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Generalized Lebesgue Points for Hajłasz Functions

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Let $X$ be a quasi-Banach function space over a doubling metric measure space $P$. Denote by $\alpha_X$ the generalized upper Boyd index of $X$. We show that if $\alpha_X < \infty$ and $X$ has absolutely continuous quasinorm, then quasieverypoint is a generalized Lebesgue point of a quasicontinuous Hajłasz function $u \in \dot{M}^s, X$. Moreover, if $\alpha_X < (Q+s)/Q$, then quasieverypoint is a Lebesgue point of $u$. As an application we obtain Lebesguetype theorems for Lorentz–Hajłasz, Orlicz–Hajłasz, and variable exponent Hajłasz functions.

1. Introduction and Main Results

Let $P = (\mathcal{P}, d, \mu)$ be a doubling metric measure space. By the Lebesgue differentiation theorem, almost everypoint of a locally integrable function is a Lebesgue point. As expected, for smoother functions, the set of non-Lebesgue points is smaller. In [1], Kinnunen and Latvala showed that, for a quasicontinuous Hajłasz–Sobolev function $u \in \dot{M}^{1,p}(P)$, $p > 1$, there exists a set $E$ of $\dot{M}^{1,p}$-capacity zero such that

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(y) \, d\mu(y) = u(x)$$

(1)

for every $x \in \mathcal{P} \setminus E$. The case $p = 1$ was studied in [2, 3]. Recently, in [4] and independently in [5], it was shown that if

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq c \left( \frac{r}{R} \right)^Q$$

(2)

for every $x \in \mathcal{P}, 0 < r \leq R$ and $y \in B(x, R)$, then for every quasicontinuous $u \in \dot{M}^{1,p}$ with $p > Q/(Q+s)$, (1) holds true outside a set of $\dot{M}^{1,p}$-capacity zero.

If we replace integral averages in (1) by medians, then the result holds true also for small $p > 0$. For $0 < \gamma < 1$, $A \subset \mathcal{P}$, and $u \in L^0$, denote

$$m^u_{\gamma}(A) = \inf \{ a \in \mathbb{R} : \mu(\{ x \in A : u(x) > a \}) < \gamma \mu(A) \}.$$ 

(3)

If $p > 0$ and $u \in \dot{M}^{s,p}$ is quasicontinuous, then by [4, Theorem 1.2], there exists a set $E \subset \mathcal{P}$ of $\dot{M}^{s,p}$-capacity zero such that

$$\lim_{r \to 0} m^u_{\gamma}(B(x,r)) = u(x),$$

(4)

for every $x \in \mathcal{P} \setminus E$ and $0 < \gamma \leq 1/2$.

In this paper, we will study the existence of (generalized) Lebesgue points for functions $u$ whose Hajłasz gradient belongs to a general quasi-Banach function space $X$. This approach allows us to simultaneously cover, for example, Orlicz–Hajłasz, Lorentz–Hajłasz, and variable exponent Hajłasz functions.

For $0 < \gamma < 1, 0 < R \leq \infty, u \in L^0(\mathcal{P})$, and $x \in \mathcal{P}$, denote

$$\mathcal{M}^\gamma u(x) = \sup_{0 < r < R} m^u_{\gamma}(B(x,r)).$$

(5)

Operator $\mathcal{M}^\gamma = \mathcal{M}^\gamma_{\infty}$ and its variants have turned out to be useful in harmonic analysis and in the theory of function spaces; see, for example, [4, 6–24].

Theorem 1. Let $\mu$ be doubling. Suppose that $X$ has absolutely continuous quasinorm and that, for every $0 < \gamma < 1$ and for every ball $B \subset \mathcal{P}$, there exists a constant $C$ such that

$$\| (\mathcal{M}^\gamma g) \chi_B \|_X \leq C \| g \|_X$$

(6)
for every \( g \in X \). Let \( 0 < s \leq 1 \). Then, for every quasicontinuous \( u \in \mathcal{M}^{1-s}(\mathcal{P}) \), there exists a set \( E \subset \mathcal{P} \) with \( C_{M^{1-s}}(E) = 0 \) such that

\[
\lim_{r \to 0} m_{\mu}(\mathcal{B}(x, r)) = 0 \quad \text{for every } x \in \mathcal{P} \setminus E \quad \text{and } 0 < \gamma < 1.
\]

We say that a point \( x \in \mathcal{P} \) satisfying (7) for every \( 0 < \gamma < 1 \) is a generalized Lebesgue point of \( u \).

The (restricted) Hardy–Littlewood maximal function of a locally integrable function \( u \) is

\[
\mathcal{M}_\mu u(x) = \sup_{0<r \leq R} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u(y)| \, d\mu(y).
\]

As usual, we denote \( \mathcal{M} = \mathcal{M}_\mu \).

**Theorem 2.** Let \( 0 < s \leq 1 \) and let \( \mu \) satisfy (2). Denote \( q = Q/(Q+s) \). Suppose that \( X \) has absolutely continuous quasi-norm and that, for every ball \( B \subset \mathcal{P} \), there exists a constant \( C \) such that

\[
\left\| \left( \mathcal{M}_\mu g \right)^{1/q} \right\|_X \leq C \|g\|_X
\]

for every \( 0 \leq g \in X \). Then, for every quasicontinuous \( u \in \mathcal{M}^{1-s}(\mathcal{P}) \), quasievery point is a Lebesgue point of \( u \).

For any quasi-Banach function space \( X \) over \( \mathcal{P} \), define

\[
\Phi_X(y) = \sup_{1/\|f\| \leq 1} \|\mathcal{M}_\mu f\|_X.
\]

The generalized upper Boyd index of \( X \) is

\[
\alpha_X = \lim_{\gamma \to 0} \frac{\log \Phi_X(y)}{\log(1/y)}.
\]

The generalized upper Boyd index was introduced in [20], where it was shown that the Hardy–Littlewood maximal operator is bounded on \( X(\mathbb{R}^n) \) if and only if \( \alpha_X < 1 \). As a corollary of Theorems 1 and 2 we have the following result.

