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Modeling of Resonating Closed Impedance Bodies with Surface Integral Equation Methods

Pasi Ylä-Oijala, Beibei Kong and Seppo Järvenpää

Abstract—Numerical solutions of various surface integral equation formulations in modeling resonating (lossless) closed impedance bodies are investigated. It is demonstrated that for certain values of purely imaginary surface impedances very strongly resonating field solutions can appear. Some of the considered formulations that are known to work well outside these resonances, e.g., for lossy surfaces, can lead to very poor accuracy or even diverging solutions at these resonances. Among the considered formulations only the combined source integral equations, discretized with a mixed scheme, both avoid the non-physical spurious internal resonances and work reasonably well at the physical (impedance) resonances.

Index Terms—Electric field integral equation, combined field integral equation, combined source integral equation, impedance boundary condition, impedance integral equation, resonance, self-dual formulation, spurious solution.

I. INTRODUCTION

CONVENTIONALLY the impedance boundary condition (IBC) has been applied to model imperfectly conducting metallic surfaces [1], [2]. Employing IBC, the solution process can be significantly simplified since modeling of the fields inside a closed body can be avoided. IBC provides reasonably good approximation as long as the skin depth is sufficiently small compared to the curvature of the surface. Recently IBC has attained increased interest in the analysis and design of metasurfaces [3]. These surfaces are thin layers of metamaterials where structure details are typically much smaller than the wavelength of the incident wave and hence the wave phenomena on these structures can be approximated with IBC.

Among the full-wave numerical methods for solving time-harmonic Maxwell's equations, the surface integral equation (SIE) based methods are particularly well suited for open-region scattering and radiation problems when the field behavior on the surface of the scatterer is described with a boundary condition, such as perfect electric conductor (PEC) or IBC.

The numerical solution of the SIEs for IBC, however, poses challenges that do not appear in the case of PEC objects. Firstly, IBC involves a rotation (cross product with the unit normal vector) which, when incorporated into certain integral operators, and discretized with conventional Galerkin-RWG approach, leads to hyper-singular non-integrable double line integrals [5]. Second problem is associated with lossless IBC bodies with purely imaginary surface impedance. These structures may support very strong resonating field solutions

[6] that give rise to ill-conditioned MoM matrices [3] and, as is demonstrated in this article, poorly converging or even diverging solutions. These resonances resemble plasmonic resonances and resonances on dielectric scatterers with high permittivities [6].

The first problem can be avoided by enforcing the boundary condition in the weak sense as an extra equation rather than directly incorporating it into the integral equations [4], [7], [8]. Several alternative approaches based on this idea have been proposed, utilizing either both the electric and magnetic surface currents [4], [8], or by eliminating the magnetic current from the equations [7], [9]. Similar ideas can also be applied for anisotropic (dyadic) surface impedances [10].

In this paper, we demonstrate that at certain resonance conditions scattering by closed IBC objects becomes extremely difficult to solve with the SIE methods and that the solution accuracy is very sensitive to the choice of the SIE formulation and discretization procedure. Some of the formulations that work well outside these resonances can lead to poor accuracy at the resonances, and produce non-zero absorption, a quantity that should vanish if the surface is lossless. The presented results suggest that to avoid non-physical spurious internal resonances and to obtain converging results at the physical (impedance) resonances, combined integral equations discretized with a mixed scheme utilizing both primary RWG and dual Buffa-Christiansen (BC) functions are needed.

II. FORMULATION

Let us consider electromagnetic scattering by a closed impenetrable object in a homogeneous background medium that is characterized with constant permittivity ϵ and permeability μ . The time factor is $e^{-i\omega t}$. Let $\mathbf{E}^{\text{inc}}$, $\mathbf{H}^{\text{inc}}$ denote an incident wave having sources outside the object. On the surface of the object we assume the impedance boundary condition (IBC). Defining tangential and rotated tangential components of a vector field on the surface S with $\gamma_t \mathbf{F} = -\mathbf{n} \times \mathbf{n} \times \mathbf{F}|_S$ and $\gamma_r \mathbf{F} = \mathbf{n} \times \mathbf{F}|_S$, respectively, IBC can be expressed as

$$\gamma_t \mathbf{E} = \eta Z_s \gamma_r \mathbf{H}. \quad (1)$$

Here Z_s is a constant normalized surface impedance, $\eta = \sqrt{\mu/\epsilon}$ is the wave impedance of the background medium, $\mathbf{E}$ and $\mathbf{H}$ are the total fields in the background medium and $\gamma_t \mathbf{E}$ and $\gamma_r \mathbf{H}$ are their tangential and rotated tangential traces on the surface.

