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An extended ice failure model to improve the fidelity of icebreaking pattern in numerical simulation of ship performance in level ice

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ABSTRACT
The modelling of the ice failure including icebreaking pattern and ice bearing capacity is an important issue in numerical simulations of ships going through level ice, in order to predict ship performance and ice loads. Previous studies model the shape of ice cusps assuming a simplified geometry, e.g. circular or triangular. According to the observations during full-scale ship trials, the geometry of the ice cusps is more elliptical rather than circular, with larger breaking length at the edges than that at the center. In this paper, a new ice failure model is developed which results in more realistic cusp shapes compared to existing approaches. The model is based on an analytically-derived differential equation, which is solved numerically via the Finite Difference Method (FDM). The predictions of ice cusps geometry are validated against full-scale measurement of ice cusps, obtained with an on-board stereo camera system. Satisfying agreement is shown. The ice failure model is incorporated into a numerical model for the prediction of ship performance in level ice. The predictions are compared with ship speed record obtained from a full-scale trial. It is shown that the model gives reasonable results for ship speed.

1. Introduction

For the design of ice-going ships and safe operation through ice-covered areas, various models have been proposed for the prediction of ship performance in ice. These methods can be classified as analytical and empirical formulas (e.g. Lindqvist, 1989; Riska et al., 1997), numerical models (e.g. Valanto, 2001; Lubbad and Løset, 2011; Sawamura et al., 2009) and probabilistic methods (e.g. Montewka et al., 2015; Li et al., 2017). Several numerical models have been developed during recent years because of their capability to account for more details about the hull shape (e.g. Su, 2011; Tan et al., 2013; Zhou et al., 2018b).

One of the key issues in the numerical approach is to determine the bearing capacity of the ice sheet and the ice cusp geometry. The most advanced methods applied in current numerical models apply numerical methods such as Finite Difference Method (FDM) to solve differential equations for this purpose (Valanto, 2001). These are more physically realistic compared to the other methods, by taking e.g. the hydrodynamic effects into account. However, the high computational demand makes this approach very time-consuming when simulating a ship voyage in an ice field over a considerable distance. The complex properties of ice also decrease the accuracy if the applied material model does not capture the ice behavior accurately. The second group of models for bearing capacity and ice cusp geometry adopts an empirical approach (Su, 2011; Liu et al., 2006; Zhou et al., 2017). These models typically contain a number of empirical coefficients, which introduce uncertainties about the resulting ice loads and ship performance when applying these models (Kuuliala, 2015). The remainder of the models implement existing analytical methods (Lubbad and Løset, 2011), which usually simplify the problem as two-dimensional. This gives physically grounded basis compared to the empirical approach, but deviates from reality due to the simplifications. Most of the models in the latter two groups treat the ice cusp geometry as simple shape, usually circular (Sawamura et al., 2009; Zhou et al., 2018b) or triangular (Lubbad and Løset, 2011). The icebreaking pattern has been recorded from some model scale tests in ice tanks, e.g. Myland and Ehlers (2017), Zhou et al. (2018a) and Su et al. (2014). As shown by Li et al. (2018b), the shape of ice cusps in reality is more elliptical rather than circular, with a longer breaking length at the edges and a shorter one at the center. The ice cusp shape is an important issue because it affects the force of the next contact and determines the channel geometry made by the ship (Li et al., 2018b). The former influences the ice loads and ship resistance, while the latter is important if there is relative lateral motion between the ship and ice sheet.
The research objective of this paper is to present an engineering model to predict the ice cusp geometry and bearing capacity with good accuracy, so that it can be implemented into numerical models for ship performance and ice loads simulation in level ice. For this aim, the analytical solution of Nevel (1958) for a semi-infinite narrow wedge beam resting on elastic foundation is further developed to be applicable for cases with large wedge angles, considering forces in angular direction in addition to the axial direction. The solution gives the stress field in the wedge plate, which yields the bearing capacity and cusp geometry. As an example, the proposed ice failure model is incorporated into a numerical model for ship performance in level ice, which simulates the ship motion and ice forces through a set of coupled equations of motion. This model is then able to predict ship speed in a level ice field.

The results of the ice failure model are validated with photos obtained through a stereo camera system installed during a full-scale measurement campaign. The validation indicates that the proposed cusp failure model can predict the ice cusp geometry reasonably well. The results of the numerical model for ship performance in ice are compared to the data measured from the same full-scale measurement campaign. Good agreement is achieved.

The critical issues regarding numerical simulation of ship performance in level ice are first presented in Section 2 to clarify the purpose and scope of this paper. In Section 3, the derivation of the proposed ice failure model is presented and then the approach to incorporate the method into numerical model is described in Section 4. The data applied in this paper are introduced in Section 5. The results and comparison with full-scale measurement are presented in Section 6. Section 7 discusses the benefits and deficiencies of the proposed method and gives suggestions for further research. Section 8 concludes.

2. Ship performance in level ice: numerical approach

It is important to clarify the scope of ‘level ice’ before entering the problem. Despite of the fact that perfect level ice rarely exists in reality, ‘level ice’ is an important concept which is representative for ice conditions where the ice sheet is continuous and covers a large area. In this paper, ‘ship in level ice’ refers to the circumstance when the ship is going through a continuous ice field where the edges of the ice sheet is far from the ship-ice interaction area, so that the radial crack arise from the interaction is not likely to propagate to the free edges. In this case, the ice can be regarded as semi-infinite field where the ice failure modes around the ship are mostly crushing and bending.

Ship performance in ice has been investigated for decades due to the need of designing ice-going ships and making regulations for safe operation. Ship performance in level ice is an engineering problem involving a large number of physical phenomenon. These include e.g. ice crushing, bending and splitting; modelling of ship-ice contact; turning of ice cusps with hydrodynamic effect, ice sliding along the hull; propeller-ice interaction. Unfortunately, most of these problems are not yet well-solved, some due to the lack of fundamental knowledge on the material level (e.g. crushing and bending) and some due to insufficient research and validation (e.g. turning of ice cusps). Consequently, the current available models inevitably contain considerable amount of assumptions and simplifications when dealing with these problems. These usually include e.g. idealizing ice material as perfect elastic, assuming the ice cusp geometry as circular or triangular and considering water as elastic foundation. An assumption or simplification has certain reasonability and is considered to be useful as long as it a) produces similar phenomenon as observed in full-scale test; and b) gives reasonable approximation to the quantity that it is supposed to predict. These are purpose-oriented criteria which may vary according to different aims. Due to these reasons, the observed information and the validation on corresponding phenomenon are of vital importance in developing models for ship performance in ice.