**Theorem 3.** Suppose that \( X \) has absolutely continuous quasi-norm and that \( \mu \) satisfies (2). Let \( 0 < s \leq 1 \) and let \( u \in \mathcal{M}^{1-s}(\mathcal{P}) \) be quasicontinuous.

(i) If \( \alpha_X < \infty \), then quasievery point is a generalized Lebesgue point of \( u \).

(ii) If \( \alpha_X < (Q+s)/Q \), then quasievery point is a Lebesgue point of \( u \).

**2. Preliminaries**

**2.1. Basic Assumptions.** In this paper, \( \mathcal{P} = (\mathcal{P}, d, \mu) \) is a metric measure space equipped with a metric \( d \) and a Borel regular, doubling outer measure \( \mu \), for which the measure of every ball is positive and finite. The doubling property means that there exists a constant \( c_\mu > 0 \) such that

\[
\mu(B(x, 2r)) \leq c_\mu \mu(B(x, r))
\]

for every ball \( B(x, r) = \{ y \in \mathcal{P} : d(x, y) < r \} \), where \( x \in \mathcal{P} \) and \( r > 0 \).

The doubling condition is equivalent to the existence of constants \( c \) and \( Q \) such that (2) holds for every \( 0 < r \leq R \), \( x \in \mathcal{P} \), and \( y \in B(x, R) \).

**The integral average of a locally integrable function \( u \) over a set \( A \) of positive and finite measure is**

\[
u_A = \int_A u \, d\mu = \frac{1}{\mu(A)} \int_A u \, d\mu.
\]

By \( \chi_E \), we denote the characteristic function of a set \( E \subset \mathcal{P} \) and by \( \mathbb{R} \), the extended real numbers \([-\infty, \infty]\).

A quasi-Banach function space \( X \) over \( \mathcal{P} \) is the set of all measurable, almost everywhere finite functions \( u : \mathcal{P} \to \mathbb{R} \). In general, \( C \) and \( c \) are positive constants whose values are not necessarily same at each occurrence. When we want to stress that the constant depends on other constants or parameters \( a, b, \ldots \), we write \( C = C(a, b, \ldots) \).

**2.2. Quasi-Banach Function Spaces.** A quasi-norm on a subspace of \( L^0(\mathcal{P}) \) is a functional \( \| \cdot \| \) such that

(i) \( \|f\| = 0 \iff f = 0 \) a.e.;

(ii) \( \|af\| = \|a\| \|f\| \) for every \( a \in \mathbb{R} \);

(iii) there exists a constant \( c_a \) such that \( \|f + g\| \leq c_a (\|f\| + \|g\|) \).

A quasi-Banach function space \( X \) over \( \mathcal{P} \) is a subspace of \( L^0(\mathcal{P}) \) equipped with a complete quasi-norm \( \| \cdot \|_X \) that has the following properties:

(i) \( g \in X \) and \( |f| \leq |g| \) a.e. \( \implies f \in X \) and \( \|f\|_X \leq \|g\|_X \);

(ii) \( \mu(E) < \infty \implies \|\chi_E\|_X < \infty \);

(iii) \( 0 \leq f_k \uparrow f \) a.e. \( \implies \|f_k\|_X \uparrow \|f\|_X \).

By the Aoki–Rolewicz theorem ([25, 26]), there exists a constant \( 0 < \rho \leq 1 \) such that

\[
\|\sum_{k=1}^{\infty} f_k\|_X \leq 4^{1/\rho} \left( \sum_{k=1}^{\infty} \|f_k\|_X^{\rho} \right)^{1/\rho}
\]

for all \( f_1, f_2, \ldots \in X \).

A quasi-norm \( \| \cdot \|_X \) on \( X \) is absolutely continuous if \( \|f\|_X \to 0 \) whenever \( f \in X \) and \( (E_k)_{k=1}^{\infty} \) is a decreasing sequence of sets such that \( \bigcap_{k=1}^{\infty} E_k = \emptyset \).

**2.3. Hajłasz Spaces.** Let \( 0 < s < \infty \). A measurable function \( g : \mathcal{P} \to [0, \infty] \) is an \( s \)-gradient of a function \( u \in L^0(\mathcal{P}) \) if there exists a set \( E \subset \mathcal{P} \) with \( \mu(E) = 0 \) such that, for all \( x, y \in \mathcal{P} \setminus E \),

\[
|u(x) - u(y)| \leq d(x, y)^s (g(x) + g(y)).
\]

The collection of all \( s \)-gradients of \( u \) is denoted by \( \mathcal{D}(u) \).
The homogeneous Hajłasz space \( M^{s,X} = M^{s,X}(\mathcal{P}) \) consists of measurable functions \( u \) for which
\[
\|u\|_{M^{s,X}} = \inf_{g \in \mathcal{D}'(u)} \|g\|_X
\]
is finite. The Hajłasz space \( M^{s,X} = M^{s,X}(\mathcal{P}) \) is \( M^{s,X} \cap X \) equipped with the norm
\[
\|u\|_{M^{s,X}} = \|u\|_X + \|u\|_{M^{s,X}}.
\]

Hajłasz spaces \( M^{s,P}(\mathcal{P}) = M^{s,L^P}(\mathcal{P}) \) were introduced in [27] for \( s = 1, p \geq 1 \) and in [28] for fractional scales. Recall that, for \( p > 1, M^{1,P}(\mathbb{R}^n) = W^{1,P}(\mathbb{R}^n) \) (see [27]), whereas for \( n/(n+1) < p \leq 1, M^{1,P}(\mathbb{R}^n) \) coincides with the Hardy–Sobolev space \( H^{1,P}(\mathbb{R}^n) \) by [29, Thm 1].

Next two lemmas for \( s \)-gradients follow easily from the definition; see [30, Lemmas 2.4] and [1, Lemma 2.6].

**Lemma 4.** Let \( u, v \in L^1(\mathcal{P}), g \in \mathcal{D}'(u), \) and \( h \in \mathcal{D}'(v) \). Then \( \max\{g, h\} \) is an \( s \)-gradient of \( \max\{u, v\} \) and \( \min\{u, v\} \).

