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General Boundary Conditions in Electromagnetics

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Abstract—The most general form of linear and local boundary conditions (General Boundary Conditions, GBC) is considered in this paper. The conditions are defined in terms of four vectors tangential to the boundary surface and two scalars. The number of parameters needed to define the conditions is at most 8. Well-known examples of boundary conditions are found to emerge for various choices of the vectors and scalars. The reflection dyadic yielding the reflected wave for a given incident plane wave is found and its eigenproblem is solved with examples considered in the paper.

I. INTRODUCTION

The general linear and local boundary conditions (GBC) at a boundary surface have been defined in [1] by two scalar conditions

$$\begin{pmatrix} \mathbf{a}_1 & \mathbf{b}_1 \\ \mathbf{a}_2 & \mathbf{b}_2 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \eta_o \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (1)$$

when the medium above the boundary surface is assumed isotropic with parameters ϵ_o , μ_o and $\eta_o = \sqrt{\mu_o/\epsilon_o}$. For a given boundary, the representation (1) is not unique. In fact, the matrix of four vectors can be replaced by another one defined by

$$\begin{pmatrix} \mathbf{a}'_1 & \mathbf{b}'_1 \\ \mathbf{a}'_2 & \mathbf{b}'_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 & \mathbf{b}_1 \\ \mathbf{a}_2 & \mathbf{b}_2 \end{pmatrix}, \quad (2)$$

in terms of any scalars satisfying $AD - CB \neq 0$. To obtain a more unique representation for the GBC conditions, we can expand the four vectors in normal and tangential components, denoted by $\mathbf{n} \cdot \mathbf{a} = a_n$ and $\mathbf{a}_t = \mathbf{a} - \mathbf{n}a_n$, where $\mathbf{n}$ is the unit vector normal to the boundary surface. Assuming

$$\Delta = a_{1n}b_{2n} - b_{1n}a_{2n} \neq 0, \quad (3)$$

we can successively eliminate $\mathbf{n} \cdot \mathbf{E}$ and $\mathbf{n} \cdot \mathbf{H}$ from the two scalar equations of (1). Changing the definitions of the vectors, the general form of GBC conditions can be written as

$$\mathbf{n} \cdot \begin{pmatrix} \alpha \mathbf{E} \\ \beta \eta_o \mathbf{H} \end{pmatrix} + \begin{pmatrix} \mathbf{a}_{1t} & \mathbf{b}_{1t} \\ \mathbf{a}_{2t} & \mathbf{b}_{2t} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_t \\ \eta_o \mathbf{H}_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (4)$$

where α and β are some scalars. In the general case with nonzero α and β , we can set $\alpha = \beta = 1$ in which case, the representation (4) for a given GBC boundary is unique. The number of parameters required for the definition is thus $4 \times 2 = 8$. For $\Delta = 0$, (1) can be reduced to a simpler form where at least one of α and β is zero.

When the medium is not isotropic, the condition (4) must be written in the form

$$\mathbf{n} \cdot \begin{pmatrix} \alpha \eta_o c \mathbf{D} \\ \beta c \mathbf{B} \end{pmatrix} + \begin{pmatrix} \mathbf{a}_{1t} & \mathbf{b}_{1t} \\ \mathbf{a}_{2t} & \mathbf{b}_{2t} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_t \\ \eta_o \mathbf{H}_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (5)$$

with $c = 1/\sqrt{\mu_o \epsilon_o}$.

II. SPECIAL CASES OF GBC

Many familiar linear boundary conditions can be recognized as special cases of the GBC conditions. A few of them are listed below.

- $\alpha = \beta = 0$

For $\mathbf{a}_{1t} \times \mathbf{a}_{2t} \neq 0$ (4) reduces to the impedance-boundary conditions of the form [2]

$$\mathbf{n} \times \mathbf{E} = \bar{\bar{Z}}_s \cdot \mathbf{H}, \quad \bar{\bar{Z}}_s = \eta_o \frac{\mathbf{a}_{1t} \mathbf{b}_{2t} - \mathbf{a}_{2t} \mathbf{b}_{1t}}{\mathbf{n} \cdot \mathbf{a}_{1t} \times \mathbf{a}_{2t}}. \quad (6)$$

- $\mathbf{a}_{1t} = \mathbf{a}_{2t} = \mathbf{b}_{1t} = \mathbf{b}_{2t} = 0$

Equation (4) reduces to the DB conditions [3] which in the isotropic medium can be written as

$$\mathbf{n} \cdot \mathbf{E} = 0, \quad \mathbf{n} \cdot \mathbf{H} = 0. \quad (7)$$

- $\beta = 0, \mathbf{b}_{1t} = \mathbf{b}_{2t} = 0$

Equation (4) reduces to the E-boundary conditions [1], which can be expressed as

$$\mathbf{a}_1 \cdot \mathbf{E} = 0, \quad \mathbf{a}_{2t} \cdot \mathbf{E} = 0, \quad \mathbf{a}_1 \times \mathbf{a}_{2t} \neq 0. \quad (8)$$

For $\mathbf{a}_1 = \mathbf{a}_{1t}$, these coincide with the PEC conditions. Similar result is valid for the H-boundary $\alpha = 0, \mathbf{a}_{1t} = \mathbf{a}_{2t} = 0$, and its special case, the PMC boundary.

