = 1/\sqrt{\mu_0 \varepsilon_0} \).

II. SPECIAL CASES OF GBC

Many familiar linear boundary conditions can be recognized
as special cases of the GBC conditions. A few of them are
listed below.

\begin{itemize}
  \item \( \alpha = \beta = 0 \)
    For \( a_{1t} \times a_{2t} \neq 0 \) (4) reduces to the impedance-boundary
    conditions of the form [2]
    \[
    \mathbf{n} \times \mathbf{E} = \mathbf{Z}_s \cdot \mathbf{H}, \quad \mathbf{Z}_s = \eta_0 a_{1t} \mathbf{a}_{2t} - a_{2t} \mathbf{a}_{1t},
    \]
    (6)
  \item \( a_{1t} = a_{2t} = b_{1t} = b_{2t} = 0 \)
    Equation (4) reduces to the DB conditions [3] which in
    the isotropic medium can be written as
    \[
    \mathbf{n} \cdot \mathbf{E} = 0, \quad \mathbf{n} \cdot \mathbf{H} = 0.
    \]
    (7)
  \item \( \beta = 0, \quad b_{1t} = b_{2t} = 0 \)
    Equation (4) reduces to the E-boundary conditions [1],
    which can be expressed as
    \[
    a_1 \cdot \mathbf{E} = 0, \quad a_2 \cdot \mathbf{E} = 0, \quad a_1 \times a_2 \neq 0.
    \]
    (8)
    For \( a_1 = a_{1t} \), these coincide with the PEC conditions.
    Similar result is valid for the H-boundary \( \alpha = 0, a_{1t} = a_{2t} = 0 \),
    and its special case, the PMC boundary.
  \item \( b_{1t} = a_{2t} = 0 \)
    Equation (4) reduces to the EH-boundary conditions [1]
    \[
    a_1 \cdot \mathbf{E} = 0, \quad b_2 \cdot \mathbf{H} = 0.
    \]
    (9)
    For \( a_1 = a_{1t} \), \( b_2 = b_{2t} \), they equal the generalized soft-
    and-hard (GSH) conditions and, for \( b_2 = a_{1t} \), the soft-
    and-hard (SH) conditions [4].
  \item \( b_{2t} = a_{1t} = 0 \)
    Equation (4) reduces to the form
    \[
    \alpha \mathbf{n} \cdot \mathbf{E} + b_{1t} \cdot \eta_0 \mathbf{H} = 0, \quad \beta \mathbf{n} \cdot \eta_0 \mathbf{H} + a_{2t} \cdot \mathbf{E} = 0,
    \]
    (10)
    which are of the form of generalized soft-and-hard/DB
    (GSHDB) conditions [5].
\end{itemize}
III. PLANE-WAVE REFLECTION

Let us consider a plane wave incident to and reflected from the GBC surface:
\[
E^{i,r}(\mathbf{r}) = E^{i,r} \exp(-j k \cdot \mathbf{r}) \tag{11}
\]
\[
k_\circ \eta_\circ \mathbf{H}^{i,r}(\mathbf{r}) = k \cdot \mathbf{E}^{i,r} \exp(-j k \cdot \mathbf{r}) \tag{12}
\]
with
\[
k^i = -k_\circ n + k_t, \quad k' = k_\circ n + k_t, \tag{13}
\]
\[
k^i \cdot k^i = k^r \cdot k^r = k_\circ^2. \tag{14}
\]

Applying the plane-wave conditions
\[
\mathbf{n} \cdot (k_\circ / \eta_\circ)(\mathbf{E}^i + \mathbf{E}^r) = -\mathbf{n} \cdot (k^i \times \mathbf{H}^i + k'^i \times \mathbf{H}^r) \tag{15}
\]
\[
\mathbf{n} \cdot (k_\circ \eta_\circ)(\mathbf{H}^i + \mathbf{H}^r) = \mathbf{n} \cdot (k^i \times \mathbf{E}^i + k'^r \times \mathbf{E}^r), \tag{16}
\]
which can be expressed for the sum fields as
\[
k_\circ n \cdot \mathbf{E} = -(n \times k_t) \cdot \eta_\circ \mathbf{H}_t, \tag{17}
\]
\[
k_\circ n \cdot \eta_\circ \mathbf{H} = (n \times k_t) \cdot \mathbf{E}_t, \tag{18}
\]
the GBC conditions (4) for the tangential fields take the form
\[
\begin{pmatrix}
  a_{1t}
  \\
  b_{1t}
\end{pmatrix}
+ \begin{pmatrix}
  (E^i_t + E^r_t)
  \\
  \eta_\circ (H^i_t + H^r_t)
\end{pmatrix} = \begin{pmatrix}
  0
  \\
  0
\end{pmatrix}, \tag{19}
\]
where we define
\[
b_{1t}' = b_{1t} - \alpha n \times k_t/k_\circ, \tag{20}
\]
\[
a_{2t}' = a_{2t} + \beta n \times k_t/k_\circ. \tag{21}
\]
Let us apply the plane-wave conditions
\[
\eta_\circ \mathbf{H}_t^i = -\mathbf{j}_t \cdot E^i_t, \tag{22}
\]
\[
\eta_\circ \mathbf{H}_t^r = \mathbf{j}_t \cdot E^r_t, \tag{23}
\]
where the dyadic \( \mathbf{j}_t \) stands for
\[
\mathbf{j}_t = \frac{1}{k_\circ k_n}(n \times k_t)k_t + k_\circ n \times \mathbf{l}), \tag{24}
\]
and is known to satisfy certain simple properties \[5\]. Defining
\[
s_{1t}^+ = a_{1t} \pm b_{1t}' \cdot \mathbf{j}_t, \tag{25}
\]
\[
s_{2t}^+ = a_{2t}' \pm b_{2t} \cdot \mathbf{j}_t, \tag{26}
\]
the conditions (19) can be written as
\[
s_{1t}^+ \cdot E^i_t + s_{1t}^+ \cdot E^r_t = 0 \tag{27}
\]
\[
s_{2t}^+ \cdot E^i_t + s_{2t}^+ \cdot E^r_t = 0. \tag{28}
\]