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For the surface integral equation (SIE) solution of the problem, we define the following electric and magnetic field integral operators (EFIO and MFIO)

$$\mathcal{T}[\mathbf{F}](\mathbf{r}) = \frac{-1}{ik} \nabla \mathcal{S}[\nabla_s \cdot \mathbf{F}](\mathbf{r}) + ik \mathcal{S}[\mathbf{F}](\mathbf{r}), \quad (2)$$

$$\mathcal{K}[\mathbf{F}](\mathbf{r}) = \nabla \times \mathcal{S}[\mathbf{F}](\mathbf{r}). \quad (3)$$

Here the single layer integral operator is given by

$$\mathcal{S}[\mathbf{F}](\mathbf{r}) = \int_S G(\mathbf{r}, \mathbf{r}') \mathbf{F}(\mathbf{r}') dS', \quad (4)$$

and $G(\mathbf{r}, \mathbf{r}') = e^{ik|\mathbf{r}-\mathbf{r}'|}/(4\pi|\mathbf{r}-\mathbf{r}'|)$ with $k = \omega\sqrt{\varepsilon\mu}$ is the Green's function of the background medium. On the surface the integral operators including differential operators are considered in the principal value sense.

With these operators we define two sets of SIEs, tangential field integral equations

$$\begin{bmatrix} \gamma_t \mathcal{T} & -\gamma_t \mathcal{K}^+ \\ \gamma_t \mathcal{K}^+ & \gamma_t \mathcal{T} \end{bmatrix} \begin{bmatrix} \eta \mathbf{J} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} -\gamma_t \mathbf{E}^{\text{inc}} \\ -\eta \gamma_t \mathbf{H}^{\text{inc}} \end{bmatrix}, \quad (5)$$

and rotated tangential current integral equations

$$\begin{bmatrix} \gamma_r \mathcal{K}^- & \gamma_r \mathcal{T} \\ -\gamma_r \mathcal{T} & \gamma_r \mathcal{K}^- \end{bmatrix} \begin{bmatrix} \eta \mathbf{J} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} -\eta \gamma_r \mathbf{H}^{\text{inc}} \\ \gamma_r \mathbf{E}^{\text{inc}} \end{bmatrix}. \quad (6)$$

Here $\mathbf{J} = \mathbf{n} \times \mathbf{H}$ and $\mathbf{M} = -\mathbf{n} \times \mathbf{E}$ are the equivalent electric and magnetic surface current densities, and

$$\gamma_t \mathcal{K}^\pm = \gamma_t \mathcal{K} \pm \frac{1}{2} \mathcal{R} \quad \text{and} \quad \gamma_r \mathcal{K}^\pm = \gamma_r \mathcal{K} \pm \frac{1}{2} \mathcal{I}, \quad (7)$$

where $\mathcal{I}[\mathbf{F}] = \mathbf{F}$ and $\mathcal{R}[\mathbf{F}] = \mathbf{n} \times \mathbf{F}$ are the identity and rotation operators. In the following these equations are used to develop various SIEs for solving scattering by IBC objects.

A. Electric and Magnetic Field Integral Equations

Let us start with the electric field integral equation (EFIE). Writing IBC for the currents as

$$Z_s \eta \mathbf{n} \times \mathbf{J} + \mathbf{M} = 0, \quad (8)$$

and combining it with the tangential EFIE, i.e., the first row of (5), yields [4]

$$\begin{bmatrix} \gamma_t \mathcal{T} & -\gamma_t \mathcal{K}^+ \\ Z_s \mathcal{R} & \mathcal{I} \end{bmatrix} \begin{bmatrix} \eta \mathbf{J} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} -\gamma_t \mathbf{E}^{\text{inc}} \\ 0 \end{bmatrix}. \quad (9)$$