The following lemma is essential [4, Lemma 7.2].

**Lemma 6.** Let \( 0 < s \leq 1 \) and let \( S \subset \mathcal{P} \) be a measurable set. Let \( u : \mathcal{P} \to \mathbb{R} \) be a measurable function with \( g \in \mathcal{D}'(u) \) and let \( \varphi \) be a bounded \( L \)-Lipschitz function supported in \( S \). Then
\[
h = (\|\varphi\|_\infty + 2 \|\varphi\|_\infty^{1-s}) L^s|u| \chi_S \in \mathcal{D}'(\varphi).
\]
Consequently, there exists a constant \( C = C(s, \|\varphi\|_\infty, L) \) such that
\[
\|u\varphi\|_{M^{s,X}} \leq C \left( \|u\chi_S\|_X + \inf_{g \in \mathcal{D}'(u)} \|g\chi_S\|_X \right)
\]
for every \( u \in M^{s,X} \).

**Lemma 7.** Let \( 0 < s \leq 1 \) and suppose that \( X \) has absolutely continuous quasinorm. Then \( s \)-Hölder continuous functions are dense in \( M^{s,X} \).

Proof. Let \( u \in M^{s,X}, g \in \mathcal{D}'(u) \cap X \) and let \( E \) be the exceptional set for (15). Then \( u \) is \( s \)-Hölder continuous with constant \( \lambda \) in the set \( E_{\lambda} = \{ x \in \mathcal{P}\setminus E : g(x) \leq \lambda \} \). By [31], there is an extension \( u_\lambda \) of \( u_{E_{\lambda}} \) to \( \mathcal{P} \) such that \( u_\lambda \) is \( s \)-Hölder continuous with constant \( \lambda \). It is easy to see that
\[
(g + \lambda) \chi_{E_{\lambda}} \in \mathcal{D}'(u - u_\lambda); \quad \text{see [32, Proposition 4.5].}
\]
By the absolute continuity of \( \|g\chi_S\|_X \), \( \|g + \lambda\chi_{E_{\lambda}}\|_X \to 0 \) and
\[
\|u - u_\lambda\|_X \leq 2\|ux\chi_{E_{\lambda}}\|_X \to 0 \text{ as } \lambda \to 0.
\]

2.5. **Discrete Maximal Functions.** In this subsection, we define "discrete" versions of the Hardy–Littlewood maximal function
\[
M^u(x) = \sup_{0 < r < R} |u|_{B(x,r)}.
\]
and the median maximal function
\[
M^{\mu}_m(x) = \sup_{0 < r < R} |m^\mu_r(B(x,r))|.
\]
We first recall the definition of a discrete convolution. Discrete convolutions are basic tools in harmonic analysis in homogeneous spaces; see, for example, [33, 34]. The following lemma is well known.

**Lemma 9.** For every \( r > 0 \), there exists a collection of balls \( \{ B_i = B(x_i, r) : i \in I \} \), where \( I \subset \mathbb{N} \) and functions \( \varphi_i : \mathcal{P} \to [0,1], i \in I, \) with the following properties:
(a) The balls \( B(x_i, r/2), i \in I, \) cover \( \mathcal{P} \).
(b) \( \sum_i \chi_{B_i} \leq C. \)
(c) For every \( i \in I, \varphi_i \) is \( C/r \)-Lipschitz, \( \varphi_i \geq C^{-1} \) on \( B_i \) and \( \varphi_i = 0 \) outside \( 2B_i \).
(d) \( \sum_i \varphi_i = 1. \)

Here, the constant \( C \) depends only on the doubling constant \( c_d. \)
For each scale \( r > 0 \), we choose a collection of balls \( \{B_i : i \in I\} \) and a collection of functions \( \{\varphi_i : i \in I\} \) satisfying conditions (a)–(d) of Lemma 9.

**Definition 10.** The discrete convolution of a locally integrable function \( u \) at the scale \( r \) is

\[
u_r = \sum_{i \in I} u_{r,i} \varphi_i,
\]

where \( \{B_i : i \in I\} \) and \( \{\varphi_i : i \in I\} \) are the chosen collections of balls and functions for the scale \( r \).

The discrete maximal function of \( u \) is \( M^* u : \mathcal{P} \to \mathbb{R} \),

\[
M^*_R u(x) = \sup_{q \in \mathbb{Q}, 0 < q < R} |u|_q(x).
\]

The discrete maximal function, which can be seen as a smooth version of the Hardy–Littlewood maximal function, was introduced in [1].

As a supremum of continuous functions, the discrete maximal functions are lower semicontinuous and hence measurable.

**Lemma 12.** Let \( 0 < R \leq \infty \). There exists a constant \( C = C(c_d) \geq 1 \) such that

\[
C^{-1} M_{3,R}^* u \leq M^*_R u \leq C M_{3,R}^* u
\]

for every \( u \in L^1_{\text{loc}} \) and

\[
M^{\gamma}_{R/2} u \leq C M^{\gamma/C}_{R} u,
\]

\[
M^{\gamma}_{R} u \leq C M^{\gamma/C}_{3,R} u
\]

for every \( u \in L^1 \) and \( 0 < \gamma < 1 \).

**Proof.** We will prove (30). The proof of (29) is similar. Let \( x \in \mathcal{P} \), \( r > 0 \), and let \( u^r = \sum_{i \in I} m^r_i(B_i) \varphi_i \) be as in Definition 11. If \( x \in 2B_i \), then \( B_i \subset B(x, 3r) \). By the doubling property, \( \mu(B(x, 3r)) \leq C \mu(B(x, r)) \), and so, by Lemma 8(c), \( m^r_i(B_i) \leq m^{\gamma/C}_u(B(x, 3r)) \). Since \( \sum_{i \in I} X_{B_i}(x) \leq C \), it follows that

\[
u^r_i(x) \leq C m^{\gamma/C}(B(x, 3r)).
\]

On the other hand, since the balls \((1/2)B_i, i \in I\) cover \( \mathcal{P} \), there is \( i \in I \) such that \( B(x, r/2) \subset B_i \). By the doubling property, \( \mu(B_i) \leq C \mu(B(x, r/2)) \), and so, by Lemma 8(c), \( m^r_i(B(x, r/2)) \leq m^{\gamma/C}_u(B(x, r/2)) \). Since \( \varphi_i \geq C^{-1} \) on \( B_i \), we have that

\[
m^r_u(B(x, r/2)) \leq C m^{\gamma/C}(x).
\]

The claim (30) follows immediately from these estimates. \( \square \)

### 3. Capacity

In this section, we define the \( M^{s, \mathcal{X}} \)-capacity and prove some of its basic properties.