Numerical simulation of ship performance in level ice is achieved by discretizing the process temporally and spatially and solving motion and force-related equations step by step. It is developed for better representation of the physical process and thus potentially higher accuracy compared to analytical or empirical formulas. However, most of the available numerical models are still semi-numerical as empirical or analytical approaches are mostly deployed to model certain phenomenon, e.g. ice bending failure and ice submersion resistance. One of the reasons is the consideration of computational demand. The detection of the contact between ship and ice sheet typically requires small time step. Since there could be many contacts along the ship hull at each time step, the total amount of contact cases to be analyzed for a several-minute voyage could be huge, making it inefficient to spend much computational power on each single case. The aforementioned unsolved fundamental questions on ice behavior also make it questionable whether the result could be greatly improved with the price of heavy computation. Consequently, the computational cost should be balanced with the improvement in accuracy which it can lead to.

Therefore at the current stage, in order to improve the accuracy of numerical models, the key issue is to make reasonable assumptions and simplifications to derive equations for relevant phenomenon, which describes the phenomenon realistically and is easy to implement. This is the principle which this paper is developed upon.

3. Ice failure modelling

As mentioned in the introduction, this paper aims to propose an engineering model for the ice cusp failure by bending due to the crushing force caused by the ship-ice contact. The intended model use is in a numerical simulation model for ship speed and ice loads prediction in level ice. One of the main expectations of the model is to obtain a more realistic ice cusp geometry compared to the state-of-art models. Furthermore, in order to efficiently and accurately simulate a ship’s motion covering a considerable distance, the computation of the ice failure should have the following features:

1. The model should be able to take account as many influencing factors as possible, e.g. the geometry of the contact area. It should determine whether a failure occurs in the ice sheet, according to certain failure criterion. When the ice sheet fails, a realistic geometry of the broken ice cusp is obtained.
2. The bearing capacity and ice cusp size resulting from the model should be statistically reasonable. In the context of ship performance in level ice, the global resistance is the most important output. For this aim, it is preferable to predict correct mean values in terms of the bearing capacity and ice cusp size rather than the extreme values.
3. As stated in Section 2, the result should be easy to be implemented into the numerical model for ship performance in ice.

The first and the third features seem to contradict with each other, since the first requires the ice failure model to be comprehensive and three-dimensional, which is computationally demanding unless a closed form solution is given analytically, while the third feature requires the calculation to be simple and straightforward. A possible way to achieve both is via meta-modelling. This is done by firstly characterizing the problem with certain parameters and then calculating the desired outputs with each combination of the input parameters. Similar approach has been done by Sawamura et al. (2008) and Sawamura et al. (2009), where they simulate the ice failure process with Finite Element Method (FEM) and built a database for the easy implementation in their numerical model for ship in level ice. This paper adopts the similar methodology of meta-modelling but with different approach for the calculation of ice failure, in order to predict more realistic ice cusps geometry.
The governing equation of this problem is written as
\[
\frac{dy}{dx} + 2 \left( \frac{dy}{dx} \right)^2 + y = 0
\]

(1)

Nevel presents the solution to this equation as:
\[
y(x) = A^x dx(x) + B^x dx(x)
\]

(2)

where \( y \) is the vertical deflection; \( x \) the beam's non-dimensional location coordinate; \( A \) and \( B \) coefficients determined from the boundary conditions; and \( dx \) and \( du \) two derived functions of \( x \) expressed by series (see Appendix A). \( x \) is the non-dimensional coordinate expressed as follows:
\[
x = \frac{x}{l_1}, \text{ where } l_1 = \left( \frac{E h^3}{12 \rho g \sigma_{grav}} \right)^{0.25} \]

(3)

where \( x \) is the location coordinate; \( l_1 \) the characteristic length; \( \rho_g \) the water density; and \( \sigma_{grav} \) the gravitational acceleration.

The boundary conditions are that the force and moment at the edge of the contact area balance with the internal moment and shear force in the wedge beam, which is expressed by
\[
\begin{align*}
E h^3 y'(x) \frac{d}{l_1} & = M_0; \\
E h^3 y'(x) \frac{d}{l_1} & = P_0
\end{align*}
\]

(4)

where \( t_r \) is the non-dimensional crushing depth expressed by \( \frac{b}{t_r} \) (see Fig. 1); \( M_0 \) and \( P_0 \) are the moment and shear force at the crushing line position \( (x = t_r) \), with the relationship \( M_0 = P_0 d_c \). The coefficients \( A \) and \( B \) then can be calculated according to Eq. (2).

Therefore, Eq. (1) can be rewritten as
\[
y(t_r, d_c, x) = A(t_r, d_c) dx(x) + B(t_r, d_c) dx(x)
\]

(5)

3.3. Extend Nevel’s solution to a wedge plate

Nevel’s solution describes stress field in the radial direction, while in reality the angular variation introduces another dimension. Consider a wedge plate with loading conditions sketched in Fig. 2. If the wedge plate is discretized into a large number of small wedge beams as illustrated in Fig. 2, initially neglecting the interaction between neighboring beams, each wedge beam can be considered as a ‘narrow’ wedge beam in line with the Nevel formulation. The non-dimensional crushing depth \( t_r \) in each wedge then becomes a function of \( \theta \), i.e. \( t_r = \tau (\theta) \). Bearing in mind that \( A \) and \( B \) also depend on \( d_c \) (Eq. (5)), the deflection field then can be expressed as.

Fig. 2b shows a special case where the crushing line inclination angle \( \alpha \) equals \( 0 \) and thus the crushing area is symmetric. Refer to the general case in Fig. 1, \( \tau \) is a function of the crushing line inclination angle \( \alpha \), the crushing depth \( r_c \), and the wedge opening angle \( \phi_w \). A simple derivation according to the geometry of the crushing area gives:
\[
\tau(\theta) = \frac{\theta}{\cos \theta \tan(\theta - \alpha) + \sin \theta}, \text{ when } \frac{-\phi_w}{2} < \theta < \frac{\phi_w}{2} \]

\[
0, \text{ when } \theta < \frac{-\phi_w}{2} \text{ or } \theta > \frac{\phi_w}{2}
\]

(7)

Due to the change of crushing depth \( \tau \) in angular direction, the deflection of the discretized narrow wedges varies with \( \theta \). Close to the edges, the crushing depth is larger than that in the middle, therefore larger deflections and stresses are associated with the wedges near the edges. An instinctive presumption is that the interaction between neighboring discretized narrow wedges will diminish the deflection and stress differences. For this purpose, the next step is then to modify the deflection field \( f(\theta, \chi) \) so that the forces can be balanced in the angular direction. The deflection field \( w(\theta, \chi) \) is approximated by defining a new function \( g(\theta) \) to deal with the interaction between neighboring narrow wedges.
This does not guarantee that the derived deflection field \( w(\theta, \chi) \) satisfies the governing equation of the Kirchhoff-Love plate theory, since \( g(\theta) \) is assumed to be independent of the location coordinate \( \chi \). As an approximation, \( g(\theta) \) is determined by a weak form of the plate governing equation \( \nabla^2 w = 0 \). Instead of exactly satisfying the governing equation of Kirchhoff-Love plate, the force is balanced on each narrow wedge with finite small wedge angle. In other words, instead of satisfying \( \nabla^2 w = 0 \), an approximation is done to satisfy \( \int \nabla^2 w \, d\chi = 0 \).