Discretizing these equations with Galerkin's method and RWG basis and test functions leads to the following matrix equation

$$\begin{bmatrix} \mathbf{T} & -\mathbf{K} - \frac{1}{2} \mathbf{R} \\ Z_s \mathbf{R} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}^J \\ \mathbf{x}^M \end{bmatrix} = \begin{bmatrix} -\mathbf{b}^E \\ 0 \end{bmatrix}. \quad (10)$$

Here $\mathbf{T}, \mathbf{K}, \mathbf{R}$ and $\mathbf{G}$ are the matrices due to the Galerkin-RWG discretization of the $\gamma_t \mathcal{T}, \gamma_t \mathcal{K}, \mathcal{R}$ and $\mathcal{I}$ operators, and $\mathbf{b}^E$ is the tangential electric excitation vector. Vectors $\tilde{\mathbf{x}}^J$ and $\mathbf{x}^M$ contain the coefficients of the basis function approximations of the currents $\eta \mathbf{J}$ and $\mathbf{M}$.

It is important to note that this approach differs from the classical IBC-EFIE [11], where the magnetic current is first eliminated from the equations using IBC and the equations are

discretized thereafter. That particular approach enforces IBC in the ‘‘strong sense’’, whereas in the approach employed here IBC is enforced in the ‘‘weak sense’’ [4], [7]. Next the magnetic unknowns are eliminated from the equations by expressing them in terms of the electric ones using the second equation of (10)

$$\mathbf{x}^M = -Z_s \mathbf{R} \mathbf{G}^{-1} \tilde{\mathbf{x}}^J. \quad (11)$$

Substituting this to the first equation of (10), gives the IBC-EFIE considered here

$$\left(\mathbf{T} + Z_s \left(\mathbf{K} + \frac{1}{2} \mathbf{R} \right) \mathbf{R} \mathbf{G}^{-1} \right) \tilde{\mathbf{x}}^J = -\mathbf{b}^E. \quad (12)$$

The magnetic field integral equation (MFIE) for the magnetic current is obtained by using a similar approach for the tangential MFIE supplemented with the IBC

$$\begin{bmatrix} \gamma_t \mathcal{T} & \gamma_t \mathcal{K}^+ \\ -1/Z_s \mathcal{R} & \mathcal{I} \end{bmatrix} \begin{bmatrix} 1/\eta \mathbf{M} \\ \mathbf{J} \end{bmatrix} = \begin{bmatrix} -\gamma_t \mathbf{H}^{\text{inc}} \\ 0 \end{bmatrix}. \quad (13)$$

Discretizing these equations with Galerkin and RWG functions, expressing the electric unknowns with the magnetic ones

$$\mathbf{x}^J = \frac{1}{Z_s} \mathbf{R} \mathbf{G}^{-1} \tilde{\mathbf{x}}^M, \quad (14)$$

and substituting this to the discretized version of the first equation of (13), gives IBC-MFIE

$$\left(\mathbf{T} + \frac{1}{Z_s} \left(\mathbf{K} + \frac{1}{2} \mathbf{R} \right) \mathbf{R} \mathbf{G}^{-1} \right) \tilde{\mathbf{x}}^M = -\mathbf{b}^H. \quad (15)$$

Here $\mathbf{x}^J$ and $\tilde{\mathbf{x}}^M$ contain the RWG coefficients of $\mathbf{J}$ and $1/\eta \mathbf{M}$.

It is also possible to develop another type of MFIE (and EFIE) by utilizing the rotated tangential current integral equations (6). These equations are used in the next section to formulate combined field integral equations (CFIE).