**Definition 13.** Let \( 0 < s \leq \infty \). The \( M^{s, \mathcal{X}} \)-capacity of a set \( E \subset \mathcal{P} \) is

\[
C_{M^{s, \mathcal{X}}}(E) = \inf \{ \|u\|_{M^{s, \mathcal{X}}(E)} : u \in \mathcal{A}_{M^{s, \mathcal{X}}}(E) \},
\]

where

\[
\mathcal{A}_{M^{s, \mathcal{X}}}(E) = \{ u \in M^{s, \mathcal{X}} : u \geq 1 \text{ on a neighbourhood of } E \}
\]

is a set of admissible functions for the capacity. We say that a property holds quasieverywhere if it holds outside a set of \( M^{s, \mathcal{X}} \)-capacity zero.

**Remark 14.** Lemma 4 easily implies that

\[
C_{M^{s, \mathcal{X}}}(E) = \inf \{ \|u\|_{M^{s, \mathcal{X}}(E)} : u \in \mathcal{A}_{M^{s, \mathcal{X}}}(E) \}
\]

where \( \mathcal{A}_{M^{s, \mathcal{X}}}(E) = \{ u \in M^{s, \mathcal{X}} : 0 \leq u \leq 1 \} \).

**Remark 15.** It is easy to see that the \( M^{s, \mathcal{X}} \)-capacity is an outer capacity, which means that

\[
C_{M^{s, \mathcal{X}}}(E) = \inf \{ C_{M^{s, \mathcal{X}}}(U) : U \supset E, U \text{ is open} \}.
\]

The \( M^{s, \mathcal{X}} \)-capacity is not generally subadditive, but for most purposes, it suffices that it satisfies inequality (37) below.

**Lemma 16.** Let \( 0 < s < \infty \) and let \( p \) be the constant from (14). Then

\[
C_{M^{s, \mathcal{X}}}(\bigcup_{i \in I} E_i)^p \leq 8 \sum_{i \in I} C_{M^{s, \mathcal{X}}}(E_i)^p
\]

whenever \( E_i \subset \mathcal{P}, i \in I \subset \mathbb{N} \).

**Proof.** Let \( \varepsilon > 0 \). We may assume that \( \sum_{i \in I} C_{M^{s, \mathcal{X}}}(E_i) < \infty \). There are functions \( u_i \in \mathcal{A}'_{M^{s, \mathcal{X}}}(E_i) \) with \( g_i \in \mathcal{D}'(u_i) \) such that

\[
\left( \|u_i\|_{\mathcal{X}} + \|g_i\|_{\mathcal{X}} \right)^p < C_{M^{s, \mathcal{X}}}(E_i)^p + 2^{-i} \varepsilon.
\]
Suppose then that $u \in M^{\mathcal{A}}$. Let $x \in \mathcal{P}$. For $k \in \mathbb{N}$, let $\varphi_k$ be a Lipschitz function of bounded support such that $\varphi_k = 1$ in $B(x, k)$. Then, by Lemma 6, $u_k = u \varphi_k \in M^{\mathcal{A}}$. By the first part of the proof, there exists quasicontinuous $u^*_k$ such that $u^*_k = u_k$ almost everywhere. Since $u_{k+1} = u = u_k$ almost everywhere in $B(x, k)$, Lemma 17 implies that there exists $E_k$ with $C_{M^{\mathcal{A}}}(E_k) = 0$ such that $u_{k+1} = u^*_k$ in $B(x, k) \setminus E_k$. It follows that the limit $\lim_{k \to \infty} u_k$ exists in $\mathcal{P} \setminus E$, where, by Lemma 16, $E = \bigcup_{k=1}^{\infty} E_k$ is of $M^{\mathcal{A}}_\mathcal{P}$ capacity zero. Define $u^* = \limsup_{k \to \infty} u_k$. Then, clearly $u^*$ is almost everywhere. Let $\varepsilon > 0$. For every $k \in \mathbb{N}$, there exists $U_k$ with $C_{M^{\mathcal{A}}}(U_k) < 2^{-k} \varepsilon$ such that $u^*_k \in \mathcal{P} \setminus U_k$ is continuous. It follows that $u^*_k \in \mathcal{P}$ is continuous and, by Lemma 16, $C_{M^{\mathcal{A}}}(\bigcup_{k=1}^{\infty} U_k \cup E) < \varepsilon$. □

The following lemma gives a useful characterization of the capacity in terms of quasicontinuous functions. The proof of the lemma is essentially same as the proof of [37, Theorem 3.4]. For $E \in \mathcal{P}$, denote

$$
\mathcal{Q}_E M^{\mathcal{A}}(E) = \left\{ u \in M^{\mathcal{A}} : u \text{ is quasicontinuous and } u \geq 1 \text{ quasi-everywhere in } E \right\},
$$

$$
\overline{C}_{M^{\mathcal{A}}}(E) = \inf_{u \in \mathcal{Q}_E M^{\mathcal{A}}(E)} \left\| u \right\|_{M^{\mathcal{A}}},
$$

Lemma 17. Suppose that $u$ and $v$ are quasicontinuous. If $u = v$ almost everywhere in an open set $U$, then $u = v$ quasi-everywhere in $U$.