Considering a discretized narrow wedge as shown in Fig. 3a, according to the shear force formulation of Kirchhoff-Love plate theory, under polar coordinate system, the shear force at a point \( (\chi, \theta) \) is expressed by:

\[
V(\chi, \theta + \delta \theta) = V(\chi, \theta) + \frac{\delta \theta}{\chi} \frac{\partial V(\chi, \theta)}{\partial \theta} = V(\chi, \theta) + \frac{D}{l_c} \left( \frac{\partial^2}{\partial \theta^2} \left( 2 - \nu \right) \frac{\partial^2}{\partial \chi^2} + \nu \frac{\partial^2}{\partial \chi^2} + (1 - \nu) \frac{2}{\chi^2} \right) w + \frac{\delta ^2 \nu}{\delta \theta^2 \chi} \delta \theta \]

(10)

The total internal force acting on the narrow wedges is then derived by integrating the difference between Eqs. (9) and (10):
Finally, summing up the internal and external forces and equaling it to zero, the following equation is obtained

$$D \left( \frac{\partial^2 C_1(\theta)g(\theta)}{\partial \theta^2} + \frac{\partial^2 C_2(\theta)g(\theta)}{\partial \theta^2} \right) = (1 - g(\theta))\frac{\partial^2 P_c}{\partial \theta^2}$$  \hspace{1cm} (17)

This equation can be physically understood such that the total force on each of the discretized wedge beams in Fig. 2 equals zero. The forces on each wedge beam stem from the interaction with neighboring beams, the crushing loads, as well as the buoyancy force.

Since the ice wedge is free at both edges, the boundary conditions include two equations at each edge, which correspond to zero moment and shear force. The same weak form approach is adopted, so that the integrals of the moment and shear force along the edges equal zero. The expressions can be written as:

$$M_{\text{edge}} = D \left( C_1 g(\theta) + \frac{\partial^2 C_1(\theta)g(\theta)}{\partial \theta^2} \right) = 0$$  \hspace{1cm} (18)

$$V_{\text{edge}} = D \left( C_2 g(\theta) + \frac{\partial^2 C_2(\theta)g(\theta)}{\partial \theta^2} \right) = 0$$  \hspace{1cm} (19)

where

$$C_1 = \frac{1}{L_c} \int_0^L (\frac{\partial f(\chi, \theta)}{\partial \chi} + \frac{1}{\chi} \frac{df(\chi, \theta)}{d\chi})d\chi; \quad C_2 = \frac{1}{L_c} \int_0^L \frac{1}{\chi^2} f(\chi, \theta)d\chi$$  \hspace{1cm} (20)

Due to the complexity of the above formulations, no attempt was made to derive an explicit analytical solution for $C_1(\theta)$ to $C_2(\theta)$. Instead, the values of $C_1(\theta)$ to $C_2(\theta)$ are evaluated by numerical integration. To obtain the solution for the differential equation of Eq. (17), the Finite Difference Method (FDM) is applied. FDM is a standard method which is well established for solving differential equations (Grossmann et al., 2007). The solution process of Eq. (17) is written in Appendix B. After discretization and some manipulation, the discretized equations can be written in the following matrix form

$$C g = R$$  \hspace{1cm} (21)

where $C$ is the coefficient matrix; $g$ the vector of unknown variables $g(\theta)$; and $R$ the result matrix. This algebraic equation can be easily solved with little computational power, yielding $g(\theta)$ numerically. According to Eq. (8), the deflection field of each discretized ice wedge is then represented by

$$\chi(\theta) = f(\chi, \theta)g(\theta)$$

where $g(\theta)$ accounts for the interaction of neighboring wedges.

According to Neve’s solution, the location of maximum stress $\chi_{\text{max}}$ (where $\chi = \chi_{\text{max}}$ is maximum) within a narrow wedge depends on the crushing depth $\tau(\theta)$. The value of the maximum stress $\sigma_r$ is

$$\sigma_r = \frac{Eh}{2\sin(\theta)} \frac{\partial f(\chi, \theta)}{\partial \chi} \Big|_{\chi=\chi_{\text{max}}}$$

With the derived solution, the magnitudes of the maximum stress is then modified by the wedge interaction function $g(\theta)$ and the Poisson’s ratio. This stress is the bending stress and is in radial direction. Another stress component in the radial direction comes from the horizontal compression force $P_h$, which can be assumed to be distributed homogeneously on the cross-section of the narrow wedge. The radial stresses are sketched in Fig. 3b. Here, a superposition is applied to obtain the maximum radial stress $\sigma_{\text{m}}(\theta)$ in each discretized narrow wedge as follows

$$\sigma_{\text{m}}(\theta) = \sigma_c(\theta) + \sigma_n(\theta)$$  \hspace{1cm} (22)

where $\sigma_c$ and $\sigma_n$ are expressed by

$$\sigma_c(\theta) = \frac{Eh}{2(1-\nu^2)} \frac{\partial f(\chi, \theta)}{\partial \chi} \Bigg|_{\chi=\chi_{\text{max}}}$$  \hspace{1cm} (23)

$$\sigma_n(\theta) = \frac{1}{4\tan(\theta)} \frac{\partial P_h}{\partial \theta} \sin(\theta - \alpha)$$  \hspace{1cm} (24)

Connecting the maximum stress points in all the narrow wedges gives the maximum stress line as shown in Fig. 2a. The bending failure is assumed to occur when the maximum radial stress at a wedge equals the flexural strength of ice. When the failure is initiated, the crack is assumed to propagate instantaneously along the maximum stress line crossing each discretized wedge beam. The shape of the crack is illustrated in Fig. 1, where $l_{\text{fract}}$ and $l_{\text{fract}}$ are the breaking lengths at both edges and $l_{\text{fract}}$ denotes the breaking length at the center. The corresponding crushing force is then the bearing capacity of the wedge plate under the given loading condition.

### 3.4. Construction of database

The derived ice failure model accounts for all the influencing variables as described at the beginning of this section, and gives the geometry of the potential ice cusps and the stress in the ice field as outputs. In the next step, an ice failure database is generated with a set of reference values, where $E = 1$ GPa, $\nu = 1$ MPa and $h = 1$ m. Cases with other values of these three factors can then be calculated simply by scaling the results according to the scaling laws derived in Appendix C. It should be noted that the scaling laws here are the natural product of the theory of Kirchhoff-Love plate, instead of being the nature of ice in reality. To the authors’ knowledge, the actual scalability of ice remains as an unsolved issue which still needs systematic verification. The influence of other factors cannot be dealt with via scaling and thus have to be calculated for each combination of the values. These include the wedge angle $\theta_c$, crushing depth $\tau_c$, crushing line inclination $k$, ship flare angle $\chi$ and crushing center location $d_c$. Fig. 4 presents the generation of the database. With the discretization of each variable in Fig. 4, this yields $51 \times 11 \times 31 = 16 \times 10^6$ cases, resulting in huge diversity in ice bending failure.