B. Combined Field Integral Equations

As is well known, both IBC-EFIE and IBC-MFIE suffer from spurious internal resonances. To avoid that problem, next we develop CFIE formulations. Consider first the electric current CFIE (JCFIE). This equation is obtained by combining the tangential EFIE with the rotated tangential MFIE, i.e., by combining the first rows of (5) and (6). By combining the resulting equation with (8) gives [4]

$$\begin{bmatrix} \gamma_t \mathcal{T} + \gamma_r \mathcal{K}^- & -\gamma_t \mathcal{K}^+ + \gamma_r \mathcal{T} \\ Z_s \mathcal{R} & \mathcal{I} \end{bmatrix} \begin{bmatrix} \eta \mathbf{J} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} -\gamma_t \mathbf{E}^{\text{inc}} - \eta \gamma_r \mathbf{H}^{\text{inc}} \\ 0 \end{bmatrix}. \quad (16)$$

Using the same approach as for the IBC-EFIE above, the matrix equation presentation of the IBC-JCFIE reads [9]

$$\begin{aligned} & \left(\mathbf{T} + \mathbf{K}_r - \frac{1}{2} \mathbf{G} + Z_s \left(\mathbf{K} + \frac{1}{2} \mathbf{R} - \mathbf{T}_r \right) \mathbf{R} \mathbf{G}^{-1} \right) \tilde{\mathbf{x}}^J \\ & = -\mathbf{b}^E - \eta \mathbf{b}_r^H. \end{aligned} \quad (17)$$

Here $\mathbf{T}_r$ and $\mathbf{K}_r$ are the matrices due to RWG-Galerkin discretization of the operators $\gamma_r \mathcal{T}$ and $\gamma_r \mathcal{K}$, and $\mathbf{b}_r^H$ is the rotated tangential magnetic excitation vector.

A magnetic current CFIE (MCFIE) is obtained by combining the tangential MFIE with the rotated tangential EFIE, i.e., the second rows of (5) and (6), and then by removing the electric unknowns from the discretized equations using the IBC as in the IBC-MFIE above.

These CFIE formulations are known to avoid spurious resonances, but their solution accuracy is usually poorer than that of the EFIE, for example. The problem is that in Galerkin-RWG approach the rotated tangential equations are not tested correctly, and there is no guarantee that the solution converges in a proper energy norm [12], [13]. To avoid this, we utilize a mixed testing scheme where the tangential equations are tested with the RWG functions and the rotated tangential ones are tested with the $-\mathbf{n} \times \mathbf{BC}$ functions, the L^2 dual functions of the RWG functions [14].

In the mixed scheme, i.e., when the rotated tangential equations are tested with $-\mathbf{n} \times \mathbf{BC}$ functions, the matrices $\mathbf{T}_r$, $\mathbf{K}_r$ and $\mathbf{G}$ are replaced with matrices $\tilde{\mathbf{T}}$, $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{R}}$. These matrices are due to operators $\gamma_t \mathcal{T}$ and $\gamma_t \mathcal{K}$, and the rotation operator, tested with BC functions.

C. IBC Integral Equation

Another type of a combined equation for IBC is the IBC integral equation (IBC-IE) [15]. In this equation tangential EFIE is combined with $-Z_s \eta$ times the rotated tangential MFIE. Combining the resulting equation with (8), discretizing the equations with RWG functions, and eliminating the magnetic unknowns, the matrix equation for IBC-IE reads

$$\left(\mathbf{T} - Z_s (\mathbf{K}_r - \frac{1}{2} \mathbf{G}) + Z_s \left(\mathbf{K} + \frac{1}{2} \mathbf{R} + Z_s \mathbf{T}_r \right) \mathbf{R} \mathbf{G}^{-1} \right) \tilde{\mathbf{x}}^J = -\mathbf{b}^E + Z_s \eta \mathbf{b}_r^H. \quad (18)$$

This equation, although being free of the spurious internal PEC cavity resonances for $Z_s > 0$, suffers from other type of spurious resonances for lossless impedance surfaces [17]. Another problem is that the rotated tangential MFIE is not tested correctly with the RWG functions, similarly as in the CFIE formulations above. It is also important to notice that IBC-IE reduces to the PEC-EFIE as $Z_s \rightarrow 0$.