Lemma 18. Suppose that $X$ has absolutely continuous quasi-norm. Then, for every $u \in M^{X}$, there exists quasicontinuous $u^*$ such that $u = u^*$ almost everywhere.

Proof. Suppose first that $u \in M^{X}$. By Lemma 7, there are continuous functions $u_i \in M^{X}$ converging to $u$ in $M^{X}$ such that

$$
\left\| u_i - u_{i+1} \right\|_{M^{\mathcal{A}}} < 2^{-2i}
$$

for every $i \in \mathbb{N}$. Moreover, by [36, Lemma 3.3], we may assume that $u_i \longrightarrow u$ pointwise almost everywhere. Denote $E_i = \{ x \in \mathcal{P} : |u_i(x) - u_{i+1}(x)| > 2^{-i} \}$ and $F_j = \bigcup_{i=j}^{\infty} E_i$. Then

$$
\left| u_j - u_k \right| \leq \sum_{i=j}^{k-1} \left| u_i - u_{i+1} \right| \leq \sum_{i=j}^{k-1} 2^{-i} = 2^{-j}
$$

in $\mathcal{P} \setminus F_j$ for every $k > j$. Hence $(u_i)$ converges pointwise in $\mathcal{P} \setminus \bigcup_{j=0}^{\infty} F_j$ and the convergence is uniform in $\mathcal{P} \setminus F_j$. By continuity, $2^{-j} \left| u_j - u_{i+1} \right| \in \mathcal{A}(E_i)$ and so

$$
C_{M^{\mathcal{A}}}(E_i) \leq 2^j \left\| u_j - u_{i+1} \right\|_{M^{\mathcal{A}}} < 2^{-i}
$$

for every $i \in \mathbb{N}$. Hence, by Lemma 16,

$$
C_{M^{\mathcal{A}}}(F_j) \leq 8 \sum_{i=j}^{\infty} C_{M^{\mathcal{A}}}(E_i) \leq C 2^{-jp},
$$

which implies that $C_{M^{\mathcal{A}}}(\bigcup_{j=0}^{\infty} F_j) = 0$. It follows that the function $u^* = \limsup_{t \to \infty} u_t$ is quasicontinuous. Moreover, $u^* = \lim_{t \to \infty} u_t = u$ almost everywhere.
4. Generalized Lebesgue Points

In this section, we prove the first main result of the paper, Theorem 1. The main ingredient of the proof of is a capacitary weak type estimate, Theorem 21.

**Lemma 20.** Let $0 < s \leq 1$, $0 < \gamma \leq 1/2$, and $0 < R < \infty$. Let $u \in L_0^s$ and $g \in \mathcal{D}(u) \cap L_1^s$. Then there exists a constant $C \geq 1$ such that $C \mathcal{M}_{3R}^{1/3} g$ is an $s$-gradient of $\mathcal{M}_R^{1/3} u$.

**Proof.** Let $r > 0$. By the definition of the discrete $\gamma$-median convolution $u_i^\gamma$ and by the properties of the functions $\varphi_i$,

$$u_i^\gamma = u + \sum_{i \in I} (m_i^\gamma(B_i) - u) \varphi_i.$$  

By Lemma 6, function

$$(g + C r^{-\gamma} |u - m_i^\gamma(B_i)|) \chi_{2B_i},$$  

is an $s$-gradient of function $(m_i^\gamma(B_i) - u) \varphi_i$ for each $i$.

Let $x \in 2B_i \setminus E$, where $E$ is the exceptional set for (15). Using Lemma 8 and the facts that $B_i \subset B(x, 3r)$ and $\mu(B(x, 3r)) \leq C \mu(B_i)$, we obtain

$$|u(x) - m_i^\gamma(B_i)| \leq m_i^\gamma|_{\text{outside } B_i}(B_i).$$

By (30) and (6),

$$\|M_{1/3}^{1/3} (u \varphi)\|_X \leq C \|M_{1/3}^{1/3} (u \varphi)\|_X \leq C \|u\|_X$$  

and, by Lemmas 20 and 6 and (6),

$$\|M_{1/3}^{1/3} (u \varphi)\|_{M^s, x} \leq C \inf_{g \in \mathcal{D}(u \varphi)} \|M_{1/3}^{1/3} g\|_X \leq C \inf_{g \in \mathcal{D}(u \varphi)} \|g\|_X \leq C \|u\|_{M^s, x}.$$

Hence, $M_{1/3}^{1/3} (u \varphi) \in M^{s, x}$. Since $M_{1/3}^{1/3} (u \varphi)$ is lower semicontinuous, $\lambda^{-1} M_{1/3}^{1/3} (u \varphi) \in \mathcal{A}_{M^{s, x}} \{|x \in \mathcal{P} : M_{1/3}^{1/3} (u \varphi) > \lambda|\}$. Thus,

$$\mathcal{C}_{M^{s, x}} \{|x \in \mathcal{P} : M_{1/3}^{1/3} (u \varphi) > \lambda|\}$$

and the claim follows.

**Lemma 22.** Suppose that $X$ has absolutely continuous quasi-norm and that $B$ is a ball such that, for every $0 < \gamma < 1$, $\lambda > 0$,

$$\lim_{k \to \infty} \mathcal{C}_{M^{s, x}} \left( \left\{ x \in B : \limsup_{r \to 0} \frac{m_{B(r)}^\gamma(B(x, r))}{\lambda} \right\} \right) = 0.$$  

whenever $\lim_{k \to \infty} \|u_k\|_{M^{s, x}} = 0$. Then, for every quasicontinuous $u \in M^{s, x}$, quasievery point in $B$ is a generalized Lebesgue point of $u$.