### 4. A numerical model for ships going through level ice

The database provides a new option for the icebreaking module of numerical simulation models for ship performance in level ice, for example Lubbad and Løset (2011) and Su (2011). In this paper, a numerical model for ship performance in level ice is developed based on the database generated by the proposed ice failure model to determine the icebreaking pattern and the bearing capacity. This section describes the modules of the numerical model.

#### 4.1. Preliminary set-up and thrust modelling

Fig. 5 presents the flow chart of this numerical model. In the initiation step, the ship waterline and the level ice sheet are discretized into points (800 points along ship waterline and 0.05 m interval for the ice sheet). At each time step, the net thrust is calculated by the modified net thrust model proposed by Li et al. (2018a). This model takes ship power, propeller pitch and ship speed as input and calculates the thrust available to overcome ice resistance. The net thrust $T_{\text{net}}$ is calculated by

$$T_{\text{net}} = T_b \left( P, P_1 \left( 1 - \frac{v}{3v_{\text{max}}(P, P_1)} - \frac{2}{3} \frac{v}{v_{\text{max}}(P, P_2)} \right) \right)^2$$  \hspace{1cm} (25)

where $T_b$ and $v_{\text{max}}$ are the achievable bollard pull and open water speed respectively as a function of ship power $P$ and propeller pitch $P_1$, $v$ the ship speed at the current step.
4.2. Contact modelling

The contact area at the current step is defined in a similar way as was done by Lubbad and Løset (2011), which is the overlapping area between the ship position at the current step and the ice profile in the previous step. The time step is taken as 0.001s to achieve good accuracy for explicit simulation. An algorithm is designed to detect the contact area according to the discretized field. A typical detected contact area is demonstrated in Fig. 6. The quadrilateral ABCD is the detect contact area and the point G is the geometric center of it. The five factors in Fig. 4 are then analyzed and used to extract stress information from the database. Specifically, $\psi$ is obtained from the ship hull geometry at the contact location; $r_c$, $\theta$ and $k$ can be calculated according to the geometry of the triangle ABV; $d_c$ is calculated by the distance between point G and line AB.

As demonstrated in Fig. 4, the proposed model is developed for ice wedges with wedge angle below 150°, which is most relevant for both sides of the ship bow, where splitting is less likely to occur. In the stem part of the bow, the wedge angle is typically larger than 150° and is then similar to a semi-infinite plate. Considerable splitting is seen to occur prior to ice bending and therefore becomes an issue which should not be simply neglected (Lubbad and Løset, 2011; Lu et al., 2015). To

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**Fig. 4.** Database generation: input and output.

**Fig. 5.** Flow chart of the numerical model for ship performance in ice.
account for this phenomenon, the same method developed by Lubbad and Løset (2011) is adopted to calculate the load to initiate splitting. If the wedge angle is larger than 150°, the splitting stress \( \sigma_f \) denotes the in-plane stress which tends to split the ice plate by Mode I fracture. If \( \sigma_s(0,0) \) is larger than the flexural strength of ice, the ice sheet is assumed to split into several wedge edges with small wedge angles and therefore can be dealt with according to the method described in the previous paragraph. The five factors in Fig. 4 are then analyzed and the maximum stress \( \sigma_{\text{max}} \) is extracted from the database.

In the next step, the extracted stress \( \sigma_{\text{max}} \) is compared with the flexural strength \( \sigma_f \) to determine if bending failure occurs. If \( \sigma_{\text{max}} \) is larger than \( \sigma_f \), an ice cusp with the shape described in the database is deleted from the ice sheet. The icebreaking module updates the ice sheets and returns the contact forces on the ship hull. This process is same with that applied by Su (2011) and therefore is not elaborated.

In the current stage, the other components of ice resistance are calculated via state-of-the-art formulas as outlined in Li et al. (2018a), which adopts Lindqvist’s empirical formula to calculate the submersion of the ice cusps. This is not the focus of this paper and is therefore to be updated by future research to make this component also numerical. The ship motion is then calculated according to rigid body dynamics (Su, 2011). The algorithm continues until the ship reaches the end of the input ice field.

5. Data

The full-scale measurement data were collected on a measurement campaign with the South African polar supply and research vessel S.A. Agulhas II (see Table 1 for the specifications). During her sea trial in March 2012 in the Baltic Sea, a stereo camera system installed at the ship bow shoulder area was used to take photos of the icebreaking process, from which broken ice cusps can be identified (Suominen et al., 2013). In addition, ship navigational data including the ship speed, and propeller and engine settings were recorded. The ice thickness was measured by an electromagnetic (EM) device and the stereo camera system. This section presents an introduction to the ice cusp measurements as well as the ship performance measurements. For further details, the reader is referred to Suominen et al. (2013).

5.1. Full-scale photos of ice cusps

Photographs obtained from the stereo camera are used to measure the dimensions of the ice cusps. The stereo camera was located at the bow shoulder where it recorded photos of the ice sheet directly underneath, from a vertical distance of 7.7 m. The stereo camera was initially installed for the measurement of ice thickness semi-automatically (Kulovesi and Lehtiranta, 2014). In this paper, a Matlab algorithm was designed to measure the geometry of the ice cusps manually. Using known conversion factors between the measured distances in pixels and the actual distances, the dimensions of the ice cusps can be determined.

Fig. 7 shows two examples of the ice cusps photos. It clearly shows that the breaking crack does not follow a circular track. Instead, it is elliptical with smaller breaking length in the middle and larger breaking length at the edges. Fig. 7 also provides examples of the ice cusp measurements. These are done so that the coordinates of five points on the ice cusps are selected, including four vertices (points 1, 3, 4, 5 in Fig. 7) and the middle point of the crack (points 2 in Fig. 7). The tip of the wedge is determined by extending the edge lines (line 1–5 and line 3–4 in Fig. 7) and obtaining the intersection point. Then the breaking length at the edges and at the center are calculated according to the coordinates. Thus, the size and geometry of the ice cusps can be measured.

The selection of ice cusps is conducted manually. An ice cusp is selected for analysis if the crack and edges are clearly shown in the photos. Special attention should be paid to cases where the ice cusps are re-broken by external forces after it breaks off from the intact level ice field. Fig. 8 demonstrates an example of such a case, where the ice cusp is re-broken during the rotation of the ice cusp relative to the ship hull. The two separated cusps in (c) were together at the moment of breaking off from the ice sheet, therefore they should be measured as one whole ice cusp. The case shown in Fig. 8 is relatively easy to identify, while there are many other cases where it is not so clear if the cusp has been re-broken. These unclear cases are therefore not selected for analysis in order to accurately reflect the breaking scene. From the photos taken on 21st and 22nd March 2012, 100 photos were selected and the geometry of these ice cusps was determined according to the procedure describe above. The results are presented and compared with the calculation and simulation results in Section 6.