D. Self-Dual Integral Equation

Recently a new type of SIE for IBC objects was proposed in [8]. The idea in this *self-dual integral equation* (SDIE) is to use both tangential EFIE and MFIE combined with IBC as

$$\left(\begin{bmatrix} \gamma_t \mathcal{T} & -\gamma_t \mathcal{K}^+ \\ \gamma_t \mathcal{K}^+ & \gamma_t \mathcal{T} \end{bmatrix} - \begin{bmatrix} Z_s \mathcal{I} & -\mathcal{R} \\ \mathcal{R} & 1/Z_s \mathcal{I} \end{bmatrix} \right) \begin{bmatrix} \eta \mathbf{J} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} -\gamma_t \mathbf{E}^{\text{inc}} \\ -\eta \gamma_t \mathbf{H}^{\text{inc}} \end{bmatrix}. \quad (19)$$

To improve the balance between the unknowns, particularly as $Z_s \rightarrow 0$, the equations can be further scaled with $\sqrt{Z_s}$ [8]

$$\left(\begin{bmatrix} \gamma_t \mathcal{T} & -\sqrt{Z_s} \gamma_t \mathcal{K}^+ \\ \gamma_t \mathcal{K}^+ & \sqrt{Z_s} \gamma_t \mathcal{T} \end{bmatrix} - \begin{bmatrix} \sqrt{Z_s} \mathcal{I} & -\mathcal{R} \\ \sqrt{Z_s} \mathcal{R} & \mathcal{I} \end{bmatrix} \right)$$

$$\begin{bmatrix} \eta \mathbf{J} \\ 1/\sqrt{Z_s} \mathbf{M} \end{bmatrix} = \begin{bmatrix} -\gamma_t \mathbf{E}^{\text{inc}} \\ -\eta \gamma_t \mathbf{H}^{\text{inc}} \end{bmatrix}. \quad (20)$$

This formulation is referred in this article as the SDIE-2. If the magnitude of the normalized surface impedance Z_s is larger than one, it would be beneficial to scale the equations differently with the surface admittance $Y_s = 1/Z_s$ [8]. That formulation, however, is not used here. The SDIEs can be discretized with RWG functions and Galerkin's method as shown in [8].

E. Combined Source Integral Equations

Still another type of SIEs exists that can be used to find solutions for scattering by IBC objects. These equations are based on the source approach [16], where the fields are expressed in terms of arbitrary source currents $\mathbf{j}$ and $\mathbf{m}$

$$\mathbf{E} = \mathbf{E}^{\text{inc}} + \eta \mathcal{T}[\mathbf{j}] - \mathcal{K}[\mathbf{m}], \quad (21)$$

$$\mathbf{H} = \mathbf{H}^{\text{inc}} + \frac{1}{\eta} \mathcal{T}[\mathbf{m}] + \mathcal{K}[\mathbf{j}]. \quad (22)$$

Taking the tangential component of the electric field on the surface and combining it with the rotated tangential component of the magnetic field, as in (1), gives

$$\begin{aligned} & \left(\gamma_t \mathcal{T} - Z_s \left(\gamma_r \mathcal{K} + \frac{1}{2} \mathcal{I} \right) \right) [\eta \mathbf{j}] \\ & + \left(-\gamma_t \mathcal{K} + \frac{1}{2} \mathcal{R} - Z_s \gamma_r \mathcal{T} \right) [\mathbf{m}] \\ & = -\gamma_t \mathbf{E}^{\text{inc}} + Z_s \eta \gamma_r \mathbf{H}^{\text{inc}}. \end{aligned} \quad (23)$$

By applying the combined source condition [18]

$$\mathbf{m} = \eta \mathbf{n} \times \mathbf{j}, \quad (24)$$

and discretizing the equations with Galerkin's method and RWG functions, gives the following matrix equation

$$\begin{bmatrix} \left(\mathbf{T} - Z_s \left(\mathbf{K}_r + \frac{1}{2} \mathbf{G} \right) \right) & -\mathbf{K} + \frac{1}{2} \mathbf{R} - Z_s \mathbf{T}_r \\ -\mathbf{R} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}^j \\ \mathbf{x}^m \end{bmatrix} = \begin{bmatrix} -\mathbf{b}^E + Z_s \eta \mathbf{b}_r^H \\ 0 \end{bmatrix}. \quad (25)$$