**Proof.** By Lemma 7, continuous functions are dense in $M^{s, x}$. Let $u \in M^{s, x}$ be quasicontinuous and let $v_k \in M^{s, x}$, $k = 1, 2, \ldots$, be continuous such that

$$\|u - v_k\|_{M^{s, x}} \to 0$$

as $k \to \infty$. Denote $w_k = u - v_k$. Fix $0 < \gamma < 1$ and $\lambda > 0$. By Lemma 8,

$$\limsup_{r \to 0} m_{B(r)}^\gamma \left( \frac{|B(x, r)|}{\lambda} \right) \leq \|u - v_k\|_{M^{s, x}} \to 0$$

as $k \to \infty$. Denote $w_k = u - v_k$. Fix $0 < \gamma < 1$ and $\lambda > 0$. By Lemma 8,

$$\limsup_{r \to 0} m_{B(r)}^\gamma \left( \frac{|B(x, r)|}{\lambda} \right) \leq \|u - v_k\|_{M^{s, x}} \to 0$$

as $k \to \infty$. Denote $w_k = u - v_k$. Fix $0 < \gamma < 1$ and $\lambda > 0$. By Lemma 8,
Hence, by Lemma 16,
\[
C_{M^\infty_x}(\left\{ x \in B : \limsup_{r \to 0} m_{\nu-u(x)}^r(B(x,r)) > \lambda \right\})^\rho 
\leq 8C_{M^\infty_x}(\left\{ x \in B : \limsup_{r \to 0} m_{\nu-u(x)}^{\rho/2}(B(x,r)) > \frac{\lambda}{2} \right\})^\rho \tag{61}
+ 8C_{M^\infty_x}(\left\{ x \in B : w_k(x) > \frac{\lambda}{2} \right\})^\rho.
\]
By assumption,
\[
C_{M^\infty_x}(\left\{ x \in B : \limsup_{r \to 0} m_{\nu-u(x)}^r(B(x,r)) > \lambda \right\})^\rho 
\to 0 \quad (62)
\]
as $k \to \infty$. Since $|w_k|$ is quasicontinuous, Lemma 19 gives
\[
C_{M^\infty_x}(\left\{ x \in B : w_k(x) > \lambda \right\}) \leq C_{M^\infty_x}(\left\{ x \in B : w_k(x) > \frac{\lambda}{2} \right\}) \tag{63}
\leq C2\lambda^{-1} \|w_k\|_{M^\infty_x} \to 0
\]
as $k \to \infty$. It follows that
\[
C_{M^\infty_x}(\left\{ x \in B : \limsup_{r \to 0} m_{\nu-u(x)}^r(B(x,r)) > \lambda \right\}) = 0
\]
for every $0 < \rho < 1$ and $\lambda > 0$. Denote
\[
E = \left\{ x \in B : \limsup_{r \to 0} m_{\nu-u(x)}^r(B(x,r)) > \lambda \right\}
\]
for some $0 < \rho < 1$. Then
\[
E = \bigcup_{n,m \geq 2} \left\{ x \in B : \limsup_{r \to 0} m_{\nu-u(x)}^{1/m}(B(x,r)) > \frac{1}{n} \right\} \tag{66}
\]
and so, by Lemma 16, $C_{M^\infty_x}(E) = 0$. Since, by Lemma 8,
\[
|m_{\nu-u(x)}^r(B(x,r)) - u(x)| = |m_{\nu-u(x)}^{1/m}(B(x,r)) - u(x)| \tag{67}
\]
the claim follows.

**Proof of Theorem 1.** Let $u \in M^{r,\infty}$ be quasicontinuous. Fix $x \in \mathcal{B}$ and, for $k \in \mathbb{N}$, let $\varphi_k : \mathcal{B} \to [0, 1]$ be a Lipschitz function of bounded support such that $\varphi_k = 1$ in $B(x,k)$. Then, by Lemma 6, $u_k = u \varphi_k \in M^{r,\infty}$. By Theorem 21 and Lemmas 12 and 22, for every $k$, quasievery point is a generalized Lebesgue point of $u_k$. Hence, for every $k \in \mathbb{N}$, quasievery point in $B(x,k)$ is a generalized Lebesgue point of $u$. Thus, by Lemma 16, quasievery point is a generalized Lebesgue point of $u$.

**5. Lebesgue Points**

In this section we prove Theorem 2.

**Lemma 23.** Suppose that $\mu$ satisfies (2). Let $0 < s \leq 1, R > 0, u \in L^s$, and $g \in \mathcal{D}(u) \cap L^q_{\text{loc}}$ where $q = Q/(Q + s)$. Then there exists a constant $C$ such that $C(\mathcal{M}_a g^q)^1/q$ is an s-gradient of $\mathcal{M}_a u$.

**Proof.** By the Sobolev–Poincaré inequality ([8, Lemma 2.2]), $u$ is locally integrable and there exists a constant $C$ such that
\[
\inf_{x \in B \setminus 4B} |u - c| \leq Cr \left( \frac{g}{R} \right)^{1/q} \tag{68}
\]
for every $x \in \mathcal{B}$ and $r > 0$.

Fix $r > 0$. By the definition of the discrete convolution $u_r$ and by the properties of the functions $\varphi_k$,
\[
u_r = u + \sum_{i \in N} (u_{R_i} - u) \varphi_i. \tag{69}
\]
By Lemma 6, function
\[
g + Cr^{-s} \left| u - u_R \right| \chi_{2B}, \tag{70}
\]
and by the properties of the functions $\varphi_k$, $\nu_r = u + \sum_{i \in N} (u_{R_i} - u) \varphi_i$. (71)

The claim follows by Lemma 5.
Hence
\[
\|u\|_X = \|u_\alpha\|_X \leq C \left( \|g\chi_\alpha\|_X + \left( \text{ess inf}_{g\in\mathcal{L}^2(B)} \|\chi_\alpha\|_X \right) \right) \\
\leq C \left( \|g\chi_\alpha\|_X + \frac{\|\chi_\alpha\|_X}{\|g\chi_4\|_X} \|g\chi_4\|_X \right) \\
\leq C \|g\chi_4\|_X .
\] (77)

**Lemma 26.** Suppose that \( \| \cdot \|_X \) is absolutely continuous and that \( B \subset \mathcal{P} \) is ball such that, for every \( \lambda > 0 \),
\[
\lim_{i\to\infty} C_{M^{i,X}} \left( \left\{ x \in B : \limsup_{r\to0} \|u_i\|_{B(x,r)} > \lambda \right\} \right) = 0
\] (84)
whenever \( \lim_{i\to\infty} \|u_i\|_{M^{i,X}} = 0 \). Then, for every quasicontinuous \( u \in M^{i,X} \), quasievery point in \( B \) is a Lebesgue point of \( u \).