5.2. Ship performance

The ship performance and ice measurement data in level ice used in this paper are the same as used in Li et al. (2018a), and hence, these are described here only briefly. The ship speed of S.A. Agulhas II is recorded via the navigational system as well as by the Automatic Identification System (AIS). The propeller motor power as well as propeller pitch and revolution are obtained from the machinery log. Ice thickness data are obtained through an electromagnetic (EM) device and a stereo camera system, so the accuracy is guaranteed (Li et al., 2018a). The ice properties have also been measured during this voyage. Suominen et al. (2013) summarize the measurement results of ice flexural strength and crushing strength by conducting a series of three-point bending and uniaxial crushing tests. The results give the mean values and standard deviation of the ice properties (see Table 2), which are implemented in the numerical simulation of the ship performance model to reflect the random nature of ice.

6. Results

This section first shows the results of the icebreaking pattern
obtained by the proposed ice failure model. After that, the ship speed prediction by the numerical simulation model for ship performance in ice are presented. Comparisons with the ice photos are conducted to validate the ice cusp size and geometry by the proposed method. The comparison on ship speed aims to test the reasonability of the proposed method to be incorporated into numerical simulation models for ship performance in ice. Thus, these comparisons give a comprehensive view on the validity of the proposed ice failure model for use in numerical models of ship performance in level ice.

6.1. Results of the new ice failure model

The proposed ice failure model shows some new features which are not provided by previous approaches. The following results are presented for a case where ice wedges of 0.28 m thickness are crushed gradually from the moment it contacts with a ship. The ice crushing and flexural strength are set as the mean values in Table 2 (1.28 MPa and 404 kPa). The maximum stress and the failure crack geometry are calculated and presented.

Fig. 9 presents an example of the location and geometry of the maximum stress lines when the wedge angle equals $\frac{\pi}{3}$ and $\frac{\pi}{1.3}$, with crushing depth $\tau$ varying from 0.01 to 0.05 and crushing line slope $k$ equaling 0 and 0.2. As shown in this figure, the location of the maximum stress line moves further away from the wedge tip when $\tau$ increases. The potential breaking lengths at the edges are obviously larger than that at the center. This is more evident when the wedge angle equals $\frac{\pi}{2}$, which is featured with larger angular variation. When the wedge angle is small ($\frac{\pi}{1.3}$), the geometry of the crushing line is similar to a circular arc centered at the wedge tip. This is consistent with the observation of Nevel (1958) that the angular variation is negligible when the wedge angle is smaller than $\frac{\pi}{1.3}$.

Fig. 10 presents the variation of the maximum stress in relation to the crushing depth $\tau$ with wedge angles $\theta_w$ equaling $\frac{\pi}{1.3}$ and $\frac{\pi}{2}$, flare angles $\psi$ equaling $\frac{\pi}{6}$ and $\frac{\pi}{3}$, and crushing line slopes $k$ equaling 0 and 0.2 (refer to Fig. 1 for the geometry). The green dashed horizontal line is the flexural strength of the ice (404 kPa), which is the limit point when the ice is broken by bending. The intersection points of this line and the other lines corresponds to the critical crushing depth $\tau_{critical}$ when bending failure occurs, which combined with Fig. 9 gives the geometry of the broken ice cusps under each crushing scenario. The following aspects are indicated by these two figures:

a) The maximum stress increases with larger wedge angles, smaller flare angles, and higher crushing line slopes;
b) The sensitivity of maximum stress on the crushing line slope increases with the wedge angle;
c) The increase in ice flexural strength or decrease in crushing strength leads to larger ice cusps.

Table 2

<table>
<thead>
<tr>
<th>Ice thickness and strength properties.</th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.28 m</td>
<td>0.08 m</td>
</tr>
<tr>
<td>$c_t$</td>
<td>1.28 MPa</td>
<td>0.38 MPa</td>
</tr>
<tr>
<td>$c_f$</td>
<td>404 kPa</td>
<td>59.17 kPa</td>
</tr>
</tbody>
</table>

Fig. 11 presents the critical crushing depths $\tau_{critical}$ and the breaking lengths at the edges and the center ($l_{edge_1}$, $l_{edge_2}$ and $l_{center}$; see Fig. 1), as a function of the combined effect of wedge angle $\theta_w$ and ship flare angle $\psi$. The results show that when the crushing line slope $k$ is 0, the smallest $\tau_{critical}$ and $l_{center}$ occurs at the highest $\theta_w$, while the smallest $l_{edge_1}$ and $l_{edge_2}$ occurs at the lowest $\theta_w$. When the crushing line slope $k$ equals 0.2, $l_{edge_2}$ (the one with shorter crushing depth) becomes smaller for the higher $\theta_w$. The trends with varying flare angle $\psi$ are the same for both $k = 0$ and $k = 0.2$. With increasing $\psi$, the breaking lengths $l_{center}$, $l_{edge_1}$...
critical refers to the mean of > − variances between 0.015 and 0.016 when increases. This also means that the ice will break within shorter change from 0 to 1.

and \( \tau_{\text{critical}} \) is not realistic (\( \tau_{\text{critical}} \) is insensitive to the change in crushing \( \alpha \) and \( \theta_w \).

Fig. 10. Stress evolution with crushing depth under various crushing conditions. The purple line is the flexural strength measured in the full-scale test. (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

and \( l_{\text{edge}} \) increase, which means that the sizes of the broken ice cusps grows. This can be physically explained by Eq. (22), that when the flare angle is larger, the in-plane crushing stress increases, which reduces the maximum stress in the wedge and thus delays the bending failure, leading to larger ice cusps. Since the flare angle of an ice-going ship is typically smaller at the bow compared to that in the bow shoulder area, the results indicate that the ice cusps are larger in the bow shoulder area than those at the bow. Given the limitations of the measurement set-up, where photos are only taken at one ship frame, this finding cannot be empirically confirmed with the available evidence, but it could be validated in future full-scale measurements.

