Berg, Kimmo

Set-valued games and mixed-strategy equilibria in discounted supergames

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Abstract

This paper examines the subgame-perfect mixed-strategy equilibria in discounted supergames. We present a method that finds all the equilibrium payoffs without public correlation in $2 \times 2$ games. The method makes it possible to solve and analyze the set of equilibria in more detail. This is the first time that the mixed-strategy payoff set is solved in the repeated prisoner’s dilemma. We find that the players can obtain higher payoffs in mixed strategies, there are more Pareto efficient outcomes, and the set of equilibria can be dramatically larger. We show that the equilibrium payoffs can be efficiently computed by finding certain extreme points of the set. This result relies on the classification of games, the monotonicity properties of the problem and splitting the sets into X-Y convex parts. We also introduce set-valued games, where the players’ payoffs are chosen from given sets, and show that these games can be solved with the same methodology.

Keywords: repeated games, mixed strategy, subgame perfection, set-valued games, public correlation, orthogonal convexity

2000 MSC: 91A20

1. Introduction

Games are used in modeling multiplayer decision making problems or multiagent systems. The prominent solution concept in non-cooperative games is the Nash equilibrium. It describes a solution where no player wishes to deviate alone from the prescribed strategy if the others play according to
their strategies. The game models have been applied in many fields of science; e.g., in modeling how animals behave, how data packets move in the telecommunications networks, or how to defend against terrorist attacks. It is important to develop numerical methods to solve and analyze properly the different types of game models.

This paper examines finding all the equilibrium payoffs in infinitely repeated games. These games impose some additional restrictions and properties on the solution concept. For example, it is reasonable to assume that the equilibrium strategies are subgame perfect. This means that the players cannot use threats that are against their own interest to sustain cooperation. This requirement of sequential rationality creates a self-reference to the set of equilibria, which makes solving equilibria difficult; note that the problem is easier to solve without subgame perfection \[1, 2\]. To check that the players do not have any profitable deviations at any stage, one needs to know the players’ minimum equilibrium payoffs. These payoffs are needed in general since the response to a deviation, i.e., the punishment, must be equilibrium behavior as well and not some unrealistic actions that nobody believes would be implemented. But the problem is that the minimum equilibrium payoffs and the corresponding strategies are not typically known in advance, and they may depend on each other \[3, 4\]. Mathematically, the characterization of equilibria can be formulated as a set-valued fixed-point equation, which was originally presented in the seminal work of Abreu; see \[1, 3, 4, 29\].

This paper studies what difference does it make if the players have different types of strategies available. For example, we examine how much larger the set of mixed-strategy equilibria is compared to pure strategies, and can the players obtain better outcomes using mixed strategies. It is also interesting to study if the mixed strategies are more complicated for the players and is it more difficult to find these strategies.

Typically, it is assumed that the players use pure strategies and they have a public correlating device available \[20, 26, 19, 6, 2\]. Such device can, e.g., be organized by a trusted third party, and it allows the players to coordinate and randomize between the different pure strategies. For example, two players can randomize between playing (left, left) or (right, right), and the combinations (left, right) and (right, left) are never realized. This kind of randomization is not possible with the normal mixed strategies. The public correlating device makes the set of equilibria convex and the task of computation much easier. The question then arises if the assumption is reasonable and what difference does it make. When the players are arbitrarily patient, i.e., the discount
factor is close to one, there is no difference and the folk theorem holds \cite{24, 25}. But in general, the difference can be dramatic. For example, Yamamoto \cite{36} shows that the set of equilibria is not convex nor monotone without public correlation, no matter how large the discount factor is. Moreover, the model without public correlation has been studied in \cite{11, 13, 14, 9}, and Berg and Kittit \cite{12} show that the set of pure-strategy equilibria is a specific type of fractal. We think that the public correlation is not reasonable assumption in all situations, and the players may not agree nor trust the party that organizes the public lottery.

This paper examines solving the set of equilibria in mixed strategies without public correlation. The mathematical model is similar to the one in pure strategies, and the model is also a special case of imperfect monitoring or stochastic games; see Section 7 in \cite{29} and \cite{30}. One important issue is whether the players observe the mixed strategies or not \cite{24, 25}. Here, we examine the model where the players only observe the realized pure actions and not the actual probabilities that the other players are using. This paper builds upon Berg and Schoenmakers \cite{16} where the mixed-strategy model is examined and they discuss what the computation of the payoff set requires.

This paper presents a method for finding all the mixed-strategy equilibria in $2 \times 2$ games. The approach is based on enumerating and classifying all the possible stage games that emerge in the repeated game. In essence, the repeated games are solved using suitable stage games where the continuation payoffs, i.e., the payoffs that the players obtain in the future rounds, are included in the stage game payoffs. This is what the recursive characterization of equilibrium implies \cite{11, 14, 30}. It is also the way how the multistage decision making problems are solved using dynamic programming; i.e., the future payoffs are taken into account using the value or cost-to-go function. Since we are looking for the entire set of equilibria, and not only one equilibrium, we have to solve the equilibria in all the stage games that produce different payoffs. The set of possible stage games is huge, but we find that only certain extreme payoffs have to be examined, which enhances the method dramatically.

The main subproblem of solving repeated games is equivalent to solving Nash equilibria of a stage game with set-valued payoffs. Consider, e.g., the coordination game shown in Figure \ref{fig:1}(a), where each of the payoff pairs $K_a$ to $K_d$ are chosen from the corresponding sets. For example, the payoffs of the pure strategy $d = (B, R)$ are chosen from the set $K_d$, i.e., between 2 and 3 for both players. We are then interested in finding all the equilibria for all the
Figure 1: Two set-valued games and their set of equilibria in grey. The payoffs of the pure-action profiles are chosen from the sets $K_x$, $x \in A$. For example, the payoff of pure action $c = (B, L)$ is chosen from the set $K_c$.

possible stage games that can be formed from the sets $K_a-K_d$, and this set is shown in grey in the figure, as we shall show later on. The pure strategies give the payoffs in the sets $K_a$ and $K_d$, and the mixed strategies the square between 2.5 and 4.5. These set-valued games can, e.g., represent situations where the players are uncertain about the payoffs and they are interested in finding all the possible equilibria when the payoffs are realized from the sets. Thus, the ideas presented in this paper can be used in solving other game models as well; e.g., games of incomplete information, stochastic games, or games with incomplete preferences [8].