By eliminating the magnetic unknowns from the equation by using the second equation and substituting the result to the first one, gives the electric combined source integral equation (JCSIE) used here

$$\begin{aligned} & \left(\mathbf{T} - Z_s \left(\mathbf{K}_r + \frac{1}{2} \mathbf{G} \right) - \left(\mathbf{K} - \frac{1}{2} \mathbf{R} + Z_s \mathbf{T}_r \right) \mathbf{R} \mathbf{G}^{-1} \right) \tilde{\mathbf{x}}^J \\ & = -\mathbf{b}^E + Z_s \eta \mathbf{b}_r^H. \end{aligned} \quad (26)$$

A dual formulation, magnetic CSIE (MCSIE), is obtained by combining the tangential component of the SIE representation of the magnetic field, $\gamma_t \mathbf{H}$, with the rotated tangential component of the electric field, $\gamma_r \mathbf{E}$, as follows

$$\gamma_t \mathbf{H} + \frac{1}{\eta Z_s} \gamma_r \mathbf{E}. \quad (27)$$

The combined source condition is then used to express the electric source current in terms of the magnetic one.

In the mixed scheme, for both JCSIE and MCSIE, the rotated tangential equations are tested with the $-\mathbf{n} \times \mathbf{BC}$ functions, similarly as in JCFIE and MCFIE above.

III. NUMERICAL EXPERIMENTS

A. Mie Results

Let us start by illustrating the problem, resonances of lossless closed impedance bodies. We consider a sphere with the size parameter $kr = 1$, where r is the radius of the sphere and k is the wave number of the background medium (vacuum). Fig. 1 shows the scattering efficiency Q_{sca} , defined as the scattering cross section divided by the geometrical cross section, for a plane wave incidence using the generalized Mie expansion [6].

Fig. 1: Scattering efficiency for an IBC sphere with size parameter $kr = 1$ versus the surface impedance. Mie solution.

The results of Fig. 1 show hot spots, “resonances”, close to the imaginary axis. To further analyze these resonances, Fig. 2 shows Q_{sca} for purely imaginary Z_s , i.e., for a lossless impedance surface. We may identify two “smooth” resonances close to $Z_s = 1.63$ and $Z_s = -0.61i$, and two “sharp” ones near $Z_s = 0.62i$ and $Z_s = -1.62i$, and one extremely sharp one at $Z_s = 0.36i$. By studying the Mie solutions, we find that these resonances correspond to the magnetic and electric (TE_1 and TM_1) dipole modes, magnetic and electric (TE_2 and TM_2) quadrupole modes, and a magnetic TE_3 mode, respectively. A more detailed analysis indicates that there exists several higher order magnetic resonances on the range $0 < X_s < 0.35$, where X_s is the imaginary part of Z_s . Similarly, there are infinitely many electric high-order resonances for larger negative values of X_s [6].

Fig. 3 shows the absorption efficiency Q_{abs} defined as the ratio of the absorption and geometrical cross sections. For pure imaginary Z_s Q_{abs} vanishes. As the real part of Z_s is increased losses are introduced. Close to the resonances Q_{abs} may obtain rather high values even for small losses.

Fig. 2: Scattering efficiency for an IBC sphere with $kr = 1$ versus purely imaginary surface impedance. Mie solution.

Fig. 3: Absorption efficiency for an IBC sphere with size parameter $kr = 1$ versus the surface impedance. Mie solution.

B. Numerical Results

Next the computations are repeated using the SIE formulations introduced in the previous section. The surface mesh of a sphere contains 720 planar triangular elements and 1080 edges. This agrees with the average edge length of about $\lambda/30$, where λ is the wavelength of vacuum. For numerical solutions, scattering and absorption cross sections are computed by first computing the scattering and total (extinction) power

$$P^{\text{sca}} = \frac{1}{2} \text{Re} \int_S (\mathbf{E}^{\text{sca}} \times \bar{\mathbf{H}}^{\text{sca}}) \cdot \mathbf{n} dS, \quad (28)$$

$$P^{\text{ext}} = -\frac{1}{2} \text{Re} \int_S (\mathbf{E}^{\text{sca}} \times \bar{\mathbf{H}}^{\text{inc}} + \mathbf{E}^{\text{inc}} \times \bar{\mathbf{H}}^{\text{sca}}) \cdot \mathbf{n} dS, \quad (29)$$