Fig. 12 presents the joint effect of the crushing line slope \( k \) and wedge angle \( \theta_w \) on the critical crushing depth \( \tau_{\text{critical}} \) and breaking lengths. The black areas correspond to conditions where the combination of the \( k \) and \( \theta_w \) is not realistic (\( \alpha > \frac{-\theta_w}{2} \), see Fig. 1). When \( \theta_w \) is small, the magnitude of \( \tau_{\text{critical}} \) is insensitive to the change in crushing line slope \( k \). For example, in Fig. 12 (a), when the wedge angle is \( \frac{\pi}{2} \), \( \tau_{\text{critical}} \) varies between 0.015 and 0.016 when \( k \) change from 0 to 1. However, when the wedge angle is around \( \frac{\pi}{2} \), \( \tau_{\text{critical}} \) varies between 0.005 and 0.013 when \( k \) changes between 0 and 0.5. A similar difference in sensitivity is also evident for the breaking lengths. Generally, with increasing crushing line slope, \( \tau_{\text{critical}}, l_{\text{center}} \) and \( l_{\text{edge}} \) decrease while \( l_{\text{edge2}} \) increases. This also means that the ice will break within shorter time with larger crushing line slope.

For the comparison with the cusp measurement results, the dependence of \( l_{\text{edge}}, l_{\text{center}} \) and \( l_{\text{edge}}/l_{\text{center}} \) on \( \theta_w \) calculated by the proposed ice failure model are plotted in Fig. 13. Here \( l_{\text{edge}} \) refers to the mean of \( l_{\text{edge1}} \) and \( l_{\text{edge2}} \). The flare angle \( \theta \) is set as 75° which approximates those in the measurement. Ice thickness, crushing strength and flexural strength are set as the mean values shown in Table 2. Two sets of \( k \) (0 and 0.2) are used. Linear regressions are plotted along with the raw data in order to demonstrate the correlation between the cusp geometry and wedge angle. Since the data are gathered at bow shoulder area, the wedge angles of most ice cusps are above 90°. The calculation clearly shows similar trends as indicated by the data, where \( l_{\text{edge}} \) and \( l_{\text{edge}}/l_{\text{center}} \) increase with larger \( \theta_w \) while \( l_{\text{center}} \) on the opposite. It also shows that \( k \) has little influence on \( l_{\text{edge}} \), and moderate influence on \( l_{\text{center}} \) and \( l_{\text{edge}}/l_{\text{center}} \). To evaluate the accuracy of the calculation compared to the measurement, the Mean Signed Deviation (MSD) is calculated for each curve in Fig. 13. The MSD is defined in this case as where \( C_l \) and \( M_l \) define the calculated and measured quantities. \( N \) is the number of samples. The reason that MSD is used instead of standard deviation or other error measures is that the measurement contains the variation of ice thickness and strength properties, while the calculation uses the mean values. Therefore, MSD gives an overall measure of the calculation accuracy compared to measurement. The MSD for \( l_{\text{edge}}, l_{\text{center}} \) and \( l_{\text{edge}}/l_{\text{center}} \) are +15.6%, +8.0%, +7.9% respectively for \( k = 0 \) and +17.1%, +14.2%, +3.4% respectively for \( k = 0.2 \), which indicates that the calculated quantities have certain underestimations, but the deviations are relatively small. Therefore, the ice failure model can give good estimation on the ice cusp geometry.
6.2. Results of the numerical model for the ship going through level ice

6.2.1. Ship speed in ice

As an example of the numerical model, the first three S.A. Agulhas II cases investigated by Li et al. (2018a) are re-simulated using the numerical model developed in this paper. The fourth case is not applied because there might be ice ridges highly influencing the ship speed. The ice properties are set and randomized according to the full-scale measurement (see Table 2). Normal distributions are assumed during the randomization. The ice thickness conditions of these three cases are summarized in Table 3. For detailed information on the measurement system, the readers are referred to Li et al. (2018a). The only assumed parameter is the elastic modulus, which is set to 5 GPa due to lack of measurement (Li et al., 2018a).

Fig. 14 presents the ice channel created by the ship, which seems to be realistic. The fact that the icebreaking pattern is not symmetrical is due to the randomization of ice thickness and properties according to Table 2. For detailed information on the measurement system, the readers are referred to Li et al. (2018a). The only assumed parameter is the elastic modulus, which is set to 5 GPa due to lack of measurement (Li et al., 2018a).

Fig. 15 presents a comparison between the simulated ship speed and the recorded ship speed. The mean and final speeds are summarized in Table 4. In the first two scenarios, the simulation predicts a higher speed while in the last scenario it underestimates the speed. The deviations both in terms of mean and final speeds are relatively small compared to recorded speed, mostly below 5%. Compared to other methods investigated by Li et al. (2018a), the new model predicts less breaking resistance thus higher speed. As presented in Table 3, at around 350 m in the first scenario and around 230 m in the second scenario, there are ice rubbles below the ice sheet, which is possibly ice ridges. This introduces additional resistance and leads to speed drops in the corresponding location, as shown in Fig. 15. Taking this into consideration, the deviation between simulation and measurement in the first two scenarios are likely to be smaller. Therefore, the simulation results give reasonable approximation on the ship speed in level ice.

6.2.2. Breaking pattern

Fig. 16 presents the histograms of the ice cusp geometry both from the measurement and from the simulation. The ice wedge angle $\theta_w$, breaking lengths at the center $l_{center}$ and at the edge $l_{edge}$ (defined as $l_{edge} = \frac{\sqrt{l_{center}^2 + l_{edge}^2}}{2}$), and the ratio between $l_{edge}$ and $l_{center}$ are plotted and then summarized in Table 5. The simulation results are obtained through a simulation run with the ice properties set in Table 2. 100 ice cusps information are extracted from the contact events which occurred at the
bow shoulder region where the ship flare angle $\psi$ is in the range of 70°–80°, in order to approximate the ice condition of the measured cases. In general, the simulation results show good similarities to the measurement, with satisfying estimation of the mean values and standard deviation in terms of the measured quantities. The estimation for $\theta_w$ and $l_{cent}$ are particularly good, which deviates from the measured mean values by less than 10%. The larger difference comes from $l_{edge}$, which underestimates the mean value by 18.2%, and results in 11.0% underestimation in the mean value of $l_{edge}/l_{center}$. The simulation generally predicts more scatter than those shown in the measurement, which possibly relates to the randomization of the ice properties. The larger difference in $l_{edge}$ and $l_{edge}/l_{center}$ may be explained by the hydrodynamic effect. According to the FEM simulation results by Sawamura et al. (2008), the hydrodynamic force may change the pressure distribution below the ice sheet and thus lead to elliptical ice cusps. Therefore, the ice cusps can be more elliptical if the hydrodynamic effect is considered. A method to effectively account for the hydrodynamic effect could be an interesting topic for future research.

7. Discussions

As mentioned in the introduction, the objective of this paper is to provide an engineering tool with a physical basis to predict a realistic icebreaking pattern with fast calculation in a simulation model for ship performance in level ice. Therefore, icebreaking patterns of a considerable number of cases with good similarity to the reality are generated, in order to implement it into numerical models. In this section, the comparisons between full-scale measurement and simulation are first discussed, after which the promising features of the proposed method as well as the limitations and possible improvements are discussed.

