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ADAPTIVE FINITE ELEMENT METHOD FOR THE MAXWELL EIGENVALUE PROBLEM*

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Abstract. In this paper we prove the optimal convergence of a standard adaptive scheme based on edge finite elements for the approximation of the solutions of the eigenvalue problem associated with Maxwell's equations. The proof uses the known equivalence of the problem of interest with a mixed eigenvalue problem.

Key words. edge finite elements, eigenvalue problem, Maxwell's equations, adaptive finite element method

AMS subject classifications. 65N30, 65N25, 35Q61, 65N50

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1. Introduction.

In this paper we present an adaptive scheme, based on standard three-dimensional edge elements, for the approximation of the Maxwell eigenvalue problem and analyze its convergence.

A posteriori error estimates for Maxwell's equations have been studied by several authors for the source problem (see, in particular, [32, 3, 37, 19, 36, 20, 38, 22, 14, 23, 43, 15, 21, 18] and the references therein). The eigenvalue problem has been studied only recently in [12, 13] where residual type error estimators are considered and proved to be equivalent to the actual error in the standard framework of efficiency and reliability. The analysis relies on the classical equivalence with a mixed variational formulation [10]. The numerical results presented in [13] confirm that the adaptive scheme driven by our error estimator converges in three dimensions with optimal rate with respect to the number of degrees of freedom.

Reference [11] presented the first convergence analysis for an adaptive scheme applied to the Laplace eigenvalue problem in mixed form. The main tools for the analysis originate from various papers related to adaptive finite elements, in particular from [41, 17, 29]. Thanks to the well-known isomorphism between the spaces $\mathbf{H}(\mathrm{curl};\Omega)$ and $\mathbf{H}(\mathrm{div};\Omega)$ in two space dimensions, the result for the Laplacian in mixed form implies the convergence of the 2D adaptive scheme for Maxwell's eigenproblem: actually, the isomorphism carries over to the corresponding mixed formulation as well as to the error estimators. In this paper we extend the results of [11] to the mixed formulation associated with Maxwell's eigenproblem in three dimensions; as we will notice, such extension is not trivial: several technical details have to be checked and suitably designed interpolation operators are used to complete the analysis. Useful results in this direction are reported in [38, 44].

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It is well understood that the convergence analysis of the adaptive scheme for eigenvalue problems has to consider multiple eigenvalues and clusters of eigenvalues in order to prevent suboptimal convergence. In particular, degeneracy of the convergence may be observed when the error estimator is computed by taking into account only a subset of the discrete eigenmodes approximating the eigensolutions we are interested in (multiple or belonging to a cluster) [40, 28, 9].

Starting from this remark, the analysis performed in [11] has been carried out for generic clusters of eigenvalues. This approach has the inconvenience of adding heavy notation dealing with deep technicalities. For this reason, we decided in this paper to develop our theory in the case of simple eigenvalues. We believe that the presentation in the case of a simple eigenvalue better highlights the novelties with respect to the previous results for the mixed Laplacian that would be hidden by the technical machinery related to clusters of eigenvalues. Nevertheless, the general case can be dealt with by using arguments similar to those in [11].

In section 2 we recall Maxwell's eigenvalue problem, its standard variational formulation, and the equivalent mixed formulation, together with their finite element discretizations. Section 3 defines our error estimator and describes the adaptive scheme. Reliability and efficiency from [13] are recalled and the theory concerning the convergence of the adaptive method is described. The auxiliary results needed for the convergence proof are collected in section 4. These include in particular discrete reliability, quasi-orthogonality, and the contraction property.

2. Maxwell's eigenvalue problem and its finite element discretization.

In this paper we deal with the well-known eigenvalue problem associated with the Maxwell equations (see, for instance, [30, 33, 7]).

Given a polyhedral domain Ω, after eliminating the magnetic field, the problem reads as follows: find $\omega \in \mathbb{R}$ and $\boldsymbol{u} \neq 0$ such that

$$\begin{aligned} &\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}) = \omega^2\varepsilon\boldsymbol{u} && \text{in } \Omega,\\ &\operatorname{div}(\varepsilon\boldsymbol{u}) = 0 && \text{in } \Omega,\\ &\boldsymbol{u}\times\boldsymbol{n} = 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\boldsymbol{u}$ represents the electric field, μ and ε represent the magnetic permittivity and electric permeability, respectively, and $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$, the boundary of Ω. For general inhomogeneous, anisotropic materials, μ and ε are 3-by-3 positive definite and bounded matrix functions. We are considering for simplicity the case when Ω is simply connected: more general situations will be described in Remark 3.

A standard variational formulation of our eigenvalue problem is obtained by considering the functional space $\mathbf{H}_0(\mathbf{curl};\Omega)$ of vector fields in $L^2(\Omega)^3$ with $\mathbf{curl}$ in $L^2(\Omega)^3$ and vanishing tangential component along $\partial\Omega$. The formulation reads as follows: find $\omega \in \mathbb{R}$ with $\omega > 0$ and $\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ with $\boldsymbol{u} \neq 0$ such that

$$(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}) = \omega^2(\varepsilon\boldsymbol{u},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega). \tag{2.2}$$

It is well known, in particular, that the condition $\omega^2 \neq 0$ is equivalent to the divergence condition $\operatorname{div}(\varepsilon\boldsymbol{u}) = 0$ due to the Helmholtz decomposition (see also Remark 1). We assume that the domain Ω and the coefficients ε, μ are such that the problem is associated with a compact solution operator. The eigenvalues can then be numbered in an increasing order as follows:

$$0 < \omega_1 \leq \omega_2 \leq \cdots \leq \omega_j \leq \ldots,$$

where the same eigenvalue is repeated as many times as its algebraic multiplicity. The associated eigenfunctions are denoted by $\boldsymbol{u}_j$ and normalized according to the L^2 norm, that is, $\|\varepsilon^{1/2}\boldsymbol{u}_j\|_0 = 1$.

A powerful tool for the analysis of this problem is a mixed formulation introduced in [10]. With the notation $\boldsymbol{\sigma} = \omega\boldsymbol{u}$, $\boldsymbol{p} = -\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{u}/\omega$, and $\lambda = \omega^2$, the variational formulation (2.2) is equivalent to the following mixed eigenproblem: find $\lambda \in \mathbb{R}$ and $(\boldsymbol{\sigma}, \boldsymbol{p}) \in \mathbf{H}_0(\mathbf{curl};\Omega) \times \boldsymbol{Q}$ with $\boldsymbol{p} \neq 0$ such that

$$
\begin{aligned}
&(\varepsilon\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\tau}, \boldsymbol{p}) = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl};\Omega),\\
&(\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\sigma}, \boldsymbol{q}) = -\lambda(\boldsymbol{p}, \boldsymbol{q}) \qquad \forall \boldsymbol{q} \in \boldsymbol{Q},
\end{aligned}
\tag{2.3}
$$

where $\boldsymbol{Q} = \mu^{-1/2}\,\mathbf{curl}(\mathbf{H}_0(\mathbf{curl};\Omega))$.

The eigenvalues of (2.3) are denoted by

$$
0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \dots .
$$

Given j, we use the notation $\boldsymbol{p}_j = -\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{u}_j/\omega_j$ and $\boldsymbol{\sigma}_j = \omega_j\boldsymbol{u}_j$ with $\lambda_j = \omega_j^2$, so that $(\lambda_j, \boldsymbol{\sigma}_j, \boldsymbol{p}_j)$ solves (2.3) and the following normalization holds true for the eigenfunction: $\|\boldsymbol{p}_j\|_0 = 1$.

The finite element approximation of (2.2) is usually performed with edge elements. Given a tetrahedral decomposition of Ω, we consider Nédélec edge elements introduced in [34, 35]. More general families of finite elements could be considered in the spirit of [2]. More precisely, the general situation can be described by adopting the following standard notation related to the de Rham complex:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{H}_0^1(\Omega) & \xrightarrow{\boldsymbol{\nabla}} & \mathbf{H}_0(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \mathbf{H}_0(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & \mathbb{R}\\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & N_h & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{\Sigma}_h & \xrightarrow{\mathbf{curl}} & \boldsymbol{F}_h & \xrightarrow{\mathrm{div}} & DG_h & \longrightarrow & \mathbb{R}.
\end{array}
\tag{2.4}
$$

In the case when $\boldsymbol{\Sigma}_h$ is a sequence of tetrahedral edge finite elements, the remaining finite element spaces will be composed by nodal Lagrange elements N_h, face elements $\boldsymbol{F}_h$, and discontinuous elements DG_h, respectively. The corresponding diagrams in the case of Nédélec elements of the first and second families can be found in (2.5.58) and (2.5.59) of [8], respectively.