where S is any surface enclosing the scatterer and $\bar{\mathbf{F}}$ denotes complex conjugate of $\mathbf{F}$. These quantities can be obtained directly from the solution vector $\mathbf{x} = [\mathbf{x}^J, \mathbf{x}^M]^T$ as [19]

$$P^{\text{sca}} = -\frac{1}{2} \text{Re}(\mathbf{x}^H \mathbf{A} \mathbf{x}), \quad P^{\text{ext}} = -\frac{1}{2} \text{Re}(\mathbf{b}^H \mathbf{x}), \quad (30)$$

where the operator (matrix)

$$\mathbf{A} = \begin{bmatrix} \eta\mathbf{T} & -\mathbf{K} \\ \mathbf{K} & 1/\eta\mathbf{T} \end{bmatrix} \quad (31)$$

gives the scattered field due to the non-scaled surface currents, $\mathbf{b}$ is the excitation vector with both electric and magnetic incident fields and H denotes Hermitian transpose. The absorbed power is then obtained as

$$P^{\text{abs}} = P^{\text{ext}} - P^{\text{sca}}. \quad (32)$$

The scattering and absorption cross sections are given by

$$\sigma^{\text{sca}} = \frac{P^{\text{sca}}}{P^{\text{inc}}} \quad \text{and} \quad \sigma^{\text{abs}} = \frac{P^{\text{abs}}}{P^{\text{inc}}}, \quad (33)$$

where, for a plane wave incidence, $\mathbf{E}^{\text{inc}}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k}\hat{\mathbf{k}}\cdot\mathbf{r}}$, the incident power is given by

$$P^{\text{inc}} = \frac{\|\mathbf{E}_0\|^2}{2\eta}, \quad (34)$$

Fig. 4 shows the error in the numerical solutions of Q_{sca} for the same case as in Fig. 2 when the equations are discretized with Galerkin-RWG scheme. Obviously, the numerical solutions agree rather well with the analytical one, except close to the above-mentioned (sharp) resonances. This becomes evident also from Fig. 5 where Q_{abs} is displayed. Since the surface is lossless, Q_{abs} should vanish and thus Fig. 5 shows the error in Q_{abs} . As these results indicate, among the considered formulations, SDIE-2, JCFIE, and JCSIE produce particularly high absorption at the resonances of the TE_2 , TE_3 and TM_2 modes.

Fig. 4: Error in scattering efficiency for an IBC sphere with size parameter $kr = 1$ versus purely imaginary Z_s .

Figs. 6 and 7 show the condition numbers of the matrices versus the purely imaginary surface impedance. The condition numbers are expected to show high values near the physical resonances, i.e., as $Z_s = 1.63i, -0.61i, 0.62i, -1.62i, 0.36i$, and on the region $0 < X_s < 0.35i$. In addition to these resonances, EFIE shows additional resonances for positive X_s , MFIE for negative X_s , and SDIE for both positive and negative X_s . In addition, MFIE, MCFIE and SDIE formulations become ill-conditioned as $Z_s = 0$.

Fig. 5: Absorption efficiency for an IBC sphere with size parameter $kr = 1$ versus purely imaginary surface impedance.

Fig. 6: Condition numbers of the matrices for an IBC sphere with $kr = 1$ versus surface impedance. EFIE, MFIE and SDIE.

Fig. 7: Condition numbers of the matrices for an IBC sphere with $kr = 1$ versus surface impedance. CFIE and CSIE.

The numerical results shown in Figs. 4-7 are obtained using the standard RWG-Galerkin discretization with RWG basis and test functions. To further investigate the solutions of SDIE, CFIE and CSIE formulations, Figs. 8 and 9 show Q_{abs} with four different SDIE formulations and with both the standard and mixed RWG test functions for tangential operators and $-\mathbf{n} \times \mathbf{BC}$ test functions for rotated tangential equations) discretizations of the CFIE and CSIE. The two new SDIE formulations are obtained by using (19) where either the first IBC equation (SDIE-3), or alternatively the second one (SDIE-4), is removed. From these results we may conclude that by removing one of the boundary conditions leads to zero absorption efficiency, and that the additional normalization with $\sqrt{Z_s}$ (SDIE-2) significantly increases absorption. It is worth of noticing that among the considered SDIE formulations only SDIE-2 is found to be free of the spurious internal resonances. For the CFIE and CSIE formulations mixed scheme reduces the absorption by a couple of orders of magnitude compared to the standard approach.