7.1. Discussions on the comparisons with full-scale measurement

The features of the proposed icebreaking model have been presented and compared with full-scale measurement in the previous section. The dependency on various influencing variables results in a

![Fig. 13. Compare the calculation result with measurement.](image)

![Fig. 14. Ice channel created by the ship.](image)
large variety in icebreaking patterns with a physical basis. The comparison with full-scale ice photo measurements shows that the predicted icebreaking pattern is more realistic compared to the state-of-art approaches. The shape of ice cusps is elliptical rather than circular when the wedge angle is relatively large. This is one of the key aspects to improve the fidelity in numerical simulation models. The size of the ice cusps by this approach is also close to the measurement, which is very important in resistance calculations, according to the findings by Li et al. (2018a). It is interesting to see in Fig. 13 that the correlation between the ice cusp geometry parameters and the wedge angle is consistent with the calculation, which gives evidence to the soundness of the proposed ice failure model. However, due to the assumptions (e.g. ice material) made in the derivation process, the predicted geometry values has certain underestimation compared to the measurement (see Table 5). As a conclusion, the proposed ice failure model predicts acceptably realistic ice cusp geometry for the implementation into numerical models, while further calibration and validation are still needed.

The comparison with ship speed data provides insights into the implementation of the ice failure model into numerical models for ship performance in level ice. As discussed in Li et al. (2018a), the submersion resistance component calculated by Lindqvist’s formula takes up a major part of the total resistance, and therefore dominates the magnitude and deviation of resistance prediction. Consequently, it is insufficient to claim a numerical model is superior to other models without looking into each resistance component. According to the results, the predicted ship speed is close to the recorded speed, which implies that the method is applicable for ship performance in ice. While the proposed ice failure model presents a step forward in terms of the fidelity of the icebreaking module, improving the representation of other force components in numerical simulation models for ship performance in ice certainly is an important area of future research. In particular, the rotation and submersion components should be further developed.

7.2. New features of the proposed ice failure model

The first new feature of the proposed ice failure model is that it improves the fidelity by realistic ice cusps geometry, which is elliptical with larger breaking length along the edges than along the centerline. The geometry of the ice cusps is similar to the statistics of the measured cusps geometry, both in terms of shape and size. This gives evidence to the validity of this method. The breaking length varies in the angular
direction and is linked to the breaking capacity of an ice wedge. Since the icebreaking pattern is one of the key issues in numerical modelling of a ship transiting level ice, the results indicate an improvement in terms of a more accurate prediction of ship performance and ice loads. In addition, it results in a more realistic ice channel geometry, which is important for analyzing scenarios where a ship breaks out of a channel, or is being escorted or in a convoy following an icebreaker.

The second new feature of the proposed method is that it accounts for most of the potential influencing factors (except for speed and hydrodynamic effects) which could have influence to the breaking pattern and bearing capacity. The dependency from the ship hull flare angle $\psi$ influences ice cusp sizes so that the cusps are smaller in the bow than in the bow shoulder area. This is a new feature which has not been captured in previous models for ice cusps. The influence of the crushing line inclination $k$ leads to large variations in the icebreaking pattern. The influence of the crushing center location (denoted by $d_0$) accounts for different crushing area geometry. The wedge angle $\delta_w$ has considerable influence on the breaking length and cusps size. These parameters are linked together in this model as the input for different loading cases, and result in large variety in icebreaking pattern.

The third important new feature of the proposed method is the fast calculation while retaining the complexity. The database approach applied in this paper, where results from a physics-based ice failure model are stored in a database for a large number of cases, allows fast numerical simulation with much more details in the local failure. Sawamura et al. (2009) adopted similar database approach by FEM to generate the database. Since the deflection field is influenced by many factors, the number of possible variable combinations is very large (about 106, see Section 4). This makes the generation of an ice failure database very time-consuming. With the proposed ice failure model, the generation of the database consumes much less time while retaining the huge variety in cases. Thus, this method provides a computationally inexpensive way to calculate the icebreaking pattern while retaining the influence of multiple factors based on a physics-based formulation.

7.3. Limitations and improvements

One of the major limitations of the proposed approach at current stage is that it does not consider hydrodynamic effects in the bending process. With hydrodynamic effect considered, the simulated ship speed in Table 4 is expected to be lower. According to Valanto (2001) and Varsta (1983), the cusp size and the bearing capacity of ice will vary at higher speed due to the interaction between water and ice. The state-of-art methods mostly consider the hydrodynamic effect via empirical formulas, e.g. Tan et al. (2014); Lindqvist (1989); Wang (2001). It can also be accounted for in the physical model via similar approach, where a different formulation of the elastic force by the water could be made dependent on ship speed. This however is not elaborated in this paper and is left for future research.

Appendix A. $d_u$ functions in Nevel’s solution

According to Nevel (1958) and Nevel (1961), $d_{u2}$ and $d_{u3}$ can be expressed by

$$d_{u2} = \frac{\pi}{2} \frac{\sqrt{\pi}}{\Gamma (\frac{3}{2})} \nev_0(\chi) - \frac{\pi}{2} \nev_1(\chi) - \frac{1}{4 \sqrt{\pi}} \nev_2(\chi)$$

$$d_{u3} = \frac{\pi}{4} \frac{\sqrt{\pi}}{\Gamma (\frac{3}{2})} \nev_0(\chi) - \ln \left(\frac{2\sqrt{2}}{\gamma}\right) \nev_1(\chi) - \frac{\Gamma (\frac{1}{2})}{4 \sqrt{\pi}} \nev_2(\chi) + n_0(\chi)$$

where $\gamma$ is Euler's constant; $\Gamma$ the gamma function. The functions $\nev_0$, $\nev_1$, $\nev_2$ and $n_0$ can be expressed by

$$\nev_m(\chi) = \chi^m + \sum_{k=1}^{\infty} \frac{(-1)^k \chi^{m+4k}}{\prod_{n=1}^{k} ((m+4n)(m+4n-1)(m+4n-2))} \quad \text{where } m = 0, 1, 2$$

8. Conclusions

This paper presents a novel ice failure model which can be used in numerical models for ship transiting level ice to determine the ship speed and ice loads on the hull. The ice failure model depends on various input variables, and is based on force-balance equations for coupled narrow wedges. The model is physics-based, and results in realistic cusp geometries without the need to resort to semi-empirical approximations using simplified cusp geometries, as applied in the current state of art. The model has a number of features which are not included in previous cusp models, e.g. the variation of breaking length along angular direction; and the connection between bearing capacity and the size of ice cusps. A comparison with full scale data gives evidence to the validity of the predictions by this model. It also shows that it is suitable to implement this model into existing numerical simulation models for ship performance in ice, improving the formulation for the icebreaking pattern. While further improvements can be made, e.g. related to the treatment of rotation and sliding forces in the ship performance simulation model, the improved formulation for the ice failure model is a step towards more realistic models for performance in level ice.

Declaration of interest

None.