The paper is structured as follows. Section 2 introduces the classification of stage games and the set-valued games. Section 3 examines the equilibria in the repeated games. The numerical method is presented in Section 4. A prisoner’s dilemma is examined in Section 5, and Section 6 concludes.
2. Stage games and set-valued games

We present the main ideas in the most simple setting of $2 \times 2$ games. We give a classification of the games and observe that the set of equilibrium payoffs are different between the classes.

A stage game can be defined with a tuple $(N, A, u)$, where $N = \{1, \ldots, n\}$ is the finite set of players, $A_i$ is the finite set of pure actions of player $i \in N$, and $u$ is the mapping that gives the players’ payoffs. The set of pure-action profiles is $A = \times_{i \in N} A_i$. A pure action of player $i$ is $x_i \in A_i$ and a pure-action profile is $x \in A$. The payoffs are given for each pure-action profile, i.e., $u : A \mapsto \mathbb{R}^n$. In $2 \times 2$ games, the four pure-action profiles are denoted by $a, b, c$ and $d$.

The players are allowed to randomize between the pure actions, i.e., use a mixed action $q_i$. It holds that $q_i(x_i) \geq 0$ for each $x_i \in A_i$ and $\sum_{x_i \in A_i} q_i(x_i) = 1$. The set of probability distributions over $A_i$ is called $Q_i$ and similarly $Q = \times_{i \in N} Q_i$. A mixed-action profile is denoted by $q = (q_1, \ldots, q_n) \in Q$. The support of a mixed action is the set of pure actions that is played with a strictly positive probability: $\text{Supp}(q_i) = \{x_i \in A_i | q_i(x_i) > 0\}$. We also define $\text{Supp}(q) = \times_{i \in N} \text{Supp}(q_i)$ and for each $x \in \text{Supp}(q)$, we let $\pi_q(x)$ be the probability that the action profile $x$ is realized if the mixed-action profile $q$ is played: $\pi_q(x) = \prod_{j \in N} q_j(x_j)$. Moreover, a strategy is truly mixed if it is not a pure strategy.

The payoff function can now be extended to mixed strategies. If the players use a mixed-action profile $q \in Q$, then player $i$ receives an expected payoff of

$$u_i(q) = \sum_{x \in A} u_i(x) \pi_q(x). \tag{1}$$

Player $i$’s opponents’ action profile is denoted by $q_{-i} \in Q_{-i} = \times_{j \in N, j \neq i} Q_j$. An action profile $q$ is a Nash equilibrium in the stage game if

$$u_i(q) \geq u_i(q'_i, q_{-i}) \text{ for all } i \in N, \text{ and } q'_i \in Q_i, \tag{2}$$

and this means that no player has a profitable deviation.

2.1. Classification of games and equilibrium payoffs

The $2 \times 2$ games can be classified many ways \[33\] based on the problem in hand. For example, Borm \[18\] distinguishes the games based on the players’ best responses and shows that there are 15 classes of games (denoted by $c1$-$c15$). Each game can be labeled with a four-digit number, each digit
telling how a certain pair of payoffs is ordered. For example, the row player compares the payoffs on the same column, and it can either be that the top action is smaller (denoted by 0), higher (1) or they are the same (2). All the games in the same class behave strategically the same way. For example, Game 2 in Figure 2 belongs to the class c4 and it can be represented by the number 2201 as the row player is indifferent (2) between the actions on both columns, and the column player prefers right (0) on the first row and left (1) on the second row. It should be noted that several four-digit games belong to the same class since we can switch the order of players without affecting the equilibria; e.g., the numbers 2211, 2200, 0022 and 1122 all belong to class c2; see Table 1 in [18].

We can classify the games to those that have (i) pure equilibria (c5-6,c8-9), (ii) truly-mixed equilibria (c14-15), and (iii) the games with ties (c1-c4,c7,c10-c13), i.e., when certain payoffs are exactly the same. This helps us distinguish games where the equilibria are easily determined by pure strategies, determined by Eq. (1), and where the equilibria are combinations of points, lines and rectangles.

Games 1-3 in Figure 2 represent a variety of equilibria that can appear in $2 \times 2$ games. The set of equilibria in Game 1 is simply the square between
In Game 2, the best reply of the column player depends on the action of the row player. The set of equilibria consists of three lines: from (1,1) to (1,2/3) to (2,2/3) to (2,2). Game 3 is one realization of the set-valued game in Figure 1(a). This stage game gives the largest payoff (4.5,4.5) in truly-mixed strategies in the set-valued game. Game 3 has three equilibria: two in pure strategies and one in mixed strategies with the probabilities (3/4,1/4).

2.2. Set-valued games

Let us now define the set-valued games and show how they can be solved. We note that the aim is only to find all the possible equilibrium payoffs and not all the equilibrium strategies. Indeed, there can be a lot of equilibrium strategies that give exactly the same payoff, and we are not interested in finding all of these strategies.

A set-valued game is defined by a tuple \( G' = (N,A,K) \), where \( K = \times_{x \in A} K_x \) gives the sets of possible payoffs. We assume that each set \( K_x \), \( x \in A \), is compact and non-empty. For example, \( K_b \) means that the payoff of action profile \( b = (T,R) \in A \) is chosen from the set \( K_b \) in Figure 1. Let \( M(u) \) denote the set of Nash equilibrium payoffs in a stage game with payoffs \( u \). The set of Nash equilibrium payoffs in a set-valued game \( G' \) is

\[
M(G') = \bigcup_{u \in K} M(u).
\]

Note that \( u \) contains all the payoffs in the stage game. For each \( u \), we choose the payoff vector \( u(x) \) for each action profile \( x \in A \) from the corresponding set \( K_x \). See a similar extension in games with incomplete preferences [8].