The discretization of (2.2) reads as follows: find $\omega_h \in \mathbb{R}$ with $\omega_h > 0$ and $\boldsymbol{u}_h \in \boldsymbol{\Sigma}_h$ with $\boldsymbol{u}_h \neq 0$ such that

$$
(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{v}) = \omega_h^2(\varepsilon\boldsymbol{u}_h, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{\Sigma}_h.
\tag{2.5}
$$

The corresponding mixed formulation is as follows: find $\lambda_h \in \mathbb{R}$ and $(\boldsymbol{\sigma}_h, \boldsymbol{p}_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{Q}_h$ with $\boldsymbol{p}_h \neq 0$ such that

$$
\begin{aligned}
&(\varepsilon\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\tau}, \boldsymbol{p}_h) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h,\\
&(\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\sigma}_h, \boldsymbol{q}) = -\lambda_h(\boldsymbol{p}_h, \boldsymbol{q}) \qquad \forall \boldsymbol{q} \in \boldsymbol{Q}_h,
\end{aligned}
\tag{2.6}
$$

where $\boldsymbol{Q}_h = \mu^{-1/2}\,\mathbf{curl}(\boldsymbol{\Sigma}_h)$. In particular, we have that $\boldsymbol{Q}_h$ is a subspace of $\mu^{-1/2}\boldsymbol{F}_h$ and it can be easily seen that $\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\sigma}_h = -\lambda_h\boldsymbol{p}_h$.

Following [10, Theorem 2.1], the equivalence between (2.5) and (2.6) can be proved using the definition of $\boldsymbol{Q}_h$ and the identities $\boldsymbol{\sigma}_h = \omega_h\boldsymbol{u}_h$, $\boldsymbol{p}_h = -\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{u}_h/\omega_h$, and $\lambda_h = \omega_h^2$.

With natural notation, we denote by $0 < \omega_{h,1} \le \omega_{h,2} \le \cdots \le \omega_{h,N(h)}$ the eigenvalues of (2.5) and by $0 < \lambda_{h,1} \le \lambda_{h,2} \le \cdots \le \lambda_{h,N(h)}$ those of (2.6). Analogously, the corresponding eigenfunctions are denoted by $\boldsymbol{u}_{h,j}$ and $(\boldsymbol{\sigma}_{h,j}, \boldsymbol{p}_{h,j})$, respectively ($j = 1, \dots, N(h)$), with $\|\varepsilon^{1/2}\boldsymbol{u}_{h,j}\|_0 = \|\boldsymbol{p}_{h,j}\|_0 = 1$. The number of discrete frequencies, repeated according to their multiplicity, is given by $N(h) = \dim \boldsymbol{Q}_h$. We discuss this fact in the next remark.

Remark 1. It is straightforward to check that the number of real eigenvalues of problem (2.6) is equal to $N(h) = \dim \boldsymbol{Q}_h$. Indeed, the matrix form of (2.6) is, with obvious notation,

$$\begin{pmatrix} \mathsf{A} & \mathsf{B}^\top \\ \mathsf{B} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mathsf{p} \end{pmatrix} = \lambda_h \begin{pmatrix} 0 & 0 \\ 0 & -\mathsf{M} \end{pmatrix} \begin{pmatrix} \sigma \\ \mathsf{p} \end{pmatrix}.$$

The number of real eigenvalues of this problem is equal to the size $N(h)$ of the matrix M, as is evident by its equivalent formulation written in terms of the Schur complement

$$\begin{aligned} &\mathsf{B}\mathsf{A}^{-1}\mathsf{B}^\top \mathsf{p} = \lambda_h \mathsf{M}\mathsf{p}, \\ &\sigma = -\mathsf{A}^{-1}\mathsf{B}^\top \mathsf{p}. \end{aligned}$$

The size of the matrix problem corresponding to (2.5) is equal to the dimension of the space $\boldsymbol{\Sigma}_h$. The Helmholtz decomposition in the case of simply connected domains implies that $\dim(\boldsymbol{\Sigma}_h) = \dim(\boldsymbol{\nabla}(N_h)) + N(h)$. Since the space $\boldsymbol{\nabla}(N_h)$ is the kernel of the **curl** operator, it follows that the number of eigenvalues corresponding to $\omega_h > 0$ is equal to $N(h)$. For an additional discussion about this count when the domain is multiply connected, the reader is referred to Remark 3.

Remark 2. It can be useful to recall that the mixed formulations (2.3) and (2.6) are not used for the definition of the method (nor for its implementation) but are crucial ingredients for its analysis.

Remark 3. It is well known that if the domain is not topologically trivial, then the first row of the diagram presented in (2.4) is not an exact sequence. More precisely, the following space of *harmonic forms* plays an important role:

$$\mathcal{H} = \{\mathfrak{h} \in \mathbf{H}_0(\mathbf{curl}; \Omega) : \mathbf{curl}\, \mathfrak{h} = 0,\ \operatorname{div}(\varepsilon \mathfrak{h}) = 0 \text{ in } \Omega\};$$

and it corresponds to the one form cohomology of the de Rham complex. The Helmholtz decomposition in this case has the following form:

$$L^2(\Omega)^3 = \boldsymbol{\nabla}(\mathrm{H}_0^1(\Omega)) \oplus \mathcal{H} \oplus \varepsilon^{-1}\, \mathbf{curl}(\mathbf{H}(\mathbf{curl}; \Omega)),$$

where the three components of the decomposition are ε-orthogonal; that is, they are orthogonal with respect to the scalar product $(\varepsilon\,\cdot, \cdot)$.

It turns out that in the general case the formulation (2.2) is no longer the variational formulation of (2.1). Indeed, functions in $\mathcal{H}$ are eigenfunctions of (2.1) with vanishing frequency. In this case, if we are not interested in the approximation of the space of harmonic functions $\mathcal{H}$, we can disregard the zero frequency and use formulations (2.2) and (2.3) for the analysis of the rest of the spectrum. The approximation of harmonic functions is outside the scope of this paper. We point the reader to possible approaches for the approximation of $\mathcal{H}$: a direct discretization of the space has been proposed in [1]; an adaptive algorithm has been presented in [25]; another indirect approach may be the use of the following alternative mixed formulation known as the *Kikuchi formulation* (see [31, 6]): find $\lambda \in \mathbb{R}$ such that for $\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ and $p \in \mathrm{H}^1_0(\Omega)$, with $\boldsymbol{u} \neq 0$, it holds that

$$\begin{aligned} &(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}) + (\boldsymbol{\nabla}\, p, \varepsilon \boldsymbol{v}) = \lambda(\varepsilon \boldsymbol{u}, \boldsymbol{v}) && \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega),\\ &(\boldsymbol{\nabla}\, q, \varepsilon \boldsymbol{u}) = 0 && \forall q \in \mathrm{H}^1_0(\Omega). \end{aligned}$$

It is not difficult to see that any solution of the Kikuchi formulation satisfies $p = 0$ (take $\boldsymbol{v} = \boldsymbol{\nabla}\, p$ in the first equation). Hence, it is immediate to check that the Kikuchi formulation is equivalent to the standard variational formulation (2.2) with the additional solution $\lambda = 0$ corresponding to $\boldsymbol{u} \in \mathcal{H}$.

3. Error estimator and adaptive method. We are going to study and analyze an adaptive finite element scheme in the framework of [26, 24, 17, 29, 11]. The scheme is based on the local error estimator (see [13])

$$\begin{aligned} \tilde\eta_K^2 &= h_K^2 \|\varepsilon \boldsymbol{u}_h - \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}_h)/\omega_h^2\|_{0,K}^2 + h_K^2 \|\operatorname{div}(\varepsilon \boldsymbol{u}_h)\|_{0,K}^2 \\ &\quad + \frac{1}{2} \sum_{F \in \mathcal{F}_{\mathrm{I}}(K)} \left[h_F \left\| [\![(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}_h/\omega_h^2) \times \boldsymbol{n}]\!] \right\|_{0,F}^2 + h_F \|[\![\varepsilon \boldsymbol{u}_h \cdot \boldsymbol{n}]\!]\|_{0,F}^2 \right], \end{aligned} \tag{3.1}$$

where K is an element of our triangulation $\mathcal{T}_h$, $\mathcal{F}_{\mathrm{I}}(K)$ is the set of inner faces of K, h_K and h_F are the diameters of K and F, respectively, and $[\![\cdot]\!]$ is the jump across an inner face F.