Fig. 8: Absorption efficiency for an IBC sphere with size parameter $kr = 1$ computed with different SDIE formulations.

C. Solution Convergence at Resonances

To get more insight into the problem, next we investigate convergence of the solutions close to the resonance of the TE_3 mode, i.e., as $Z_s \approx i0.36$. We use five different meshes including 480, 750, 1080, 1470, and 1920 edges. The results of Fig. 10 suggest that SDIE-2 is not able to find the resonance. It is also important to note that in Fig. 11 only the mixed discretized CFIE and CSIE formulations are considered. With standard Galerkin-RWG approach the convergence of the solutions of these formulations is much poorer. An interesting observation is that the solutions of SDIE-3 and MFIE, as well as the solutions of SDIE-4 and EFIE, for Q_{sca} , seem to be nearly identical.

To further illustrate the challenge of numerical modeling of strongly resonating IBC bodies with SIE methods, Figs. 12 - 15 show Q_{abs} close to TE_2 and TE_3 resonances computed with standard RWG-Galerkin and mixed discretized CFIE and CSIE formulations. Near the TE_2 resonance, both the standard

Fig. 9: Absorption efficiency for an IBC sphere with $kr = 1$ with standard (solid lines) and mixed (dashed lines) CFIE and CSIE formulations.

Fig. 10: Scattering efficiency for an IBC sphere with $kr = 1$ using five different meshes near the TE_3 resonance. Arrows indicate convergence of the solutions with respect to the mesh.

Fig. 11: Scattering efficiency for an IBC sphere with $kr = 1$ close to TE_3 resonance. Mixed tested CFIE and CSIE.

(Fig. 12) and mixed (Fig. 13) discretization show convergence, but at the TE_3 resonance only the mixed one gives converging solutions (Fig. 15).

Fig. 12: Absorption efficiency for an IBC sphere with size parameter $kr = 1$ close to the TE_2 resonance. Combined formulations with standard discretization.

Fig. 13: Absorption efficiency for an IBC sphere with size parameter $kr = 1$ close to the TE_2 resonance. Combined formulations with mixed discretization.

D. Solution with JCSIE

From the above experiments we may deduce that the mixed discretized JCSIE has generally fairly good performance. It is important to note that it does not necessarily give the most accurate results nor best conditioned matrix for all Z_s . To study mixed JCSIE for a wider range of Z_s , Figs. 16 and 17 show errors in Q_{sca} and Q_{abs} for the same case as in Figs. 1 and 3 with a mesh having 480 edges. Particularly in Q_{sca} the accuracy problems are mostly focused on certain purely imaginary values of Z_s .

IV. CONCLUSIONS

We have demonstrated that for certain lossless surface impedances (purely imaginary Z_s) the field solutions can sup-

Fig. 14: Absorption efficiency for an IBC sphere with size parameter $kr = 1$ close to the TE_3 resonance. Combined formulations with standard discretization.

Fig. 15: Absorption efficiency for an IBC sphere with size parameter $kr = 1$ close to the TE_3 resonance. Combined formulations with mixed discretization.

Fig. 16: Error in scattering efficiency for an IBC sphere with $kr = 1$ versus complex surface impedance. Mixed JCSIE.

Fig. 17: Error in absorption efficiency for an IBC sphere with $kr = 1$ versus complex surface impedance. Mixed JCSIE.

port very strong resonances. At these resonances the solution accuracy of the surface integral equations is found to depend strongly on the adopted integral equation formulation and discretization procedure. Some of the considered formulations, such as self-dual and combined field and source ones, discretized with standard Galerkin-RWG scheme, may give very poor accuracy and non-zero absorption for lossless surfaces.

Among the considered formulations, only the combined source formulations, discretized with a mixed scheme, are found to perform reasonable well on the studied parameter values. Generally, combined formulations are needed to avoid spurious internal resonances and a mathematically correct discretization scheme is required to obtain converging solutions near the physical (impedance) resonances.

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