- $\mathbf{b}_{1t} = \mathbf{a}_{2t} = 0$

Equation (4) reduces to the EH-boundary conditions [1]

$$\mathbf{a}_1 \cdot \mathbf{E} = 0, \quad \mathbf{b}_2 \cdot \mathbf{H} = 0. \quad (9)$$

For $\mathbf{a}_1 = \mathbf{a}_{1t}, \mathbf{b}_2 = \mathbf{b}_{2t}$ they equal the generalized soft-and-hard (GSH) conditions and, for $\mathbf{b}_{2t} = \mathbf{a}_{1t}$, the soft-and-hard (SH) conditions [4].

- $\mathbf{b}_{2t} = \mathbf{a}_{1t} = 0$

Equation (4) reduces to the form

$$\alpha \mathbf{n} \cdot \mathbf{E} + \mathbf{b}_{1t} \cdot \eta_o \mathbf{H} = 0, \quad \beta \mathbf{n} \cdot \eta_o \mathbf{H} + \mathbf{a}_{2t} \cdot \mathbf{E} = 0, \quad (10)$$

which are of the form of generalized soft-and-hard/DB (GSHDB) conditions [5].

III. PLANE-WAVE REFLECTION

Let us consider a plane wave incident to and reflected from the GBC surface:

$$\mathbf{E}^{i,r}(\mathbf{r}) = \mathbf{E}^{i,r} \exp(-j\mathbf{k}^{i,r} \cdot \mathbf{r}), \quad (11)$$

$$k_o \eta_o \mathbf{H}^{i,r}(\mathbf{r}) = \mathbf{k}^{i,r} \times \mathbf{E}^{i,r} \exp(-j\mathbf{k}^{i,r} \cdot \mathbf{r}), \quad (12)$$

with

$$\mathbf{k}^i = -k_n \mathbf{n} + \mathbf{k}_t, \quad \mathbf{k}^r = k_n \mathbf{n} + \mathbf{k}_t, \quad (13)$$

$$\mathbf{k}^i \cdot \mathbf{k}^i = \mathbf{k}^r \cdot \mathbf{k}^r = k_o^2. \quad (14)$$

Applying the plane-wave conditions

$$\mathbf{n} \cdot (k_o / \eta_o) (\mathbf{E}^i + \mathbf{E}^r) = -\mathbf{n} \cdot (\mathbf{k}^i \times \mathbf{H}^i + \mathbf{k}^r \times \mathbf{H}^r), \quad (15)$$

$$\mathbf{n} \cdot (k_o \eta_o) (\mathbf{H}^i + \mathbf{H}^r) = \mathbf{n} \cdot (\mathbf{k}^i \times \mathbf{E}^i + \mathbf{k}^r \times \mathbf{E}^r), \quad (16)$$

which can be expressed for the sum fields as

$$k_o \mathbf{n} \cdot \mathbf{E} = -(\mathbf{n} \times \mathbf{k}_t) \cdot \eta_o \mathbf{H}_t, \quad (17)$$

$$k_o \mathbf{n} \cdot \eta_o \mathbf{H} = (\mathbf{n} \times \mathbf{k}_t) \cdot \mathbf{E}_t, \quad (18)$$

the GBC conditions (4) for the tangential fields take the form

$$\begin{pmatrix} \mathbf{a}_{1t} & \mathbf{b}'_{1t} \\ \mathbf{a}'_{2t} & \mathbf{b}_{2t} \end{pmatrix} \cdot \begin{pmatrix} (\mathbf{E}_t^i + \mathbf{E}_t^r) \\ \eta_o (\mathbf{H}_t^i + \mathbf{H}_t^r) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (19)$$

where we define

$$\mathbf{b}'_{1t} = \mathbf{b}_{1t} - \alpha \mathbf{n} \times \mathbf{k}_t / k_o, \quad (20)$$

$$\mathbf{a}'_{2t} = \mathbf{a}_{2t} + \beta \mathbf{n} \times \mathbf{k}_t / k_o. \quad (21)$$

Let us apply the plane-wave conditions

$$\eta_o \mathbf{H}_t^i = -\bar{\mathbf{J}}_t \cdot \mathbf{E}_t^i, \quad (22)$$

$$\eta_o \mathbf{H}_t^r = \bar{\mathbf{J}}_t \cdot \mathbf{E}_t^r, \quad (23)$$

where the dyadic $\bar{\mathbf{J}}_t$ stands for

$$\bar{\mathbf{J}}_t = \frac{1}{k_o k_n} ((\mathbf{n} \times \mathbf{k}_t) \mathbf{k}_t + k_n^2 \mathbf{n} \times \bar{\mathbf{I}}), \quad (24)$$

and is known to satisfy certain simple properties [5]. Defining

$$\mathbf{s}_{1t}^\pm = \mathbf{a}_{1t} \pm \mathbf{b}'_{1t} \cdot \bar{\mathbf{J}}_t, \quad (25)$$

$$\mathbf{s}_{2t}^\pm = \mathbf{a}'_{2t} \pm \mathbf{b}_{2t} \cdot \bar{\mathbf{J}}_t, \quad (26)$$

the conditions (19) can be written as

$$\mathbf{s}_{1t}^- \cdot \mathbf{E}_t^i + \mathbf{s}_{1t}^+ \cdot \mathbf{E}_t^r = 0 \quad (27)$$

$$\mathbf{s}_{2t}^- \cdot \mathbf{E}_t^i + \mathbf{s}_{2t}^+ \cdot \mathbf{E}_t^r = 0. \quad (28)$$