Given a set of elements $\mathcal{M}$, we use the notation

$$\tilde\eta(\mathcal{M})^2 = \sum_{K \in \mathcal{M}} \tilde\eta_K^2,$$

and we write $\tilde\eta = \tilde\eta(\mathcal{T}_h)$ for the global error estimator when no confusion arises. Moreover, a subscript κ is used when $\tilde\eta_\kappa$ refers to the mesh $\mathcal{T}_\kappa$.

Given an initial mesh $\mathcal{T}_0$ and a bulk parameter $\theta \in \mathbb{R}$, with $0 < \theta \leq 1$, we compute a sequence of meshes $\{\mathcal{T}_\ell\}$, solutions $\{(\omega_\ell^2, \boldsymbol{u}_\ell)\}$, and estimators $\{\tilde\eta(\mathcal{T}_\ell)\}$ according to the standard solve/estimate/mark/refine strategy (see [26]). In particular, at a given level ℓ, the marking step consists in choosing a minimal subset $\mathcal{M}_\ell$ of $\mathcal{T}_\ell$ such that

$$\theta \tilde\eta_\ell^2(\mathcal{T}_\ell) \leq \tilde\eta_\ell^2(\mathcal{M}_\ell).$$

The new mesh $\mathcal{T}_{\ell+1}$ is given by the smallest admissible refinement of $\mathcal{T}_\ell$ satisfying $\mathcal{M}_\ell \cap \mathcal{T}_{\ell+1} = \emptyset$ according to the rules defined in [4, 42].

Considering the equivalence between the standard formulation (2.5) and the mixed formulation (2.6), the local error estimator for the mixed problem takes the following form:

$$\begin{aligned} \eta_K^2 &= h_K^2 \|\varepsilon \boldsymbol{\sigma}_h + \mathbf{curl}(\mu^{-1/2} \boldsymbol{p}_h)\|_{0,K}^2 + h_K^2 \|\operatorname{div}(\varepsilon \boldsymbol{\sigma}_h)\|_{0,K}^2 \\ &\quad + \frac{1}{2} \sum_{F \in \mathcal{F}_{\mathrm{I}}(K)} \left(h_F \left\| [\![(\mu^{-1/2} \boldsymbol{p}_h) \times \boldsymbol{n}]\!] \right\|_{0,F}^2 + h_F \|[\![\varepsilon \boldsymbol{\sigma}_h \cdot \boldsymbol{n}]\!]\|_{0,F}^2 \right). \end{aligned} \tag{3.2}$$

It is easy to check that the following relation between the two estimators holds true:

$$\tilde\eta_K^2 = \frac{1}{\lambda_h} \eta_K^2 \quad \forall K \in \mathcal{T}_h.$$

In particular, the comments stated in Remark 2 can be extended to the error estimators: our analysis will be performed by using the mixed formulation (2.6) and the estimator (3.2) even if the scheme is originally defined in terms of the standard formulation (2.5) and the estimator (3.1).

In the rest of this section we present our main result in the case of an eigenvalue of multiplicity one, since we believe that in this case it is easier to describe the main arguments leading to the optimal convergence of the adaptive scheme. Moreover, for ease of notation, we assume from now on that ε and μ are scalar and $\varepsilon = \mu = 1$. This assumption does not reduce the relevance of our result: more general situations can be dealt with by adopting arguments similar to those in [10] or [16].

Let $\omega = \omega_j$ be a simple eigenvalue of (2.2) and $\widetilde{W} = \mathrm{span}\{\boldsymbol{u}_j\}$ the associated one-dimensional eigenspace. Let $\omega_\ell = \omega_{\ell,j}$ be the jth discrete eigenvalue of (2.5) computed with the adaptive scheme on the mesh $\mathcal{T}_\ell$ and $\widetilde{W}_\ell = \mathrm{span}\{\boldsymbol{u}_{\ell,j}\}$ the corresponding eigenspace. The gap between $\widetilde{W}$ and $\widetilde{W}_\ell$ is measured by

$$\delta(\widetilde{W}, \widetilde{W}_\ell) = \sup_{\substack{\boldsymbol{u}\in\widetilde{W} \\ \|\boldsymbol{u}\|_{\mathbf{curl}}=1}} \inf_{\boldsymbol{u}_\ell\in\widetilde{W}_\ell} \|\boldsymbol{u}-\boldsymbol{u}_\ell\|_{\mathbf{curl}}.$$

For the reader's convenience, we recall the reliability and efficiency properties proved in [13]. As is common for eigenvalue problems, the efficiency property is not local in the sense that it relies on the difference between ω and ω_h, which is a global quantity.

Proposition 3.1. *There exists $C > 0$ such that, if $(\omega, \boldsymbol{u})$ and $(\omega_h, \boldsymbol{u}_h)$ are solutions of problems* (2.2) *and* (2.5)*, respectively (the latter approximating the former as h goes to zero), we have for h small enough*

Reliability

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbf{curl}} \le C\tilde{\eta}, \qquad |\omega^2 - \omega_h^2| \le C\tilde{\eta}^2,$$

Efficiency

$$\tilde{\eta}_K \le C\left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,K'} + \|\mathbf{curl}(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,K'} + h_K\left\|\omega^2\boldsymbol{u} - \omega_h^2\boldsymbol{u}_h\right\|_{0,K'}\right),$$

where K' denotes the union of the elements sharing a face with K.

Proof. See Propositions 5 and 6 of [13]. ∎

The convergence of the adaptive scheme is usually described by making use of the nonlinear approximation classes discussed in [4]. Denoting by $\mathbb{T}(m)$ the set of admissible refinements of $\mathcal{T}_0$ whose cardinality differs from that of the initial triangulation by less than m, the best algebraic convergence rate $s \in (0, +\infty)$ for the approximation of functions belonging to a space $\boldsymbol{\mathcal{W}}$ is characterized in terms of the following seminorm:

$$|\boldsymbol{\mathcal{W}}|_{\mathcal{A}_s} = \sup_{m\in\mathbb{N}} m^s \inf_{\mathcal{T}\in\mathbb{T}(m)} \delta(\boldsymbol{\mathcal{W}}, \boldsymbol{\Sigma}_{\mathcal{T}}),$$

where $\boldsymbol{\Sigma}_{\mathcal{T}}$ is the edge finite element space on the mesh $\mathcal{T}$.

The main result of our paper, stated in the next theorem, shows that if $\widetilde{W}$ has bounded $\mathcal{A}_s$-seminorm for some s, then the optimal convergence order s is obtained by the sequence of solutions constructed by the adaptive procedure described above.

Theorem 3.2. *Provided the meshsize of the initial mesh $\mathcal{T}_0$ and the bulk parameter θ are small enough, if the eigenspace satisfies $|\widetilde{W}|_{\mathcal{A}_s} < \infty$, then the sequence of discrete eigenspaces $\widetilde{W}_\ell$ computed on the mesh $\mathcal{T}_\ell$ fulfills the optimal estimate*

$$\delta(\widetilde{W}, \widetilde{W}_\ell) \le C(\mathrm{card}(\mathcal{T}_\ell) - \mathrm{card}(\mathcal{T}_0))^{-s}|\widetilde{W}|_{\mathcal{A}_s}.$$

Moreover, the eigenvalue satisfies the double order rate of convergence

$$|\omega - \omega_\ell| \le C\delta(\widetilde{W}, \widetilde{W}_\ell)^2.$$

The proof of Theorem 3.2 is based on the corresponding result written in terms of the mixed formulations (2.3) and (2.6).

Let $\lambda = \lambda_j$ be a simple eigenvalue of (2.3) and $W = \mathrm{span}\{(\boldsymbol{\sigma}_j, \boldsymbol{p}_j)\}$ the associated one-dimensional eigenspace. Let $\lambda_\ell = \lambda_{\ell,j}$ be the jth discrete eigenvalue corresponding to the ℓth level of refinement in the adaptive scheme and $W_\ell = \mathrm{span}\{(\boldsymbol{\sigma}_{\ell,j}, \boldsymbol{p}_{\ell,j})\}$ the associated eigenspace. The gap between W and W_ℓ is measured by

$$\delta(W, W_\ell) = \sup_{\substack{(\boldsymbol{\sigma},\boldsymbol{p})\in W\\ \|\boldsymbol{p}\|_0=1}} \inf_{(\boldsymbol{\sigma}_\ell,\boldsymbol{p}_\ell)\in W_\ell} \left(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_\ell\|_0^2 + \|\boldsymbol{p} - \boldsymbol{p}_\ell\|_0^2\right)^{1/2}.$$

We recall the reliability and efficiency properties proved in [13]. It turns out that in the case of the mixed formulation it is possible to obtain a local efficiency estimate.