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\[ \text{nel}(\chi) = \text{nev}(\chi) \ln(\chi) - \sum_{k=1}^{m} \text{nev}(\chi_k) \left( \frac{1}{4r - 1} + \frac{1}{4r + 1} \right) \]  \hspace{1cm} (A.4)

where \( \text{nev}(\chi_k) \) is the \( k \)th term of \( \text{nev}(\chi) \).

**Appendix B. FDM solution of Eq. (17)**

The expression of Eq. (17) is

\[
D \left( \frac{\partial^2 C_1(\theta)g(\theta)}{\partial \theta^2} + \frac{\partial^4 C_2(\theta)g(\theta)}{\partial \theta^4} \right) = (1 - g(\theta)) \frac{\partial p}{\partial \theta} \tag{B.1}
\]

It can be simplified as

\[
C_0 \hat{g}(\theta) + \frac{\partial^2 C_1(\theta)g(\theta)}{\partial \theta^2} + \frac{\partial^4 C_2(\theta)g(\theta)}{\partial \theta^4} = R \tag{B.2}
\]

where \( C_0 = R = \frac{\partial p}{\partial \theta} \).

The following finite difference approximations are applied for the 1st to 4th order derivative of a function \( f(\theta) \):

\[
\frac{df}{d\theta_i} = \frac{f_{i+1} - f_{i-1}}{2\delta} \tag{B.3}
\]

\[
\frac{df}{d\theta^2_i} = \frac{f_{i+2} - 2f_{i+1} + f_{i-2}}{4\delta^2} \tag{B.4}
\]

\[
\frac{df}{d\theta^3_i} = \frac{f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_{i-2}}{2\delta^3} \tag{B.5}
\]

\[
\frac{df}{d\theta^4_i} = \frac{f_{i+4} - 4f_{i+3} + 6f_{i+2} - 4f_{i+1} + f_{i-2}}{4\delta^4} \tag{B.6}
\]

The reason of this form of approximations being applied is that by this way, the accuracy of differentiating Eq. (B.5) into Eq. (B.6) is guaranteed. This corresponds to Eqs. (12) and (17), which ensures the force balance on each discretized narrow edge. According to these approximations, Eq. (17) can be discretized into \( n \) equations:

\[
\begin{bmatrix}
C_{1,0} & C_{1,1} & C_{1,2} & C_{1,3} & \cdots & 0 \\
0 & \vdots & \ddots & \vdots & \ddots & \ddots \\
0 & C_{n,n-3} & C_{n,n-2} & C_{n,n-1} & C_{n,n} & C_{n,n+1} & C_{n,n+2} & C_{n,n+3}
\end{bmatrix}
\begin{bmatrix}
g_{-2} \\
g_{-1} \\
g_0 \\
g_1 \\
g_2 \\
g_3 \\
g_4 \\
g_{n+3}
\end{bmatrix}
= 
\begin{bmatrix}
R_{-2} \\
R_{-1} \\
R_0 \\
R_1 \\
R_2 \\
R_3 \\
R_4 \\
R_{n+3}
\end{bmatrix}
\tag{B.7}
\]

where \( g_i \) is the \( i \)th discretized point of \( g(\theta) \); \( g_{-2}, g_{-1}, g_0, g_{n+1}, g_{n+2}, g_{n+3} \) the fictitious nodes outside of the \( \theta \) field; \( C_{ij} \) denotes the coefficients to be multiplied with \( g_i \) to determine the force balance on the \( i \)th discretized narrow wedge. The derivation of the \( C \) coefficients is trivial, therefore not presented here. To solve the group of equations, six more equations are required to form an \( n+6 \) by \( n+6 \) matrix. However, the boundary equations at the two edges can only provide four more equations. Therefore, the first and last equation of Eq. (34) is altered through the following derivation:

\[
\frac{df}{d\theta^2_{i,0.5}} = \frac{df}{d\theta_i} \frac{df}{d\theta_{i+1}} = \frac{f_{i+2} - f_{i+1} - f_i - f_{i-1}}{2\delta^2} \tag{B.8}
\]

\[
\frac{df}{d\theta^2_{14.5}} = \frac{df}{d\theta_i} \frac{df}{d\theta_{i+1}} = \frac{f_{i+4} - 4f_{i+3} + 6f_{i+2} - 4f_{i+1} + f_{i-2}}{2\delta^4} \tag{B.9}
\]

where \( i = 1 \) or \( n-1 \). In this way, the terms containing \( g_{-1} \) and \( g_{n+3} \) are eliminated.

The boundary conditions can be processed by the same approach, which yields four additional equations regarding the moment and shear force at the edges. Combining these four equations with the above equations yields

\[
Cg = R \tag{B.10}
\]
Finally, \(g\) can be easily calculated by \(\mathbf{C}^{-1}\mathbf{R}\).

### Appendix C. Scaling laws in terms of \(E, h\) and \(\sigma_c\)

In this section, the scaling law in terms of ice thickness and ice properties are derived. Here, \(l\) represent the maximum stress location in the ice field, including \(h_{\text{ref}}\) and \(V_{\text{outer}}\). Since the results in the database are non-dimensionalized by the characteristic length \(l_e\), therefore \(l \propto l_e \propto h^{0.65} \cdot \sigma_c^{0.55} \cdot \sigma_c\), I can be scaled from the database as

\[
I = I_{\beta_0} \left( \frac{E}{E_{\text{ref}}} \right)^{0.25} \left( \frac{h}{h_{\text{ref}}} \right)^{0.75}
\]

where \(I_{\beta_0}\) is the result read from the generated database; \(E_{\text{ref}}\) and \(h_{\text{ref}}\) are the reference values used to generate the database.

The scaling law for the stress involves the crushing strength of ice \(\sigma_c\). Noticing that \(f(\chi, \theta)\) is proportional to \(\sigma_c\), according to Eq. (23),

\[
\sigma_c(\chi, \theta) \propto \frac{db}{d^2} (\chi, \theta) \propto \frac{db}{h_{\text{ref}}^{0.35}} = E^{0.45} h^{-0.5} \cdot \sigma_c.
\]

Then as \(P_h\) is a proportional to \(\sigma_c l_e^2\), according to Eq. (24), \(\alpha(\theta) \propto \frac{h^{0.25} \sigma_c^{0.75}}{h} \propto E^{0.25} \cdot h^{-0.25} \cdot \sigma_c\).

Therefore

\[
\sigma_n = \sigma_c + \sigma_c \cdot \alpha \cdot \frac{E}{E_{\text{ref}}} \left( \frac{h}{h_{\text{ref}}} \right)^{0.45} \cdot \frac{S}{S_{\text{ref}}} + \sigma_c \cdot \frac{E}{E_{\text{ref}}} \left( \frac{h}{h_{\text{ref}}} \right)^{0.25} \cdot \frac{S}{S_{\text{ref}}}.
\]

### References


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