**Proof of Theorem 2.** We may assume that \( \mathcal{P} \) contains at least two points. Then \( \mathcal{P} \) can be covered by balls \( B_k = B(x_k, r_k) \), \( k \in I \), where \( I \subset \mathbb{N} \), such that the spheres \( \{ y : d(x,y) = 6r_k \} \) are nonempty.

Let \( u \in M^{i,X} \) be quasicontinuous and, for \( k \in I \), let \( \varphi_k \) be a Lipschitz function of bounded support such that \( \varphi_k = 1 \) in \( B(x_k, r_k) \). Then, by Lemma 6, \( u_k = u \varphi_k \in M^{i,X} \).

By Theorem 25 and Lemmas 12 and 26, for every \( k \in I \), quasievery point in \( B_k \) is a Lebesgue point of \( u_k \) and hence of \( u \). Hence, by Lemma 16, quasievery point in \( \mathcal{P} \) is a Lebesgue point of \( u \). \( \square \)

### 6. Proof of Theorem 3

If \( \alpha_X < \infty \), then clearly \( \mathcal{M}^Y \) is bounded on \( X \). Hence, the first part of Theorem 3 follows from Theorem 1. The second part follows from Theorem 2 via the following lemma.

**Lemma 27.** Let \( p \leq 1 \). If \( \alpha_X < 1/p \), then the operator \( u \mapsto (\mathcal{M}|u|^p)^{1/p} \) is bounded on \( X \).

**Proof.** By assumption, there are constants \( \alpha < 1/p \) and \( C > 0 \) such that
\[
\|\mathcal{M}^Y u\|_X \leq C \gamma^{-\alpha} \|u\|_X .
\] (85)
for every \( u \in X \) and \( 0 < \gamma < 1 \). Denote by \( v^* \) the decreasing rearrangement of a function \( v \), that is,
\[
v^*(t) = \inf \{ a \geq 0 : (\{ x \in P : |v(x)| > a \}) < t \} .
\] (86)
Then, for every ball \( B \), we have
\[
\int_B |u|^p \, d\mu = \frac{1}{\mu(B)} \int_0^{\mu(B)} \left( u_{x_B}^* \right)^p (t) \, dt \\
= \int_0^1 (u_{x_B}^*)^p (y \mu(B)) \, dy \\
= \int_0^1 m_y^Y (B) \, dy .
\] (87)

Hence,
\[
M |u|^p (x) \leq \int_0^1 \mathcal{M}^0 u (x)^p \, dy \leq \sum_{i=1}^\infty 2^{-i} \mathcal{M}^2 \mathcal{L}^2 u (x)^p .
\] (88)

Let \( 0 < \varepsilon < 1 \) be such that \( \alpha < \varepsilon / p \). By the Hölder inequality, we obtain
\[
(M |u|^p (x))^{1/p} \leq C \sum_{i=1}^\infty 2^{-i} \mathcal{L}^{2-i} u (x) .
\] (89)
Thus, by (85) and (14),
\[
\left\| (M |u|^p)^{1/p} \right\|_X \leq C \left\| \sum_{j=1}^{\infty} 2^{-j/p} M^{2^{-j}} u \right\|_X \\
\leq C \left( \sum_{j=1}^{\infty} 2^{-j/p} \left\| M^{2^{-j}} u \right\|_X^{1/\sigma} \right)^{\sigma} \\
\leq C \left( \sum_{j=1}^{\infty} 2^{-j(1/p-a\sigma)} \right)^{1/\sigma} \left\| u \right\|_X \\
\leq C \left\| u \right\|_X.
\]
\[\square\]

7. Examples

7.1. Lorentz Spaces. For \(0 < p < \infty, 0 < q \leq \infty,\) and \(u \in L^0,\)

\[
\left\| u \right\|_{L^{p,q}} = \left( \int_0^\infty \lambda^q \mu \left( \{ x \in \mathcal{P} : |u(x)| > \lambda \} \right)^{\frac{q}{p}} \frac{d\lambda}{\lambda} \right)^{1/q}
\]  
(91)

when \(0 < q < \infty\) and

\[
\left\| u \right\|_{L^{p,q}} = \sup_{\lambda > 0} \lambda \mu \left( \{ x \in \mathcal{P} : |u(x)| > \lambda \} \right)^{1/p}.
\]  
(92)

Then \(L^{p,q} = \{ u \in L^0 : \left\| u \right\|_{L^{p,q}} < \infty \}\) equipped with \(\left\| \cdot \right\|_{L^{p,q}}\) is a quasi-Banach function space. If \(0 < p, q < \infty,\) quasinorm \(\left\| \cdot \right\|_{L^{p,q}}\) is absolutely continuous.

Lemma 28. There exists a constant \(C = C(g)\) such that

\[
\mu \left( \{ x \in \mathcal{P} : M u (x) > \lambda \} \right) \leq C \gamma^{-1} \mu \left( \{ x \in \mathcal{P} : |u(x)| > \lambda \} \right)
\]  
(93)

for every \(u \in L^0\) and \(\lambda > 0.\)

Proof. It follows easily from the definitions that, for every \(u \in L^0, x \in \mathcal{P}, 0 < \gamma < 1,\) and \(\lambda > 0,\) we have

\[
M u (x) > \lambda \iff M X_{1 \in \mathcal{P} : \mu(y) > \lambda} (x) > \gamma.
\]  
(94)

Hence, the claim follows from the well-known weak type inequality for the Hardy–Littlewood maximal operator. \(\square\)

The following lemma is an immediate consequence of Lemma 28.