PROPOSITION 3.3. *Let* $(\lambda, \boldsymbol{\sigma}, \boldsymbol{p})$ *and* $(\lambda_h, \boldsymbol{\sigma}_h, \boldsymbol{p}_h)$ *be solutions of problems* (2.3) *and* (2.6)*, respectively, such that the latter approximates the former as h goes to zero.*

Reliability *There exist* $\rho_{\mathrm{rel1}}(h)$ *and* $\rho_{\mathrm{rel2}}(h)$ *tending to zero as* $h \to 0$ *and positive constants* C *independent of the mesh size such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0 \le C\eta + \rho_{\mathrm{rel1}}(h)(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0),$$
$$|\lambda - \lambda_h| \le C\eta^2 + \rho_{\mathrm{rel2}}(h)(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0)^2.$$

Efficiency *For each* $K \in \mathcal{T}_h$,

$$\eta_K \le C(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,K'} + \|\boldsymbol{p} - \boldsymbol{p}_h\|_{0,K'}),$$

where K' *is the union of the tetrahedra sharing a face with* K.

Proof. See Theorems 3 and 4 of [13]. The estimate for $|\lambda - \lambda_h|$ is an immediate consequence of (4.2). ∎

The counterpart of Theorem 3.2 in the framework of the mixed formulation is stated as follows.

THEOREM 3.4. *Provided the meshsize of the initial mesh* $\mathcal{T}_0$ *and the bulk parameter* θ *are small enough, if the eigenspace satisfies* $|W|_{\mathcal{A}_s} < \infty$*, then the sequence of discrete eigenspaces* W_ℓ *corresponding to the solution computed on the mesh* $\mathcal{T}_\ell$ *fulfills the optimal estimate*

$$\delta(W, W_\ell) \le C(\mathrm{card}(\mathcal{T}_\ell) - \mathrm{card}(\mathcal{T}_0))^{-s}|W|_{\mathcal{A}_s}.$$

Moreover, the eigenvalue satisfies the double order rate of convergence

$$|\omega - \omega_\ell| \le C\delta(W, W_\ell)^2.$$

The proof of our main result has the same structure as the one presented in [11], based on [28] and [17]. For this reason, we do not repeat it here, but we conclude this section by listing some keystone properties that are essential for the proof of our main result. We refer the interested reader to [11] and to the references therein for a rigorous proof of how to combine them in order to get the result of Theorem 3.4.

The following properties involve quantities related to meshes that will be denoted by $\mathcal{T}_H$, $\mathcal{T}_h$, or $\mathcal{T}_\ell$. In general, $\mathcal{T}_h$ denotes an arbitrary refinement of a fixed mesh

$\mathcal{T}_H$, while $\mathcal{T}_\ell$ refers to the sequence of meshes designed by the adaptive procedure. The eigenmode approximating $\{\lambda, (\boldsymbol{\sigma}, \boldsymbol{p})\}$ will be indicated by $\{\lambda_\kappa, (\boldsymbol{\sigma}_\kappa, \boldsymbol{p}_\kappa)\}$, where κ may be H, h, or ℓ, respectively. We assume that the sign of $(\boldsymbol{\sigma}_\kappa, \boldsymbol{p}_\kappa)$ is chosen in such a way that the scalar product between $\boldsymbol{p}$ and $\boldsymbol{p}_\kappa$ is positive (so that the same is true for the scalar product between $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_\kappa$).

PROPERTY 1 (discrete reliability). *There exist a constant C_{drel} and a function $\rho_{\mathrm{drel}}(H)$ tending to zero as H goes to zero, such that, for a sufficiently fine mesh $\mathcal{T}_H$ and for all refinements $\mathcal{T}_h$ of $\mathcal{T}_H$, it holds that*

$$\begin{aligned}\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0 &\le C_{\mathrm{drel}}\eta_H(\mathcal{T}_H \setminus \mathcal{T}_h)\\ &+ \rho_{\mathrm{drel}}(H)(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_H\|_0).\end{aligned}$$

PROPERTY 2 (quasi-orthogonality). *There exists a function $\rho_{\mathrm{qo}}(h)$ tending to zero as h goes to zero, such that*

$$\begin{aligned}\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 &\le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_H\|_0^2 + \|\boldsymbol{p} - \boldsymbol{p}_H\|_0^2 - \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 - \|\boldsymbol{p} - \boldsymbol{p}_h\|_0^2\\ &+ \rho_{\mathrm{qo}}(h)(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_H\|_0^2 + \|\boldsymbol{p} - \boldsymbol{p}_H\|_0^2).\end{aligned}$$

PROPERTY 3 (contraction). *If the initial mesh $\mathcal{T}_0$ is sufficiently fine, there exist constants $\beta \in (0, +\infty)$ and $\gamma \in (0, 1)$ such that the term*

$$\xi_\ell^2 = \eta(\mathcal{T}_\ell)^2 + \beta(\|\boldsymbol{\sigma}_\ell - \boldsymbol{\sigma}_{\ell+1}\|_0^2 + \|\boldsymbol{p}_\ell - \boldsymbol{p}_{\ell+1}\|_0^2)$$

satisfies for all integers ℓ

$$\xi_{\ell+1}^2 \le \gamma\xi_\ell^2.$$

In the next section we will show how to prove the above properties. While in some cases these are natural extensions of the analogous results for the Laplace eigenproblem in mixed form (see [11]), we will see that in particular the *discrete reliability* property requires a more careful analysis.

4. Proof of the main results. We start this section by recalling some known results for the approximation of problem (2.3) (recall that for simplicity we deal with trivial topology and homogeneous material $\varepsilon = \mu = 1$). The first one is a superconvergence estimate which has been proved in [13, Lemma 9].

LEMMA 4.1. *Let $(\lambda, \boldsymbol{\sigma}, \boldsymbol{p})$ and $(\lambda_h, \boldsymbol{\sigma}_h, \boldsymbol{p}_h)$ be solutions of (2.3) and (2.6), respectively, with $\|\boldsymbol{p}\|_0 = \|\boldsymbol{p}_h\|_0 = 1$ and such that the latter approximates the former as h goes to zero. Then, there exists a function $\rho_{\mathrm{sc}}(h)$ tending to zero as $h \to 0$ such that*

$$\|\boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h\|_0 \le \rho_{\mathrm{sc}}(h)(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0), \tag{4.1}$$

where $\boldsymbol{P}_h$ denotes the L^2-projection onto $\boldsymbol{Q}_h$.

If $(\lambda, \boldsymbol{\sigma}, \boldsymbol{p})$ and $(\lambda_\kappa, \boldsymbol{\sigma}_\kappa, \boldsymbol{p}_\kappa)$ for $\kappa = h, H$ are as in Lemma 4.1, thanks to the definition of $\boldsymbol{Q}_\kappa$, it is not difficult to verify that the following equations hold true (see [27, Lemma 4]):

$$\begin{aligned}\lambda - \lambda_h &= \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 - \lambda_h\|\boldsymbol{p} - \boldsymbol{p}_h\|_0^2,\\ \lambda_h - \lambda_H &= \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 - \lambda_H\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2.\end{aligned} \tag{4.2}$$

Using the error estimates for $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$ and $\|\boldsymbol{p}-\boldsymbol{p}_h\|_0$, one can obtain the following bound, which will be used several times in what follows: there exists $\rho_0(H)$ tending to zero as H goes to zero such that

$$|\lambda_h - \lambda_H| \le \rho_0(H)(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0). \tag{4.3}$$

It is useful to recall the source problem associated with (2.3): given $\mathbf{g} \in L^2(\Omega)^3$, find $(\boldsymbol{\sigma}_g, \boldsymbol{p}_g) \in \mathbf{H}_0(\mathbf{curl};\Omega) \times \boldsymbol{Q}$ such that

$$
\begin{aligned}
&(\boldsymbol{\sigma}_g, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \boldsymbol{p}_g) = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl};\Omega),\\
&(\mathbf{curl}\,\boldsymbol{\sigma}_g, \boldsymbol{q}) = -(\mathbf{g}, \boldsymbol{q}) \qquad \forall \boldsymbol{q} \in \boldsymbol{Q}.
\end{aligned} \tag{4.4}
$$

Since we have taken $\mu = 1$, it turns out that $\boldsymbol{Q} = \mathbf{curl}(\mathbf{H}_0(\mathbf{curl};\Omega)) = \mathbf{H}_0(\mathrm{div}^0;\Omega)$, which is the space of vectorfields in $L^2(\Omega)^3$ with zero divergence and vanishing normal component along the boundary.