The set-valued game in Figure 1(a) is simple to solve since the class (c14) is the same no matter what payoffs are chosen from the sets \( K_a - K_d \). The games in class c14 have three Nash equilibria: the payoff \( (a_1,a_2) \in K_a \), \( (d_1,d_2) \in K_d \), and the truly-mixed-strategy equilibrium. Thus, the sets \( K_a \) and \( K_d \) belong to the set of equilibria in the set-valued game; \( K_a \cup K_d \subset M(G') \). The probability of playing T or L in the mixed-strategy equilibrium is \( (d_j - c_j)/(a_j + d_j - b_j - c_j) \), where \( (b_1,b_2) \in K_b \), \( (c_1,c_2) \in K_c \), and \( j \) is the other player’s index. Similarly, the equilibrium payoff \( (v_1,v_2) \) is given by

\[
v_i = \frac{a_i d_i - b_i c_i}{a_i + d_i - b_i - c_i}.
\]
In this set-valued game, it is enough to find the four truly-mixed strategies (shown by the black dots) that give the extreme payoffs in order to find the whole payoff set. This is because the equilibrium payoffs are continuous and monotone in the players’ stage game payoffs. By monotonicity, we mean that for all \(a_i' \geq a_i\) it holds that either \(v_i' \geq v_i\) or \(v_i' \leq v_i\), and this holds for all \(a_i, b_i, c_i, d_i\) and for \(i\) and \(j\), i.e., the mixed-strategy payoff mapping is either non-decreasing or non-increasing in each of its arguments. The monotonicity guarantees that the extreme points of the equilibrium payoffs are found from the extreme points of the \(K\) sets, and the continuity means that all the values between the extreme payoffs can be obtained. We can also see that the player \(j\)’s payoffs do not directly affect the player \(i\)’s equilibrium payoff, i.e., \(v_i\) only depends on \(a_i\) to \(d_i\) and not \(a_j\) to \(d_j\). The only indirect effect is through the sets \(K_a-K_d\). It could be so that \(a_j\) affects how large \(a_i\) can be chosen since \((a_i, a_j)\) need to belong to the set \(K_a\), but this is not the case in this game.

The following shows that the truly-mixed-strategy equilibrium payoffs are monotone in the players’ payoffs if the ordering of certain payoffs do not change.

**Proposition 1.** In 2 \(\times\) 2 set-valued games without ties, the truly-mixed-strategy equilibrium payoffs are monotone in the players’ payoffs if the signs of \(b_i - d_i, c_i - d_i, c_i - a_i,\) and \(b_i - a_i, i = 1, 2,\) remain the same.

**Proof.** The monotonicity follows from the fact that the partial derivatives do not change their signs.

\[
\begin{align*}
\frac{\partial v_i}{\partial a_i} &= \frac{(b_i - d_i)(c_i - d_i)}{(a_i + d_i - b_i - c_1)^2} \quad (< 0, \text{ in Fig. } \mathbb{P}(a)) \\
\frac{\partial v_i}{\partial b_i} &= \frac{(a_i + d_i - b_i - c_1)^2}{(b_i - a_i)(b_i - d_i)} \quad (< 0, \text{ in Fig. } \mathbb{P}(a)) \\
\frac{\partial v_i}{\partial c_i} &= \frac{(a_i + d_i - b_i - c_1)^2}{(a_i + b_i)(a_i - c_i)} \quad (> 0, \text{ in Fig. } \mathbb{P}(a)) \\
\frac{\partial v_i}{\partial d_i} &= \frac{(a_i + d_i - b_i - c_1)^2}{(a_i + d_i - b_i - c_1)} \quad (> 0, \text{ in Fig. } \mathbb{P}(a))
\end{align*}
\]

Now, it is easy to determine the payoffs that give the extreme truly-mixed-strategy equilibrium payoffs. For the minimum (maximum) payoffs, we choose \(a_i\) and \(b_i\) as large (small) as possible and \(c_i\) and \(d_i\) as small (large)
as possible from the sets $K_a-K_d$. For more general sets $K_a-K_d$, this is a bit more complicated since $a_i$ and $a_j$ may be coupled through the set $K_a$, and we have to examine the Pareto efficient frontiers of the set $K_a$, as will be explained in Section 2.3.

The set-valued game in Figure 1(b) is more complicated but still simple to solve. The game contains all the possible classes (c1-c15) since the ordering of the pairs $(a_i,c_i)$ and $(b_i,d_i)$ can be freely chosen, i.e., we can choose any of the cases: $a_i < c_i$, $a_i > c_i$, or $a_i = c_i$. However, we only need two classes: c1 and any of the classes that give $(a_1,a_2) \in K_a$ as a pure-strategy equilibrium. We get the grey square between $(1,1)$ and $(4,4)$ as the equilibrium payoffs by choosing $(4,4) \in K_a$, $(1,4) \in K_b$, $(4,1) \in K_c$, and $(1,1) \in K_d$, which makes a game of class c1. The grey area in the set $K_a$ is given by the pure strategies.

We may now argue that there are no other equilibrium payoffs than the grey area shown the figure. The pure strategies have to be inside the sets $K_a-K_d$, but the equilibrium payoffs cannot be smaller than the minimax payoffs, i.e., below 1. Thus, we cannot obtain the white areas in the sets $K_b$ and $K_c$. Similarly, it can be shown that the truly-mixed-strategy payoffs have to be between the maximum of $b_1$ and $d_1$ and the minimum of $a_1$ and $c_1$ for player 1. Thus, there cannot be mixed-strategy payoffs outside the square between $(1,1)$ and $(4,4)$; and $M(G') = K_a \cup \text{Square}((1,1),(4,4))$.

2.3. Orthogonal convexity

The following results show that under certain conditions it is possible to simplify the computation of equilibria such that we only need to find certain extreme points of the set. This enables a decomposition scheme where we 1) split the set-valued game into smaller parts, 2) process these parts separately and 3) take the union of the produced sets, as is explained in Section 4.3.

**Definition 1.** A set $S$ is X-Y convex if it is connected and contains all the vertical and horizontal lines, i.e., if for any $(x,z) \in S$ and $(y,z) \in S$ (or similarly $(z,x) \in S$ and $(z,y) \in S)$ then $(\lambda x + (1-\lambda)y,z) \in S$ (or $(z,\lambda x + (1-\lambda)y) \in S$) for all $\lambda \in (0,1)$.

We note that this is not the standard definition since these sets need not be connected in general. These sets are also called rectilinear convex, orthogonal convex, or restricted-orientation convex [23, 31, 32].