Lemma 29. Let \(0 < p < \infty\) and \(0 < q \leq \infty.\) There exists a constant \(C = C(g, p)\) such that

\[
\left\| \mathcal{M}^\gamma u \right\|_{L^{p,q}} \leq C \gamma^{-1/p} \left\| u \right\|_{L^{p,q}}
\]  
(95)

for every \(u \in L^{p,q}\) and \(0 < \gamma < 1.\) Consequently, \(\alpha_{L^{p,q}} \leq 1/p.\)

Lemma 29 and Theorem 3 imply the following result for Lorentz–Hajłasz functions.
7.3. Variable Exponent Spaces. Let \( p : \mathcal{P} \rightarrow (0, \infty) \) be a measurable function. The space \( L^{p(\cdot)} \) consisting of functions \( u \) for which
\[
\left( \int_{\mathcal{P}} \left( \frac{|u(x)|}{\lambda} \right)^{p(x)} \, d\mu(x) \right)^{1/p(x)} < \infty
\]
for some \( \lambda > 0 \) equipped with norm
\[
\|u\|_{L^{p(\cdot)}} = \left( \int_{\mathcal{P}} \left( \frac{|u(x)|}{\lambda} \right)^{p(x)} \, d\mu(x) \right)^{1/p(x)}
\]
is a quasi-Banach function space.

A measurable function \( p : \mathcal{P} \rightarrow (0, \infty) \) is locally log-Hölder continuous if there exists a constant \( C_p > 0 \) such that
\[
|p(x) - p(y)| \leq \frac{C_p}{\log(e + 1/d(x, y))}
\]
for all \( x, y \in \mathcal{P} \).

Denote
\[
p_- = \operatorname{ess \ inf}_{x \in \mathcal{P}} p(x), \quad p_+ = \operatorname{ess \ supp}(p(x)) \quad (x \in \mathcal{P})
\]
We need the following result from [19].

**Lemma 33.** Let \( p : \mathcal{P} \rightarrow (0, \infty) \) be locally log-Hölder continuous with \( p_- > 0 \) and \( p_+ < \infty \). Suppose that there exist constants \( p_{\infty} > 0 \) and \( 0 < a < 1 \) such that
\[
\int_{\mathcal{P}} a^{1/p(x) - p_-} \, d\mu(x) < \infty.
\]
Then there exists a constant \( C \) such that
\[
\|\mathcal{M}^p u\|_{L^{p(\cdot)}} \leq C \|u\|_{L^{p(\cdot)}}
\]
for every \( u \in L^{p(\cdot)} \) and \( 0 < \gamma < 1 \). Consequently, \( \alpha_{L^{p(\cdot)}} \leq 1/p_- \).

The next lemma follows from [38, Lemma 2.3], [38, Corollary 3.5] and from the fact that the function \( t \mapsto 1/t \) is bi-Lipschitz from \([a, b]\) to \([1/b, 1/a]\) whenever \( 0 < a < b < \infty \).

**Lemma 34.** Suppose that \( p : \mathcal{P} \rightarrow (0, \infty) \) is locally log-Hölder continuous and that \( p_- > 0 \) and \( p_+ < \infty \). Then, for any ball \( B_0 \subseteq \mathcal{P} \), there exists a locally log-Hölder continuous extension \( \tilde{p} \) of \( p|_B \) such that \( \tilde{p}_- = p_- \), \( \tilde{p}_+ = p_+ \), and
\[
\int_{\mathcal{P}} a^{1/\tilde{p}(x) - \tilde{p}_-} \, d\mu(x) < \infty.
\]
for some \( \tilde{p}_\infty > 0 \) and \( 0 < a < 1 \).

**Theorem 35.** Suppose that \( \mu \) satisfies (2) and that \( p : \mathcal{P} \rightarrow (0, \infty) \) is locally log-Hölder continuous with \( p_+ < \infty \). Let \( 0 < s \leq 1 \) and let \( u \in M^{p(\cdot)} := M^{p(x)} \) be quasicontinuous.

(1) If \( p_- > 0 \), then quasievery point is a generalized Lebesgue point of \( u \).

(2) If \( p_- > Q/(Q + s) \), then quasievery point is a Lebesgue point of \( u \).

**Proof.** (1) Since \( p_- < \infty \), \( L^{p(\cdot)} \) has absolutely continuous quasinorm. By Theorem 1, it suffices to show that, for every ball \( B = B(x, r) \) and \( 0 < \gamma < 1 \), there exists a constant \( C \) such that
\[
\|(M^p I) f\|_{L^{p(\cdot)}} \leq C \|f\|_{L^{p(\cdot)}}
\]
for every \( f \in L^{p(\cdot)} \). By Lemma 34, \( p|_{B(x,r+1)} \) can be extended to \( \tilde{p} \) on \( \mathcal{P} \) such that \( \tilde{p}_- < \infty \), \( \tilde{p}_- > 0 \), and (109) holds true for some \( 0 < a < 1 \) and \( \tilde{p}_\infty > 0 \). By Lemma 33, there exists a constant \( C \) such that
\[
\|\mathcal{M}^{\tilde{p}} u\|_{L^{p(\cdot)}} \leq C \|u\|_{L^{p(\cdot)}}
\]
for every \( u \in L^{p(\cdot)} \).

(2) Denote \( q = Q/(Q + s) \). By Theorem 2, it suffices to show that, for every ball \( B = B(x, r) \), there exists a constant \( C \) such that
\[
\|(M^p I) g\|_{L^{p(\cdot)}}^{1/q} \leq C \|g\|_{L^{p(\cdot)}}^{1/q}
\]
for every \( 0 \leq g \in L^{p(\cdot)} \). By Lemma 34, \( p|_{B(x,r+1)} \) can be extended to \( \tilde{p} \) on \( \mathcal{P} \) such that \( \tilde{p}_- < \infty \), \( \tilde{p}_- > q_1 \), and (109) holds true for some \( 0 < a < 1 \), \( \tilde{p}_\infty > 0 \). Hence, by Lemma 33, \( \alpha_{L^{p(\cdot)}} \leq 1/p_- \). Consequently, \( \alpha_{L^{p(\cdot)}} \leq 1/p_- \).

\[\Box\]

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares that he has no conflicts of interest.