Standard regularity results for (4.4) imply that, if Ω is a Lipschitz polyhedron, then both components of the solution of (4.4) are in $\mathbf{H}^s(\Omega)$ for some $s > 1/2$ (see, for instance, the discussion related to [12, Theorem 2.1]).

The discretization of (4.4) reads as follows: find $(\boldsymbol{\sigma}_{g,h}, \boldsymbol{p}_{g,h}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{Q}_h$ such that

$$
\begin{aligned}
&(\boldsymbol{\sigma}_{g,h}, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \boldsymbol{p}_{g,h}) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h,\\
&(\mathbf{curl}\,\boldsymbol{\sigma}_{g,h}, \boldsymbol{q}) = -(\mathbf{g}, \boldsymbol{q}) \qquad \forall \boldsymbol{q} \in \boldsymbol{Q}_h.
\end{aligned} \tag{4.5}
$$

The following error estimate is well known (see [5]):

$$
\|\boldsymbol{\sigma}_g - \boldsymbol{\sigma}_{g,h}\|_0 + \|\boldsymbol{p}_g - \boldsymbol{p}_{g,h}\|_0 \le C h^s \|\mathbf{g}\|_0, \quad s > 1/2. \tag{4.6}
$$

A special situation that will be useful in the proof of Property 1 is given by the following problem: find $(\widehat{\boldsymbol{\sigma}}_H, \widehat{\boldsymbol{p}}_H) \in \boldsymbol{\Sigma}_H \times \boldsymbol{Q}_H$ such that

$$
\begin{aligned}
&(\widehat{\boldsymbol{\sigma}}_H, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \widehat{\boldsymbol{p}}_H) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_H,\\
&(\mathbf{curl}\,\widehat{\boldsymbol{\sigma}}_H, \boldsymbol{q}) = -\lambda_h(\boldsymbol{p}_h, \boldsymbol{q}) \quad \forall \boldsymbol{q} \in \boldsymbol{Q}_H.
\end{aligned} \tag{4.7}
$$

It is classical to obtain the estimate stated in the following lemma.

LEMMA 4.2. *Let $(\boldsymbol{\sigma}_H, \boldsymbol{p}_H) \in \boldsymbol{\Sigma}_H \times \boldsymbol{Q}_H$ be the solution of* (2.6) *on the mesh $\mathcal{T}_H$ and $(\widehat{\boldsymbol{\sigma}}_H, \widehat{\boldsymbol{p}}_H) \in \boldsymbol{\Sigma}_H \times \boldsymbol{Q}_H$ be the solution of* (4.7). *Then there exists $\rho_1(H)$ tending to zero as H goes to zero such that*

(4.8)

$$
\|\boldsymbol{\sigma}_H - \widehat{\boldsymbol{\sigma}}_H\|_0 + \|\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H\|_0 \le C\|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H \boldsymbol{p}_h\|_0 + \rho_1(H)(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0).
$$

Proof. Let $\{\lambda_{H,i}, (\boldsymbol{\sigma}_{H,i}, \boldsymbol{p}_{H,i})\}$ $(i = 1, \dots, N(H))$ be the family of eigensolutions of problem (2.6) related to the mesh $\mathcal{T}_H$ (recall that $\lambda_H = \lambda_{H,j}$). We have

$$
\|\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H\|_0^2 = \sum_{i=1}^{N(H)} a_i^2, \qquad a_i = (\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H, \boldsymbol{p}_{H,i}).
$$

For $i = j$,

$$
\begin{aligned}
a_j &= (\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H, \boldsymbol{p}_H) = 1 - (\widehat{\boldsymbol{p}}_H, \boldsymbol{p}_H) = 1 + \frac{1}{\lambda_H}(\widehat{\boldsymbol{p}}_H, \mathbf{curl}\,\boldsymbol{\sigma}_H) = 1 - \frac{1}{\lambda_H}(\widehat{\boldsymbol{\sigma}}_H, \boldsymbol{\sigma}_H)\\
&= 1 + \frac{1}{\lambda_H}(\boldsymbol{p}_H, \mathbf{curl}\,\widehat{\boldsymbol{\sigma}}_H) = 1 - \frac{\lambda_h}{\lambda_H}(\boldsymbol{p}_h, \boldsymbol{p}_H) = 1 - \frac{\lambda_h}{\lambda_H} + \frac{\lambda_h}{\lambda_H}\big(1 - (\boldsymbol{p}_h, \boldsymbol{p}_H)\big)\\
&= \frac{\lambda_H - \lambda_h}{\lambda_H} + \frac{\lambda_h}{2\lambda_H}\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 = \left(1 + \frac{\lambda_h}{2\lambda_H}\right)\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 - \frac{1}{\lambda_H}\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2.
\end{aligned}
$$

For $i \neq j$, since $a_i = -(\widehat{\boldsymbol{p}}_H, \boldsymbol{p}_{H,i})$, we can proceed as follows:

$$
\begin{aligned}
\lambda_{H,i}(\widehat{\boldsymbol{p}}_H, \boldsymbol{p}_{H,i}) &= -(\mathbf{curl}\,\boldsymbol{\sigma}_{H,i}, \widehat{\boldsymbol{p}}_H) = (\widehat{\boldsymbol{\sigma}}_H, \boldsymbol{\sigma}_{H,i}) = -(\mathbf{curl}\,\widehat{\boldsymbol{\sigma}}_H, \boldsymbol{p}_{H,i})\\
&= \lambda_h(\boldsymbol{p}_h, \boldsymbol{p}_{H,i}) = \lambda_h(\boldsymbol{P}_H \boldsymbol{p}_h, \boldsymbol{p}_{H,i}),
\end{aligned}
$$

which gives

$$(\lambda_{H,i} - \lambda_h)(\widehat{\boldsymbol{p}}_H, \boldsymbol{p}_{H,i}) = -\lambda_h(\boldsymbol{p}_{H,i}, \widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h).$$

Hence,

$$\begin{aligned}\sum_{i\neq j} a_i^2 &= \sum_{i\neq j} a_i \frac{\lambda_h}{\lambda_{H,i} - \lambda_h}(\boldsymbol{p}_{H,i}, \widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h)\\ &\le \max_{i\neq j}\left|\frac{\lambda_h}{\lambda_{H,i} - \lambda_h}\right| \left(\sum_{i\neq j} a_i^2\right)^{1/2} \left(\sum_{i\neq j}(\boldsymbol{p}_{H,i}, \widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h)^2\right)^{1/2}\\ &\le \max_{i\neq j}\left|\frac{\lambda_h}{\lambda_{H,i} - \lambda_h}\right| \left(\sum_{i\neq j} a_i^2\right)^{1/2} \|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0.\end{aligned}$$

Putting things together, we get

$$\begin{aligned}\|\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H\|_0^2 &= \sum_{i=1}^{N(H)} a_i^2\\ &\le C(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2)^2 + \max_{i\neq j}\left|\frac{\lambda_h}{\lambda_{H,i} - \lambda_h}\right|^2 \|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0^2.\end{aligned} \tag{4.9}$$

If H is small enough (remember that we have assumed in Theorem 3.2 that the initial mesh is fine enough), then the denominator $\lambda_{H,i} - \lambda_h$ is bounded away from zero for all $i \neq j$ and for all h. This implies the desired estimate for $\|\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H\|_0$.