**Proposition 2.** The truly-mixed-strategy equilibrium payoffs are X-Y convex in a $2 \times 2$ set-valued game if the sets $K_a-K_d$ are X-Y convex and the payoffs are monotone.
Proof. Take any two payoffs \((v_1, v_2)\) and \((v_1', v_2')\) given by \((a_1, a_2, \ldots, d_1, d_2)\) and \((a_1', a_2', \ldots, d_1', d_2')\) that belong to the sets \(K_a-K_d\). Since the sets \(K_a-K_d\) are X-Y convex, either \((a_1, a_2, \ldots, d_1, d_2)\) or \((a_1', a_2', \ldots, d_1', d_2')\) belongs to the sets \(K_a-K_d\). Assume the first one. Then the lines from \((v_1, v_2)\) to \((v_1', v_2)\), and from \((v_1', v_2)\) to \((v_1', v_2')\) belong to the payoff set, since the mapping of mixed-strategy payoff is continuous and the mixed strategy exists for all the payoff values since the ordering of payoffs does not change. The monotonicity implies that the signs of \(a_1 - c_1, b_1 - d_1, a_2 - b_2\) and \(c_2 - d_2\) do not change. Thus, the payoff set is X-Y convex.

Note that we do not need the monotonicity but a weaker condition that the equilibrium payoffs are continuous, i.e., \(a_i + d_1 - b_i - c_i \neq 0\), and that the truly-mixed strategy remains as an equilibrium. However, we need it for the following result.

**Proposition 3.** In a 2 \(\times\) 2 set-valued game, if the sets \(K_a-K_d\) are X-Y convex and the payoffs are monotone, then the extreme points of the truly-mixed-strategy equilibrium payoffs are produced by the extreme points of the X-Y convex sets.

**Proof.** The monotonicity guarantees that the extrema are found in the extreme points of the X-Y convex sets. 

This means that we can simplify the task of computation in the following way. When we want to find the set of all truly-mixed-strategy equilibria, we can split the sets \(K_a-K_d\) based on the classes, the X-Y convexity, and the regions where the mixed-strategy payoff mapping is monotone. For each of these regions, we have to examine only the combinations of the Pareto corner points of the X-Y convex sets, and determine the X-Y convex hull of these points. This idea is demonstrated with the following example.

Consider the set-valued game in Figure [3]. Note that the set of equilibria (including pure strategies) is not X-Y convex, even though the set of truly-mixed-strategy equilibria is. We explain how to find the north-east (NE) frontier of the truly-mixed-strategy equilibria. To maximize the players’ payoffs, we want to minimize \(a_1, b_1, a_2\) and \(c_2\) and maximize \(b_2, c_1, d_1\) and \(d_2\). Thus, we want to examine the black dots shown in the left figure. We need to compute all the possible combinations; we can choose two points from each set \(K_a-K_d\), which makes a total of \(2^4 = 16\) payoffs. These payoffs are shown as black and white dots on the right figure. We can see that only
some of them, the black dots, give the extreme payoffs. It is difficult to know
in advance which combinations give the extreme points. For example, 14 out
of the 16 points are extreme points in the south-west (SW) frontier.

This process is presented in Algorithm 1. Let us explain how it works
for the NE frontier in Figure 3. In Step 1, we select the points (6.5, 6) and
(6, 6.5) as the set A, and the points (0, 5) and (0.5, 5.5) as the set B, and so
on. In Step 2, we compute the 16 combinations shown on the right of Figure
3. For example, the combination (6.5, 6) × (0, 5) × (5, 0) × (3, 3.5) gives the
payoff (4.33, 4.66) shown as the white dot. In Step 3, we form the X-Y convex
hull, which is shown on the left of Figure 3.

3. Repeated games

In a repeated game, the same stage game is played many times. The in-
finite repeated games are sometimes called as supergames. We assume that
the players only observe the realized pure actions and not the probabilities
that are used in the mixed actions. This means that the public past play
can be denoted by the set of histories $H^k = A^k = \prod_k A$ for stage $k \geq 0$,
Algorithm 1: Compute truly-mixed-strategy equilibrium payoffs (c14 or c15) for X-Y convex sets.

for each NE, SE, SW and NW frontier do
  1. Select the correct Pareto points from the sets $K$ to form the sets $A$ to $D$.
  2. Compute the mixed-strategy payoffs for all the combinations of points in sets $A$ to $D$, i.e., compute $m(u)$ for $u \in A \times B \times C \times D$, where $m(u)$ gives the mixed-strategy payoff in Eq. (3).
  3. Form the X-Y convex hull of the computed payoffs $m(u)$.

where $H^0 = A^0 = \{\emptyset\}$ is the empty set and it corresponds to the beginning of the game. A history contains all the pure-action profiles that have been realized in the previous stages. The set of all histories is $H = \bigcup_{k=0}^{\infty} H^k$. In a repeated game, a public strategy $\sigma_i$ of player $i \in N$ is a mapping that assigns a probability distribution over player $i$’s actions for each possible history $\sigma_i : H \mapsto Q_i$. The set of player $i$’s strategies is $\Sigma_i$. The players’ strategies form the strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$. A strategy profile of all players except player $i$ is denoted by $\sigma_{-i}$, and the set of strategy profiles is given by $\Sigma = \times_{i \in N} \Sigma_i$.

Player $i$ discounts the future payoffs with a discount factor $\delta_i \in (0, 1)$. The expected discounted payoff of a strategy profile $\sigma$ for player $i$ is

$$U_i(\sigma) = \mathbb{E} \left[ (1 - \delta_i) \sum_{k=0}^{\infty} \delta_i^k u_i^k(\sigma) \right], \quad (6)$$

where $u_i^k(\sigma)$ is the payoff of player $i$ at stage $k$ induced by the strategy profile $\sigma$. A profile $\sigma$ is a Nash equilibrium if no player has a profitable deviation, i.e.,

$$U_i(\sigma) \geq U_i(\sigma_i', \sigma_{-i}) \text{ for all } i \in N, \text{ and } \sigma_i' \in \Sigma_i, \quad (7)$$

and it is a subgame-perfect equilibrium (SPE) if it induces a Nash equilibrium in every subgame, i.e.,

$$U_i(\sigma|h) \geq U_i(\sigma_i', \sigma_{-i}|h) \text{ for all } i \in N, \ h \in H, \text{ and } \sigma_i' \in \Sigma_i, \quad (8)$$

where $\sigma|h$ is the restriction of the strategy profile after history $h \in H$. 