Let us denote
\[
\mathbf{n} \cdot s_{1t}^+ \times s_{2t}^+ = \Delta^+. \tag{29}
\]

Applying the expansion
\[
\mathbf{n} \times ((s_{1t}^+ \times s_{2t}^+) \times \mathbf{l}) = -\mathbf{n} \times (s_{1t}^+ s_{2t}^- - s_{2t}^+ s_{1t}^-) = -\Delta^+ \mathbf{j}_t, \tag{30}
\]
and assuming \( \Delta^+ \neq 0 \), we can write
\[
E^i_t = \frac{1}{\Delta^+} \mathbf{n} \times (s_{1t}^+ s_{2t}^- - s_{2t}^+ s_{1t}^-) \cdot E^r_t = \mathbf{R}_t \cdot E^i_t, \tag{31}
\]
where \( \mathbf{R}_t \) is the reflection dyadic for tangential fields. Its expression becomes
\[
\mathbf{R}_t = -\frac{1}{\Delta^+} \mathbf{n} \times (s_{1t}^+ s_{2t}^- - s_{2t}^+ s_{1t}^-), \tag{32}
\]
where one has yet to substitute (25), (26), (20) and (21).

IV. EIGENWAVES

Let us consider the eigenproblem of the reflection dyadic, which can be interpreted as
\[
E^i_t = \mathbf{R}_t \cdot E^i_t = \lambda E^i_t. \tag{33}
\]

From (31) we can expand
\[
E^i_t = E_1 n \times s_{1t}^+ + E_2 n \times s_{2t}^+, \tag{34}
\]
and split (33) in two scalar equations as
\[
(n \times s_{1t}^+ \cdot s_{1t}^+ + \lambda s_{1t}^+ s_{2t}^- + s_{2t}^+ s_{1t}^- E_1 = 0, \tag{35}
\]
\[
(n \times s_{1t}^+ \times s_{1t}^-)E_1 + (n \times s_{2t}^-) \cdot (s_{1t}^+ + \lambda s_{1t}^+) = 0. \tag{36}
\]
Eliminating \( E_1 \) and \( E_2 \), the equation for \( \lambda \) can be expressed as
\[
\Delta^+ \lambda^2 + A \lambda + \Delta^- = 0, \tag{37}
\]
with
\[
A = n \cdot (s_{1t}^+ \times s_{2t}^- + s_{1t}^- \times s_{2t}^+). \tag{38}
\]

Let us verify (37) by considering as a special case the GSHDB boundary restricted by (10), or
\[
a_{1t} = b_{2t} = 0, \tag{39}
\]
whence
\[
s_{1t}^+ = \pm b_{1t}' \cdot \mathbf{j}_t, \quad s_{2t}^+ = a_{2t}' \tag{40}
\]
and
\[
\Delta^+ = -\Delta^-, \quad A = 0. \tag{41}
\]

Inserting these in (37) the eigenvalue equation is reduced to
\[
(\lambda^2 - 1) \Delta^+ = 0, \tag{42}
\]
whence the eigenvalues become \( \lambda = +1 \) and \( \lambda = -1 \). This corresponds to the known fact that, for the two eigenvalues, the GSHDB boundary can be replaced by respective PEC and PMC boundaries \[5\]. For \( \Delta^+ = 0 \) the plane wave is matched to the GSHDB boundary so that there is no reflected wave for a given incident wave.

Other examples will be considered in the full paper.

REFERENCES