The estimate for $\|\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H\|_0$ can be obtained as follows:

$$\begin{aligned}\|\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H\|_0^2 &= -(\mathbf{curl}(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H), \widehat{\boldsymbol{p}}_H - \boldsymbol{p}_H) = (\lambda_h\boldsymbol{P}_H\boldsymbol{p}_h - \lambda_H\boldsymbol{p}_H, \widehat{\boldsymbol{p}}_H - \boldsymbol{p}_H)\\ &\le |\lambda_h - \lambda_H|\|\widehat{\boldsymbol{p}}_H - \boldsymbol{p}_H\|_0 + \lambda_h\|\boldsymbol{p}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0\|\widehat{\boldsymbol{p}}_H - \boldsymbol{p}_H\|_0\\ &\le C(|\lambda_h - \lambda_H|^2 + \|\widehat{\boldsymbol{p}}_H - \boldsymbol{p}_H\|_0^2 + \|\boldsymbol{p}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0^2)\\ &\le C(|\lambda_h - \lambda_H|^2 + \|\widehat{\boldsymbol{p}}_H - \boldsymbol{p}_H\|_0^2 + \|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0^2).\end{aligned}$$

Using (4.3) and (4.9), we obtain the final estimate. ∎

LEMMA 4.3. *Let* $(\boldsymbol{\sigma}_h, \boldsymbol{p}_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{Q}_h$ *be the solution of* (2.6) *and* $(\widehat{\boldsymbol{\sigma}}_H, \widehat{\boldsymbol{p}}_H) \in \boldsymbol{\Sigma}_H \times \boldsymbol{Q}_H$ *be the solution of* (4.7). *Then for* H *small enough we have*

$$\|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0 \le CH^s(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0).$$

Proof. We use a duality argument in order to get a bound for $\|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0$. Let $(\boldsymbol{\xi}, \boldsymbol{w}) \in \mathbf{H}_0(\mathbf{curl}; \Omega) \times \boldsymbol{Q}$ be the solution of

$$\begin{aligned}(\boldsymbol{\xi}, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \boldsymbol{w}) &= 0 && \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl}; \Omega),\\ (\mathbf{curl}\,\boldsymbol{\xi}, \boldsymbol{q}) &= (\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h, \boldsymbol{q}) && \forall \boldsymbol{q} \in \boldsymbol{Q},\end{aligned}$$

and let $(\boldsymbol{\xi}_h, \boldsymbol{w}_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{Q}_h$ (resp., $(\boldsymbol{\xi}_H, \boldsymbol{w}_H) \in \boldsymbol{\Sigma}_H \times \boldsymbol{Q}_H$) be the corresponding discrete solution on the mesh $\mathcal{T}_h$ (resp., $\mathcal{T}_H$). We have

$$\begin{aligned}\|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0^2 &= (\mathbf{curl}\,\boldsymbol{\xi}_H, \widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h) = (\mathbf{curl}\,\boldsymbol{\xi}_H, \widehat{\boldsymbol{p}}_H - \boldsymbol{p}_h)\\ &= -(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h, \boldsymbol{\xi}_H)\\ &= -(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h, \boldsymbol{\xi}_H - \boldsymbol{\xi}_h) - (\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h, \boldsymbol{\xi}_h)\\ &= -(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h, \boldsymbol{\xi}_H - \boldsymbol{\xi}_h) + (\mathbf{curl}(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h), \boldsymbol{w}_h)\\ &= -(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h, \boldsymbol{\xi}_H - \boldsymbol{\xi}_h) + (\mathbf{curl}(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h), \boldsymbol{w}_h - \boldsymbol{P}_H\boldsymbol{w}_h).\end{aligned}$$

By the Cauchy–Schwarz inequality, we obtain

$$\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0^2 \le \|\widehat{\boldsymbol{\sigma}}_H-\boldsymbol{\sigma}_h\|_0\|\boldsymbol{\xi}_H-\boldsymbol{\xi}_h\|_0+\|\operatorname{\mathbf{curl}}(\widehat{\boldsymbol{\sigma}}_H-\boldsymbol{\sigma}_h)\|\|\boldsymbol{w}_h-\boldsymbol{P}_H\boldsymbol{w}_h\|_0. \tag{4.10}$$

In order to bound $\|\boldsymbol{\xi}_H-\boldsymbol{\xi}_h\|_0$ in (4.10), we use the triangle inequality and the error estimates for the mixed source problem (4.6)

$$\|\boldsymbol{\xi}_H-\boldsymbol{\xi}_h\|_0 \le \|\boldsymbol{\xi}_H-\boldsymbol{\xi}\|_0+\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_0 \le C(h^s+H^s)\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0. \tag{4.11}$$

From the definition of the discrete spaces, since $\boldsymbol{Q}_h=\operatorname{\mathbf{curl}}(\boldsymbol{\Sigma}_h)$ for any choice of the mesh, it is clear that

$$\operatorname{\mathbf{curl}}(\widehat{\boldsymbol{\sigma}}_H)=-\lambda_h\boldsymbol{P}_H\boldsymbol{p}_h,\qquad \operatorname{\mathbf{curl}}(\boldsymbol{\sigma}_h)=-\lambda_h\boldsymbol{p}_h.$$

Therefore, from Lemma 4.2 we obtain

$$\begin{aligned}\|\operatorname{\mathbf{curl}}(\boldsymbol{\sigma}_h-\widehat{\boldsymbol{\sigma}}_H)\|_0 &\le \lambda_h\|\boldsymbol{p}_h-\boldsymbol{P}_H\boldsymbol{p}_h\|_0\\ &\le \lambda_h(\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0+\|\boldsymbol{p}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0)\\ &\le \lambda_h\big(\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0+C\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0\\ &\quad +C\rho_1(H)(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0)\big).\end{aligned} \tag{4.12}$$

Considering again the definition of the solution of the dual problem, the last norm appearing in (4.10) can be bounded by using (4.6) and the properties of the projection operator $\boldsymbol{P}_H$:

$$\begin{aligned}\|\boldsymbol{w}_h-\boldsymbol{P}_H\boldsymbol{w}_h\|_0 &\le \|\boldsymbol{w}_h-\boldsymbol{w}\|_0+\|\boldsymbol{w}-\boldsymbol{P}_H\boldsymbol{w}\|_0+\|\boldsymbol{P}_H(\boldsymbol{w}-\boldsymbol{w}_h)\|_0\\ &\le C\|\boldsymbol{w}-\boldsymbol{w}_h\|_0+\|\boldsymbol{w}-\boldsymbol{P}_H\boldsymbol{w}\|_0\\ &\le C(h^s+H^s)\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0.\end{aligned} \tag{4.13}$$

Collecting all the obtained estimates for the four norms in (4.10), we arrive at

$$\begin{aligned}\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0^2 \le{}& CH^s\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0\rho_1(H)\big(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0\big)\\ &+C(h^s+H^s)\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0^2,\end{aligned}$$

which implies that, for H sufficiently small, we have

$$\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0 \le CH^s(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0). \tag{4.14}$$

□

4.1. Proof of Property 1. The proof of Property 1 (*discrete reliability*) constitutes the main novelty with respect to the results present in the literature. The structure of the proof is a combination of the analogous proof in [11] and of some of the results in [13]. However, some new estimates are needed that will be detailed in this section. The presentation of the proof has been made clearer following the suggestions of an anonymous referee.

Let us start with the estimate of $\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0$. We split $\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H$ using a discrete Helmholtz decomposition as

$$\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H=\boldsymbol{\nabla}\alpha_h+\boldsymbol{\zeta}_h, \tag{4.15}$$

where $\alpha_h\in \mathrm{H}_0^1(\Omega)$ is a Lagrange finite element in N_h and $\boldsymbol{\zeta}_h$ is an edge element in $\boldsymbol{\Sigma}_h$ satisfying

$$(\boldsymbol{\nabla}\alpha_h,\boldsymbol{\nabla}\psi_h)=(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H,\boldsymbol{\nabla}\psi_h)\quad \forall\psi_h\in N_h, \tag{4.16}$$

and, for some $\boldsymbol{r}_h \in \boldsymbol{Q}_h$,

$$
\begin{aligned}
&(\boldsymbol{\zeta}_h, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \boldsymbol{r}_h) = 0 && \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h,\\
&(\mathbf{curl}\,\boldsymbol{\zeta}_h, \boldsymbol{q}) = (\mathbf{curl}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H), \boldsymbol{q}) && \forall \boldsymbol{q} \in \boldsymbol{Q}_h.
\end{aligned}
\tag{4.17}
$$

In particular, $(\boldsymbol{\zeta}_h, \boldsymbol{r}_h)$ approximates the solution of the mixed problem (4.4) with source term $\mathbf{g} = -\,\mathbf{curl}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H)$.