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The characterization of equilibrium in repeated games has been developed in [1, 3, 4], see also [21]. Many papers examine pure-strategy equilibria [26, 11, 9, 10], but the mixed-strategy model is not that different. Actually, many papers examine a more general model of imperfect monitoring or stochastic games; see, e.g., Section 7 in [29]. The presentation here follows Berg and Schoenmakers [16].

In a repeated game, the play at each stage is strategically equivalent to playing an augmented stage game, where the continuation payoffs are included in the payoffs. Similarly, the set of subgame-perfect equilibrium payoffs in the repeated game corresponds to the set of all equilibrium payoffs in all the possible augmented games, i.e., set-valued games, that can be constructed from all the possible combinations of continuation payoffs; as we will now show.

Let $V$ be the set of subgame-perfect equilibrium payoffs in the discounted repeated game. Consider an augmented stage game where the payoff of each action profile $x \in A$ is given by

$$\tilde{u}_i(x) = (I - T)u(x) + Tz(x),$$

where $I$ is an $n \times n$ identity matrix, $T$ is a diagonal matrix with $\delta_1, \ldots, \delta_n$ on the diagonal, and $z(x)$ is the continuation payoff after $x$. The set of subgame-perfect equilibrium payoffs can now be characterized as follows [29, 10].

**Theorem 1.** The set $V$ is the largest bounded fixed point of the mapping $B$:

$$W = B(W) = \bigcup_{z(x) \in W} M((I - T)u(x) + Tz(x)). \quad (9)$$

Recall that $M((I - T)u(x) + Tz(x))$ denotes the set of Nash equilibrium payoffs in a game, where player $i$ receives the payoff $(1 - \delta_i)u_i(x) + \delta_i z_i(x)$ after the pure-action profile $x \in A$. The value $u(x)$ is the vector of stage game payoffs and $z(x)$ is the vector of continuation payoffs after action profile $x$, which is chosen from the set $W$. Theorem 1 means that the payoff set $V$ corresponds to the equilibria in a set-valued game where the payoffs are chosen from $K_x = \{(I - T)u(x) + Tz(x), z(x) \in V\}$ for each $x \in A$. We note that this is not an efficient formulation in terms of computation, since there are a lot of redundant stage games that produce exactly the same payoffs.
4. Computational method

We demonstrate first how to compute the set of equilibria in the \(c_{12}\) and \(c_{14}\) classes. These classes are used in the prisoner’s dilemma for the low discount factor values. Then, we examine the classes \(c_{1}\) and \(c_{4}\) that are needed for the medium discount factor values. Finally, we present the general scheme for \(2 \times 2\) games in Section 4.3.

4.1. Algorithm for the small discount factors: classes \(c_{12}\) and \(c_{14}\)

We can simplify the computational task in the prisoner’s dilemma when the discount factor values are small, i.e., when it is not possible to play the action profiles \(b\) and \(c\) in pure strategies. This means that the sets \(K_b\) and \(K_c\) are strictly below the set \(K_d\); see the left of Figure 4. The stage games that can appear in these games are of the form \(x0y0\) where \(x\) and \(y\) are 0, 1, or 2, and they correspond to the classes \(c_5\), \(c_6\), \(c_8\), \(c_9\), \(c_{12}\) and \(c_{14}\). The classes \(c_{12}\) (2010 or 1020) and \(c_{14}\) (1010) are the only ones that involve truly-mixed strategies. The other classes \(c_5\), \(c_6\), \(c_8\) and \(c_9\) are pure strategies and they are easy to solve. Thus, the algorithm only needs to find all the equilibria in the classes \(c_{12}\) and \(c_{14}\).

The examples of \(c_{12}\) and \(c_{14}\) classes are shown in the left of Figure 4. The black dots and the circles form the following games

<table>
<thead>
<tr>
<th></th>
<th>3.51,3.51</th>
<th>0.88,3.39</th>
<th>3.39,3.51</th>
<th>0.25,3.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.25,0.41</td>
<td>1,1</td>
<td>3.39,0.88</td>
<td>1,1</td>
<td></td>
</tr>
</tbody>
</table>

and they generate the \(c_{14}\) point \((1.70, 2.99)\) and the dashed \(c_{12}\) line from \((3.39, 3.5)\) to \((3.39, 1.70)\), respectively.

Let us now explain how to compute all the \(c_{12}\) lines. The set of all \(c_{12}\) equilibria is illustrated in grey in Figure 4 for a fictitious set-valued game with \(K_b = \{(0.25, 3.25)\}, K_d = \{(1,1)\},\) and the \(K_a\) and \(K_c\) sets are shown in the figure. We only consider here the vertical lines, and the horizontal lines could be handled in the same way. The vertical \(c_{12}\) lines are possible only when the \(K_a\) and \(K_c\) sets are vertically aligned, i.e., when the payoffs of player 1 are between 1 and 4. We need to find the maximum and minimum payoffs of player 2 given the payoff of player 1 \(v_1\), i.e., the northern and the southern border of the set. The northern border is easy as it follows the set \(K_a\). This is because the \(c_{12}\) line starts from the chosen payoff \(z \in K_a\), and we can choose \(z\) as high as possible from the set \(K_a\) given the payoff \(v_1\).
Figure 4: Illustration of different classes: c12 and c14 for $\delta = 0.25$ on the left, and c1 and c4 for $\delta = 0.4$ on the right. The left shows the third iteration of the algorithm for $\delta = 0.25$. The black dots generate the c14 payoff $(1.70, 2.99)$. The circles generate the dashed c12 line from $(3.39, 3.5)$ to $(3.39, 1.70)$. The right shows the tenth iteration of the algorithm for $\delta = 0.4$. The black dots generate the c1 rectangle shown by the dashed lines. The circles generate the three dashed c4 lines shown by the z-shaped figure.