Let us denote

$$\mathbf{n} \cdot \mathbf{s}_{1t}^\pm \times \mathbf{s}_{2t}^\pm = \Delta^\pm. \quad (29)$$

Applying the expansion

$$\begin{aligned} \mathbf{n} \times ((\mathbf{s}_{1t}^+ \times \mathbf{s}_{2t}^+) \cdot \bar{\mathbf{I}}_t) &= -\mathbf{n} \times (\mathbf{s}_{1t}^+ \mathbf{s}_{2t}^+ - \mathbf{s}_{2t}^+ \mathbf{s}_{1t}^+) \\ &= -\Delta^+ \bar{\mathbf{I}}_t, \end{aligned} \quad (30)$$

and assuming $\Delta^+ \neq 0$, we can write

$$\begin{aligned} \mathbf{E}_t^r &= \frac{1}{\Delta^+} \mathbf{n} \times (\mathbf{s}_{1t}^+ \mathbf{s}_{2t}^+ - \mathbf{s}_{2t}^+ \mathbf{s}_{1t}^+) \cdot \mathbf{E}_t^i \\ &= \bar{\mathbf{R}}_t \cdot \mathbf{E}_t^i, \end{aligned} \quad (31)$$

where $\bar{\mathbf{R}}_t$ is the reflection dyadic for tangential fields. Its expression becomes

$$\bar{\mathbf{R}}_t = -\frac{1}{\Delta^+} \mathbf{n} \times (\mathbf{s}_{1t}^+ \mathbf{s}_{2t}^- - \mathbf{s}_{2t}^+ \mathbf{s}_{1t}^-), \quad (32)$$

where one has yet to substitute (25), (26), (20) and (21).

IV. EIGENWAVES

Let us consider the eigenproblem of the reflection dyadic, which can be interpreted as

$$\mathbf{E}_t^r = \bar{\mathbf{R}}_t \cdot \mathbf{E}_t^i = \lambda \mathbf{E}_t^i. \quad (33)$$

From (31) we can expand

$$\mathbf{E}_t^i = E_1 \mathbf{n} \times \mathbf{s}_{1t}^+ + E_2 \mathbf{n} \times \mathbf{s}_{2t}^+, \quad (34)$$

and split (33) in two scalar equations as

$$(\mathbf{n} \times \mathbf{s}_{1t}^+) \cdot (\mathbf{s}_{2t}^- + \lambda \mathbf{s}_{2t}^+) E_1 + \mathbf{n} \cdot \mathbf{s}_{2t}^+ \times \mathbf{s}_{2t}^- E_2 = 0, \quad (35)$$

$$\mathbf{n} \cdot \mathbf{s}_{1t}^+ \times \mathbf{s}_{1t}^- E_1 + (\mathbf{n} \times \mathbf{s}_{2t}^+) \cdot (\mathbf{s}_{1t}^- + \lambda \mathbf{s}_{1t}^+) E_2 = 0. \quad (36)$$

Eliminating E_1 and E_2 , the equation for λ can be expressed as

$$\Delta^+ \lambda^2 + A \lambda + \Delta^- = 0, \quad (37)$$

with

$$A = \mathbf{n} \cdot (\mathbf{s}_{1t}^+ \times \mathbf{s}_{2t}^- + \mathbf{s}_{1t}^- \times \mathbf{s}_{2t}^+). \quad (38)$$

Let us verify (37) by considering as a special case the GSHDB boundary restricted by (10), or

$$\mathbf{a}_{1t} = \mathbf{b}_{2t} = 0, \quad (39)$$

whence

$$\mathbf{s}_{1t}^\pm = \pm \mathbf{b}'_{1t} \cdot \bar{\mathbf{J}}_t, \quad \mathbf{s}_{2t}^\pm = \mathbf{a}'_{2t}, \quad (40)$$

and

$$\Delta^+ = -\Delta^-, \quad A = 0. \quad (41)$$

Inserting these in (37) the eigenvalue equation is reduced to

$$(\lambda^2 - 1) \Delta^+ = 0, \quad (42)$$

whence the eigenvalues become $\lambda = +1$ and $\lambda = -1$. This corresponds to the known fact that, for the two eigenwaves, the GSHDB boundary can be replaced by respective PEC and PMC boundaries [5]. For $\Delta^+ = 0$ the plane wave is matched to the GSHDB boundary so that there is no reflected wave for a given incident wave.

Other examples will be considered in the full paper.

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