Clearly, we have

$$
\|\,\boldsymbol{\nabla}\,\alpha_h\|_0 \le C\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0, \qquad \|\boldsymbol{\zeta}_h\|_{\mathbf{curl}} + \|\boldsymbol{r}_h\|_0 \le C\|\,\mathbf{curl}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H)\|_0.
$$

Let us estimate the first term of (4.15). By standard procedure, defining α_H as the Scott–Zhang interpolant of α_h on $\mathcal{T}_H$ (see [39]), we have

$$
\|\,\boldsymbol{\nabla}\,\alpha_h\|_0^2 = (\boldsymbol{\nabla}\,\alpha_h, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H) = -(\boldsymbol{\nabla}\,\alpha_h, \boldsymbol{\sigma}_H) = -(\boldsymbol{\nabla}(\alpha_h - \alpha_H), \boldsymbol{\sigma}_H),
$$

since $(\boldsymbol{\nabla}\,\alpha_h, \boldsymbol{\sigma}_h) = (\boldsymbol{\nabla}\,\alpha_H, \boldsymbol{\sigma}_H) = 0$ from the first equation of (2.6). Integrating by parts element by element, we get

$$
\begin{aligned}
\|\,\boldsymbol{\nabla}\,\alpha_h\|_0^2 &= \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big((\alpha_h - \alpha_H, \operatorname{div}\boldsymbol{\sigma}_H) - \frac{1}{2}\sum_{F\in K}\int_F (\alpha_h - \alpha_H)[\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!] \Big)\\
&\le C \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(\|\operatorname{div}\boldsymbol{\sigma}_H\|_{0,K} H_K \|\,\boldsymbol{\nabla}\,\alpha_h\|_{0,K}\\
&\qquad\qquad + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} \|[\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!]\|_{0,F} H_F^{1/2}\|\alpha_h\|_{1,K}\Big)\\
&\le C\|\,\boldsymbol{\nabla}\,\alpha_h\|_0 \Bigg(\Big(\sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} H_K^2 \|\operatorname{div}\boldsymbol{\sigma}_H\|_{0,K}^2 \Big)^{1/2}\\
&\qquad\qquad + \Big(\sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h}\ \sum_{F\in\mathcal{F}_I(K)} H_F \|[\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!]\|_{0,F}^2 \Big)^{1/2} \Bigg)\\
&\le C\|\,\boldsymbol{\nabla}\,\alpha_h\|_0\,\eta_H(\mathcal{T}_H\setminus\mathcal{T}_h).
\end{aligned}
\tag{4.18}
$$

In order to estimate the second term in (4.15), we proceed as follows:

$$
\begin{aligned}
\|\boldsymbol{\zeta}_h\|_0^2 &= (\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) = -(\mathbf{curl}\,\boldsymbol{\zeta}_h, \boldsymbol{r}_h) = -(\mathbf{curl}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H), \boldsymbol{r}_h)\\
&= (\lambda_h \boldsymbol{p}_h - \lambda_H \boldsymbol{p}_H, \boldsymbol{r}_h)\\
&= (\lambda_h - \lambda_H)(\boldsymbol{p}_h, \boldsymbol{r}_h) + \lambda_H(\boldsymbol{p}_h - \boldsymbol{p}_H, \boldsymbol{r}_h).
\end{aligned}
\tag{4.19}
$$

We bound the two terms in the last line separately.

From the classical inf-sup condition involving edge and face elements (see, for instance, [10]), we have

$$
\begin{aligned}
(\boldsymbol{p}_h, \boldsymbol{r}_h) &\le \|\boldsymbol{p}_h\|_0\|\boldsymbol{r}_h\|_0 = \|\boldsymbol{r}_h\|_0\\
&\le C \sup_{\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h} \frac{(\mathbf{curl}\,\boldsymbol{\tau}_h, \boldsymbol{r}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{curl}}} = C \sup_{\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h} \frac{(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{curl}}} \le C\|\boldsymbol{\zeta}_h\|_0.
\end{aligned}
$$

Hence, using (4.3), we conclude the estimate of the first term in (4.19) as follows:

$$
(\lambda_h - \lambda_H)(\boldsymbol{p}_h, \boldsymbol{r}_h) \le C\rho_0(H)\big(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0\big)\|\boldsymbol{\zeta}_h\|_0.
\tag{4.20}
$$

The second term of (4.19) is easily bounded by considering the previously obtained estimate $\|\boldsymbol{r}_h\|_0 \le C\|\boldsymbol{\zeta}_h\|_0$. Then we have

$$\begin{aligned}\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0 \le C\big(&\eta_H(\mathcal{T}_H\setminus\mathcal{T}_h)+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0\\&+\rho_0(H)(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0)\big).\end{aligned}\tag{4.21}$$

We now move to the term $\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0$. We consider the following auxiliary problem: find $\boldsymbol{\chi}_h\in\boldsymbol{\Sigma}_h$ and $\boldsymbol{z}_h\in\boldsymbol{Q}_h$ such that

$$\begin{aligned}&(\boldsymbol{\chi}_h,\boldsymbol{\tau})+(\mathbf{curl}\,\boldsymbol{\tau},\boldsymbol{z}_h)=0 &&\forall\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h,\\&(\mathbf{curl}\,\boldsymbol{\chi}_h,\boldsymbol{q})=(\boldsymbol{p}_h-\boldsymbol{p}_H,\boldsymbol{q}) &&\forall\boldsymbol{q}\in\boldsymbol{Q}_h.\end{aligned}\tag{4.22}$$

Therefore, we have $\mathbf{curl}\,\boldsymbol{\chi}_h=\boldsymbol{p}_h-\boldsymbol{p}_H$ and $\|\boldsymbol{\chi}_h\|_{\mathbf{curl}}\le C\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0$.

We are going to use a technical tool introduced in [44, Theorem 4.1]. More precisely, if $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$, there exists an operator $\boldsymbol{\mathcal{P}}_H:\boldsymbol{\Sigma}_h\to\boldsymbol{\Sigma}_H$ such that for all $\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h$ it holds that $\boldsymbol{\mathcal{P}}_H\boldsymbol{\tau}=\boldsymbol{\tau}$ on the elements of $\mathcal{T}_H$ that have not been refined (more precisely, on the elements of $\mathcal{T}_H$ whose closures have no intersection with the closures of any refined elements). Such an operator is stable in the $\mathrm{H}(\mathbf{curl})$-norm, i.e., $\|\boldsymbol{\mathcal{P}}_H\boldsymbol{\tau}\|_{\mathbf{curl}}\le C\|\boldsymbol{\tau}\|_{\mathbf{curl}}$ for all $\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h$.

We get

$$\begin{aligned}\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0^2&=(\boldsymbol{p}_h-\boldsymbol{p}_H,\mathbf{curl}\,\boldsymbol{\chi}_h)=-(\boldsymbol{\sigma}_h,\boldsymbol{\chi}_h)-(\boldsymbol{p}_H,\mathbf{curl}\,\boldsymbol{\chi}_h)\\&=-(\boldsymbol{\sigma}_h,\boldsymbol{\chi}_h)-(\boldsymbol{p}_H,\mathbf{curl}(\boldsymbol{\chi}_h-\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h))-(\boldsymbol{p}_H,\mathbf{curl}\,\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h)\\&=-(\boldsymbol{\sigma}_h,\boldsymbol{\chi}_h)-(\boldsymbol{p}_H,\mathbf{curl}(\boldsymbol{\chi}_h-\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h))+(\boldsymbol{\sigma}_H,\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h)\\&=-(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H,\boldsymbol{\chi}_h)-(\boldsymbol{p}_H,\mathbf{curl}(\boldsymbol{\chi}_h-\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h))-(\boldsymbol{\sigma}_H,\boldsymbol{\chi}_h-\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h).\end{aligned}\tag{4.23}$$

Let us set $\boldsymbol{\vartheta}_h=\boldsymbol{\chi}_h-\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h$ and denote by $\boldsymbol{\mathcal{S}}_H$ the operator introduced in [38, Theorem 1] mapping $\mathbf{H}_0(\mathbf{curl};\Omega)$ into the space of lowest order Nédélec elements so that there exist $\varphi\in\mathrm{H}_0^1(\Omega)$ and $\boldsymbol{s}\in\mathbf{H}_0^1(\Omega)$ satisfying

$$\begin{aligned}&\boldsymbol{\vartheta}_h-\boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h=\boldsymbol{\nabla}\,\varphi+\boldsymbol{s},\\&h_K^{-1}\|\varphi\|_{0,K}+\|\boldsymbol{\nabla}\,\varphi\|_{0,K}\le C\|\boldsymbol{\vartheta}_h\|_{0,K'},\\&h_K^{-1}\|\boldsymbol{s}\|_{0,K}+\|\boldsymbol{\nabla}\,\boldsymbol{s}\|_{0,K}\le C\|\,\mathbf{curl}\,\boldsymbol{\vartheta}_h\|_{0,K'}\end{aligned}$$

for all $K\in\mathcal{T}_h$ and with K' denoting the union of elements in $\mathcal{T}_h$ sharing at least a vertex with K.