Figure 5: A set-valued game and its c12 equilibrium set shown in grey. We only need to find the southern border of the set since the northern border follows the boundary of the set $K_a$. The southern border traces the changes in the maxima of $K_a$ and $K_c$ sets, which happen here at 1.5, 2.5, 3 and 3.5.
Algorithm 2: Compute the southern (northern) border for the vertical c12 (c4) set.

0. Find lowest (L) and highest (H) common values for $K_a$ and $K_c$.
Set $x = H$. Find minima of player 2 from $K_b$ and $K_d$.
1. Compute the mixed-strategy payoff for $a_2 = \max v_2 \in K_a$ s.t. $v_1 = x$ and
$c_2 = \max v_2 \in K_c$ s.t. $v_1 = x$.
2. If $x = L$, stop. Otherwise, set $x = \max(\text{na}, \text{nc}, L)$ and go to 1,
where \text{na} (\text{nc}) is the next value of player 1 below $x$ when the
maximum of $K_a$ ($K_c$) changes.

The only thing to compute for the c12 set is the minimum payoff of player 2 given $v_1$. To find this minimum payoff, we can use the monotonicity result, i.e., choose the player 2’s maximum payoffs from $K_a$ and $K_c$ given $v_1$, and the minimum payoffs from $K_b$ and $K_d$. We need to trace the maximum values of $K_a$ and $K_c$ for different values of $v_1$. In Figure 5, they change at 1.5, 2.5, 3 and 3.5.

Let us demonstrate Algorithm 2 for the example in Figure 4. In Step 0, we find $L = 1$ and $H = 4$, and set $x = 4$. In Step 1, we find $a_2 = 3.3$ and $c_2 = 0.3$, and compute the payoff in Eq. (3) for $(3.3, 3.25, 0.3, 1)$ which gives the payoff of 3.1 for player 2. In Step 2, we find $x = \max(3.5, 2.5, 1) = 3.5$. Thus, the minimum payoff of player 2 is 3.1 between $v_1$ values of 3.5 and 4. In Step 1, we update $a_2 = 3.6$, and compute the payoff for $(3.6, 3.25, 0.3, 1)$, which gives 2.5. In Step 2, as the maximum of $K_a$ changes next, we update $x = \max(3, 2.5, 1) = 3$. This is repeated until we get to $x = 1$.

Note that the c12 set need not be X-Y convex even if the $K_x$ sets are X-Y convex as Figure 5 demonstrates. However, the algorithm finds the correct c12 set since we find the extreme payoffs for the given values of $v_1$. This means that the $K_x$ sets need not be split into X-Y convex parts when the c12 sets are computed.

The first and third iterations in the prisoner’s dilemma for $\delta = 0.25$ are shown in Figure 4. The iteration sets are shown in grey and the c12 vertical sets are shown in darker shade. The vertical c12 payoffs are possible only when the $K_a$ and $K_c$ sets are vertically aligned, i.e., when the payoff of player 1 is between 3.25 and 3.63. On the third iteration, c12 payoffs are between 3.25 and 3.51.

Let us now explain how to compute all the c14 equilibria. When all the
Figure 6: The first and third iterations of the algorithm for $\delta = 0.25$. The c12 vertical payoffs are shown in darker shade. On the right, we can see that the c12 set jumps up at $v_1 = 3.39$ from value 1.70 to 2.56 since the maximum payoff of player 2 from $K_c$ drops from 0.88 to 0.41.

$K_x$ sets are X-Y convex, we can simply compute all the extreme points for the four (NE, NW, SE, SW) Pareto frontiers and form their X-Y convex hull; as explained in Algorithm 1, Figure 3 and Section 2.3.

When the $K_x$ sets are not X-Y convex, we cannot simply apply Algorithm 1 since the c14 set need not be X-Y convex in this case. However, we can split the $K_x$ sets into (connected) X-Y convex polygons, process all the combinations separately using Algorithm 1, and take the union of the generated sets. Figure 7 illustrates how a given set is split into three X-Y convex sets. All the $K_x$ sets are split in the same way and this creates a total of $3^4 = 81$ combinations of set-valued games that need to be solved. For example, Combination 2 means that we use the first set for $K_a-K_c$ and the second set for $K_d$. If we denote by $w_i$ the c14 set of combination $i$, then the whole c14 set is the union of the sets $w_i$.

Finally, the $B(V_i)$ set is the union of the sets c12, c14, $K_d$ and $K_a$ (the parts that are not below $K_b$ and $K_c$). Based on our experience, it seems that the c12 lines are actually redundant and always inside the c14 sets, if the eastern/northern border is generated properly for the c14 sets. The same is true for the pure strategies in $K_a$. See Figure 6 where both the $K_a$ set and the c12 set (darker grey) are inside the c14 set (both greys). Thus, we could
make the algorithm faster by computing only $K_d$ and the c14 sets.

4.2. Algorithm for the medium discount factors: classes c1 and c4

When the discount factors are medium in the prisoner’s dilemma, we only need to consider the pure equilibria and the classes c1 and c4. The other classes generate payoffs that are inside these sets. For example, when both players use truly-mixed strategies, the payoff is always inside the c1 rectangles. The class c4 is the only one that generates payoffs outside the c1 rectangles, and it uses the white space around $(3.6, 2.6) \in K_a$ that we discuss more in Section 5.

The c1 and c4 classes are illustrated on the right in Figure 4. The black dots and the circles form the following games

\[
\begin{array}{c|cc|cc|c}
3.2,3.2 & 1.3,3.2 & 2.66,3.59 & 1.42,3.59 \\
3.2,1.3 & 1.3,1.3 & 2.8,1.1 & 1,1.1
\end{array}
\]

and they generate the c1 rectangle (and all the points inside) shown by the dashed lines between the points and the three c4 lines shown by the dashed z-shaped figure.

Figure 8 shows the first two iterations of the algorithm for $\delta = 0.4$. On the first iteration, we only need to take the union of the c1 rectangle and the pure strategies $K_b$ and $K_c$ (cutted with the minimum payoff 1). On the second iteration, we also need to compute the (vertical) c4 set, which is shown with the darker shade on the right.