From the first equation in (2.6) it follows that $(\boldsymbol{\sigma}_H,\boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h)+(\mathbf{curl}\,\boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h,\boldsymbol{p}_H)=0$. This implies that (4.23) gives

$$\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0^2=-(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H,\boldsymbol{\chi}_h)-(\boldsymbol{p}_H,\mathbf{curl}(\boldsymbol{\vartheta}_h-\boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h))-(\boldsymbol{\sigma}_H,\boldsymbol{\vartheta}_h-\boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h).\tag{4.24}$$

The first term can be estimated as follows using (4.22) and the definition of $\widehat{\boldsymbol{\sigma}}_H$:

$$\begin{aligned}-(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H,\boldsymbol{\chi}_h)&=-(\boldsymbol{\sigma}_h-\widehat{\boldsymbol{\sigma}}_H,\boldsymbol{\chi}_h)-(\widehat{\boldsymbol{\sigma}}_H-\boldsymbol{\sigma}_H,\boldsymbol{\chi}_h)\\&=(\mathbf{curl}(\boldsymbol{\sigma}_h-\widehat{\boldsymbol{\sigma}}_H),\boldsymbol{z}_h)-(\widehat{\boldsymbol{\sigma}}_H-\boldsymbol{\sigma}_H,\boldsymbol{\chi}_h)\\&=-(\widehat{\boldsymbol{\sigma}}_H-\boldsymbol{\sigma}_H,\boldsymbol{\chi}_h)\\&\le\|\widehat{\boldsymbol{\sigma}}_H-\boldsymbol{\sigma}_H\|_0\|\boldsymbol{\chi}_h\|_0\\&\le\|\widehat{\boldsymbol{\sigma}}_H-\boldsymbol{\sigma}_H\|_0\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0.\end{aligned}$$

It follows from Lemmas 4.2 and 4.3 that there exists $\rho_2(H)$, tending to zero as H goes to zero, such that

$$\|\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H\|_0 \le \rho_2(H)(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0).$$

The remaining two terms in (4.24) can be bounded together.

$$\begin{aligned}
&(\boldsymbol{p}_H, \mathbf{curl}(\boldsymbol{\vartheta}_h - \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h)) + (\boldsymbol{\sigma}_H, \boldsymbol{\vartheta}_h - \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h)\\
&= \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(\int_K \mathbf{curl}\,\boldsymbol{p}_H \cdot \boldsymbol{s} + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} \int_F [\![\boldsymbol{p}_{\mathrm{H}} \times \boldsymbol{n}]\!]\cdot \boldsymbol{s}\Big) + (\boldsymbol{\sigma}_H, \boldsymbol{s}) + (\boldsymbol{\sigma}_H, \boldsymbol{\nabla}\varphi)\\
&= \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(\int_K (\boldsymbol{\sigma}_H + \mathbf{curl}\,\boldsymbol{p}_H) \cdot \boldsymbol{s} + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} \int_F [\![\boldsymbol{p}_{\mathrm{H}} \times \boldsymbol{n}]\!]\cdot \boldsymbol{s}\Big)\\
&\quad + \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(-\int_K \operatorname{div}\boldsymbol{\sigma}_H\varphi + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} \int_F [\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!]\varphi\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\big|(\boldsymbol{p}_H, \mathbf{curl}(\boldsymbol{\vartheta}_h - \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h)) + (\boldsymbol{\sigma}_H, \boldsymbol{\vartheta}_h - \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h)\big|\\
&\quad\le \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(\|\boldsymbol{\sigma}_H + \mathbf{curl}\,\boldsymbol{p}_H\|_{0,K}\|\boldsymbol{s}\|_{0,K} + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} \|[\![\boldsymbol{p}_{\mathrm{H}}\times\boldsymbol{n}]\!]\|_{0,F}\|\boldsymbol{s}\|_{0,F}\Big)\\
&\qquad + \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(\|\operatorname{div}\boldsymbol{\sigma}_H\|_{0,K}\|\varphi\|_{0,K} + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} \|[\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!]\|_{0,F}\|\varphi\|_{0,F}\Big)\\
&\quad\le C\sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(H_K\|\boldsymbol{\sigma}_H + \mathbf{curl}\,\boldsymbol{p}_H\|_{0,K}\|\mathbf{curl}\,\boldsymbol{\vartheta}_h\|_{0,K'}\\
&\qquad + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} H_F^{1/2}\|[\![\boldsymbol{p}_{\mathrm{H}}\times\boldsymbol{n}]\!]\|_{0,F}\|\mathbf{curl}\,\boldsymbol{\vartheta}_h\|_{0,K'}\Big)\\
&\qquad + C\sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(H_K\|\operatorname{div}\boldsymbol{\sigma}_H\|_{0,K}\|\boldsymbol{\vartheta}_h\|_{0,K'}\\
&\qquad + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} H_F^{1/2}\|[\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!]\|_{0,F}\|\boldsymbol{\vartheta}_h\|_{0,K'}\Big)\\
&\quad\le C\eta_H(\mathcal{T}_H\setminus\mathcal{T}_h)\|\boldsymbol{\vartheta}_h\|_{\mathbf{curl}} \le C\eta_H(\mathcal{T}_H\setminus\mathcal{T}_h)\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0.
\end{aligned}$$

Finally, (4.24) becomes

$$\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0 \le C\eta_H(\mathcal{T}_H\setminus\mathcal{T}_h) + \rho_2(H)(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0). \tag{4.25}$$

Putting things together, estimates (4.21) and (4.25) give the final result.

4.2. Proof of Property 2. The proof of Property 2 (*quasi-orthogonality*) can be obtained after appropriate modification of the analogous result in [11].

By direct computation we have

$$\begin{aligned}
\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 &= \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_H\|_0^2 - \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 - 2(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H),\\
\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 &= \|\boldsymbol{p} - \boldsymbol{p}_H\|_0^2 - \|\boldsymbol{p} - \boldsymbol{p}_h\|_0^2 - 2(\boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{p}_h - \boldsymbol{p}_H).
\end{aligned}$$

Since $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$, we have that $\boldsymbol{\sigma}_H \in \boldsymbol{\Sigma}_h$; hence the error equations relative to (2.3) and (2.6) give

$$\begin{aligned}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H) &= -(\mathbf{curl}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H), \boldsymbol{p} - \boldsymbol{p}_h)\\ &= (\lambda_h \boldsymbol{p}_h - \lambda_H \boldsymbol{p}_H, \boldsymbol{p} - \boldsymbol{p}_h)\\ &= (\lambda_h \boldsymbol{p}_h - \lambda_H \boldsymbol{p}_H, \boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h).\end{aligned}$$

Using Lemma 4.1 and the equalities in (4.2), we obtain

$$\begin{aligned}&(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H) + (\boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{p}_h - \boldsymbol{p}_H)\\ &\quad = (\lambda_h \boldsymbol{p}_h - \lambda_H \boldsymbol{p}_H, \boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h) + (\boldsymbol{p}_h - \boldsymbol{p}_H, \boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h)\\ &\quad \le \big(|\lambda_h - \lambda_H| + (1+\lambda_H)\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0\big)\|\boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h\|_0\\ &\quad \le \big(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 + \lambda_H\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 + (1+\lambda_H)\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0\big)\\ &\qquad\quad \cdot \rho_{\mathrm{sc}}(h)\big(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0\big),\end{aligned}$$

which, using Young's inequality, gives the desired result with

$$\rho_{\mathrm{drel}}(H) = C(\rho_2(H) + \rho_0(H)).$$

4.3. Proof of Property 3. The contraction property is quite standard in the framework of adaptive schemes; see [17]. It is a consequence of the following error estimator reduction property: there exist constants $\beta_1 \in (0, +\infty)$ and $\gamma_1 \in (0,1)$ such that, if $\mathcal{T}_{\ell+1}$ is the refinement of $\mathcal{T}_\ell$ generated by the adaptive scheme, it holds that

$$\eta(\mathcal{T}_{\ell+1})^2 \le \gamma_1 \eta(\mathcal{T}_\ell)^2 + \beta_1\big(\|\boldsymbol{\sigma}_\ell - \boldsymbol{\sigma}_{\ell+1}\|_0^2 + \|\boldsymbol{p}_\ell - \boldsymbol{p}_{\ell+1}\|_0^2\big).$$

In our case, the proof can be obtained with natural modifications of the one outlined in [11] and using the following notation:

$$e_\ell^2 = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_\ell\|_0^2 + \|\boldsymbol{p} - \boldsymbol{p}_\ell\|_0^2, \qquad \mu_\ell^2 = \eta(\mathcal{T}_\ell)^2.$$

5. Conclusions.

In this paper we have proved the optimal convergence of an adaptive finite element scheme for the approximation of the eigensolutions of the Maxwell system. The scheme makes use of the Nédélec edge finite element in three space dimensions and a standard residual-based error estimator. The proof is based on an equivalent mixed formulation. The most challenging part of the proof consists in showing a suitable discrete reliability property.

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