Since the iteration sets are always X-Y convex, we can reduce the task of computing $B(V_i)$ to finding certain extreme points of the different classes. Here, we only consider the payoffs above 3.5 for player 1; Theorem 6 in [16] shows that the square from $(1,1)$ to $(3.5,3.5)$ is a c1 set when $\delta \geq 2/7$. We
have three regions: the c1 region from 3.5 to 3.551, the c4 region from 3.551 to 3.628, and the pure equilibria from 3.628 to 3.821. These regions are illustrated by the vertical dotted lines in Figure 8. We get 3.551 by cutting away from $K_a$ the parts that are below 2.8 (the minimum payoff of player 2 in $K_b$); see the horizontal dotted line. We cannot form c1 for values below 2.8 since then player 2 is no longer indifferent between $a$ and $b$. The maximum payoff of player 1 in the cutted $K_a$ is 3.551.

The c1 region from 3.5 to 3.551 is easy to compute; we simply pick the corner points of the cutted set $K_a$. The same is true for the third region from 3.628 to 3.821; we pick the corner points of the cutted set $K_c$ (the parts that are above the minimum payoff 1).

The c4 region from 3.551 to 3.628 is the only one that requires processing. We can use Algorithm 2, and it generates the northern border for the c4 set. We choose the minimum values from $K_b$ and $K_c$ and trace the maximum values of $K_a$ and $K_c$ given the payoff of player 1.

### 4.3. General algorithm for $2 \times 2$ games

We present a method that is based on the iteration of the mapping $B$; we call this idea as the outer approach [26, 5]. First, a superset $V_0 \supseteq V$ is chosen. Then $V_{i+1} = B(V_i)$ is iterated until $V_{i+1} = V_i$ or they are close enough. If
\( V_{i+1} = V_i \) then \( V = V_i \) and the payoff set \( V \) has been found. If \( V_{i+1} \approx V_i \) then we only know that \( V \subseteq V_{i+1} \) and \( V_i \) is a subset of approximate equilibria, i.e., some players may gain a little (depending on how close the sets \( V_i \) and \( V_{i+1} \) are) by deviating from the approximate equilibrium strategies. There is no guarantee that \( V_{i+1} \) is close to \( V \).

The problem with the outer approach is the computation of \( B \), which corresponds to solving a set-valued game. We suggest that the set-valued game is split into regions based on the classification of games; see Algorithm 3. For example, we first check which classes are possible, and then use a specialized algorithm for each possible class with appropriate payoff values, i.e., pick only the parts of the \( K_x \) sets that satisfy the requirement of the class. The \( K_x \) sets may have to be split further based on the class. Classes c14 and c15 require that we split the sets into X-Y convex parts in order to guarantee that we find the correct payoff sets. Classes c4 and c12 require only connected sets but not X-Y convexity.

To speed up the computation, one should go through the regions in a proper order, starting from the ones that produce the largest sets of equilibria. This way one can check if the other regions need to be examined at all. If Region 1 produces a set of equilibrium payoffs and Region 2 cannot produce payoffs outside this set, then Region 2 can be skipped in this iteration. For example, all the classes, except c1 and the pure \( K_a \) set, are redundant in Figure 1(b). Moreover, Classes c12 and c14 are redundant in the prisoner’s dilemma for \( \delta \geq 2/7 \).

5. Prisoner’s dilemma

This is the first time the mixed-strategy equilibria is solved for a wide range of discount factors. For earlier research, see [7, 34, 35, 22, 28, 17, 11, 2]. We examine the following prisoner’s dilemma, when the players have the equal discount factors. We have chosen the square between \((1,1)\) and \((4,4)\) as \( V_0 \), which satisfies \( V \subseteq V_0 \).

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5,3.5</td>
<td>0,4</td>
</tr>
<tr>
<td>4,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

\(^{1}\)The Matlab codes can be found at https://users.aalto.fi/~kberg/mixed/.
Algorithm 3: Compute all mixed-strategy equilibrium payoffs

input: payoffs $u_i$, minimax values $v_i$, discount factors $\delta_i$.
output: Approximate equilibrium payoffs.

begin
  Initialize payoffs $V_0$.
  while $B(V_i)$ not close to $V_i$ do
    1. Solve the set-valued game using $V_i$ as the continuation payoffs. Split the problem into regions by the classes, X-Y convexity, and the monotonicity regions. Use a specialized algorithm for each class.
    2. Go through the regions, starting from the ones that give large sets of payoffs.
    3. Check if the region can produce payoffs that have not yet been found, and in this case, compute the payoffs produced by the region.
    4. $B(V_i)$ is the union of the sets. Update $V_{i+1} = B(V_i)$.

First, we examine the set of equilibria for $\delta = 1/4$. The value is chosen such that it is not possible to play in pure strategies the action profiles $b$ and $c$ giving payoffs $(0, 4)$ and $(4, 0)$.

The set of mixed-strategy equilibria is shown as the grey area in Figure 4. It shows the 16th iteration, which is likely to be very close to the actual payoff set. The iteration details are shown in Table 4. The area changes less than $10^{-5}$ on each of the last five iterations and $3 \cdot 10^{-8}$ on the last iteration. We can see that the payoff set is two dimensional and dramatically larger than both the pure strategy (the black dots in the figure) and the correlated pure-strategy payoffs (the diagonal line between $u(a)$ and $u(d)$). Thus, it is possible to obtain much more payoffs using mixed strategies. Moreover, the shape of the set is interesting and not something we expected to see. After 16th iteration, there are some numerical problems when performing the set intersections; this is probably related to numerical precision and rounding errors as it happens when the intersected corner points are close by.

It is a good question how close this approximation is to the actual payoff set. It could be that the payoff set keeps shrinking slowly and collapses to the set of pure-strategy payoffs, but we can show that this is not the case using the idea that we call as the inner approach [20, 6]. Here, a subset
Figure 9: Pure and mixed-strategy equilibria in the prisoner’s dilemma with $\delta = 0.25$ (left) and $\delta = 0.4$ (right). The figures are zoomed to the feasible and individually rational (FIR) payoffs, i.e., the quadrilateral between $(1,1)$, $(1,3.5)$, $(3.5,3.5)$ and $(3.5,1)$. The black dots show the pure-strategy equilibria. The black dots are always inside the $K_x$ sets, $x \in A$, which are shown on the right.

Table 1: The iteration details for $\delta = 0.25$ (left) and $\delta = 0.4$ (right). The running time of the algorithm is in seconds.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Time (s)</th>
<th>Area</th>
<th>Corner points</th>
<th>Time (s)</th>
<th>Area</th>
<th>Corner points</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.6</td>
<td>1.6924</td>
<td>28</td>
<td>0.05</td>
<td>6.9089</td>
<td>32</td>
</tr>
<tr>
<td>8</td>
<td>7.1</td>
<td>1.3938</td>
<td>168</td>
<td>0.09</td>
<td>6.8315</td>
<td>272</td>
</tr>
<tr>
<td>12</td>
<td>50</td>
<td>1.3930</td>
<td>440</td>
<td>0.23</td>
<td>6.8292</td>
<td>2560</td>
</tr>
<tr>
<td>15</td>
<td>154</td>
<td>1.3930</td>
<td>728</td>
<td>2.3</td>
<td>6.8292</td>
<td>13828</td>
</tr>
<tr>
<td>16</td>
<td>213</td>
<td>1.3930</td>
<td>840</td>
<td>6.4</td>
<td>6.8292</td>
<td>24264</td>
</tr>
</tbody>
</table>
of equilibria is generated such that the set itself contains the continuation payoffs that are required to generate these payoffs; these are so-called self-generating sets \[28\]. We do not know how to efficiently implement this idea in general, but illustrate it in this example. The advantage of the inner approach is that one has at least some subset of equilibria if such sets can be found. A similar idea was presented in \[10\], where the concept of self-supporting sets was introduced. Here, we allow the sets to depend on the other sets and the sets can be generated using different stage games, whereas a self-supporting set is given by a single stage game. Thus, the idea of the inner approach is more general but also more complicated than the idea of using self-supporting sets.

We show that the lines from \(p_3:5\); \(p_3:3\) to \(p_3:5\); \(p_3:1\) (or symmetrically to \(p_1:7\); \(p_3:5\)) are equilibrium payoffs. We construct the following game

\[
\begin{array}{|c|c|}
\hline
3.5,3.5 & 0.25,3.25 \\
\hline
3.5,0.875 & 1,1 \\
\hline
\end{array}
\]

which corresponds to the continuation payoffs \((3.5, 3.5)\), \((1, 1)\), \((2, 3.5)\), \((1, 1)\), respectively. For example, \((1 - \delta)(4, 0) + \delta(2, 3.5) = (3.5, 0.875)\). This stage game belongs to the class c12, and its Nash equilibria are the line from \((3.5, 3.5)\) to \((3.5, 1.75)\) and the point \((1, 1)\). This set is a subset of the payoff set if we can show that the continuation payoffs can be obtained as equilibrium payoffs. All the continuation payoffs, except \((2, 3.5)\), are generated by the stage game itself. However, the continuation payoff \((2, 3.5)\) belongs to the line from \((3.5, 3.5)\) to \((1.75, 3.5)\). Thus, the two lines support each other, and are equilibrium payoffs. Similarly, we could find some other equilibrium payoffs using the inner approach.

The pure (the black dots) and the mixed-strategy payoffs (the grey area) for \(\delta = 0.4\) are shown in Figure 4. The pure payoffs are much fewer and do not cover the space as widely. We can see that the mixed and correlated pure payoffs are quite close as the areas are 6.83 and \(7\frac{1}{7} \approx 7.14\) (FIR area).

An interesting observation is that the maximum payoff in mixed strategies is a little higher than in the pure strategies. In pure strategies, the maximum equilibrium payoff is 3.8, and approximately 3.82 in mixed strategies. This means that the mixed strategies are important and they do not only produce the interior payoffs that fill the gaps between the pure strategies.
Another interesting region is the white space around \((3.6, 2.6) \in K_a\). These payoffs are actually used in the stages games (chosen as a payoff from \(K_a\)) to generate mixed-equilibrium payoffs and thus can be realized in the repeated game, but they are not obtained as equilibrium payoffs. Actually, all the payoffs below the sets \(K_b\) and \(K_c\), i.e., below 2.6 cannot be obtained as pure equilibria from the set \(K_a\).

Both of these examples show that there are much more Pareto efficient outcomes compared to pure strategies. When \(\delta = 0.4\), there are basically two Pareto points in pure strategies: \((3.8, 1.4)\) and \((3.5, 3.5)\). In mixed strategies, there are many additional points, e.g., approximately \((3.62, 2.34)\), \((3.55, 2.97)\) and \((3.52, 3.31)\). This means that the players have more reasonable equilibria to choose from.

We also observe that the payoff set is a fractal for wide range of discount factors, which is not something that we expected. By fractal, we mean that there is a region that has a lot of details, i.e., more and more corner points are needed as we keep on iterating. This is something that may slow down the computation. However, we are not sure if this is due to the fact that we use X-Y convex sets in the algorithm.

5.1. Comparison of different strategies

The set of equilibria with different strategies is presented in Table 2 for different discount factor values. The question when the set of equilibria is equal to the feasible and individually rational (FIR) payoffs has been studied in [13]. The critical value for the discount factor is \(\delta = 2/7 \approx 0.286\) for the correlated pure strategies, which is dramatically lower than the value \(\delta = 8/13 \approx 0.615\) for the pure and mixed strategies. The value is the same for the pure and mixed strategies since the last payoff that is obtained lies on the line between the pure-action payoffs \(u(a)\) and \(u(b)\) (or \(u(c)\)) and this last payoff can only be obtained in pure strategies.

When \(\delta < 1/6 \approx 0.167\), it is only possible to play the stage game Nash equilibrium (action profile \(d\)) and get the payoff \(u(d) = (1, 1)\). When \(1/6 \leq \delta < 2/7 \approx 0.286\), it is only possible to play the action profiles \(a\) and \(d\) in pure strategies. The set of pure-strategy equilibrium payoffs under public correlation is the diagonal line between \(u(d) = (1, 1)\) and \(u(a) = (3.5, 3.5)\). The set of pure-strategy equilibrium payoffs is the following set of points between the payoffs \(u(d) = (1, 1)\) and \(u(a) = (3.5, 3.5)\) \([13]\): \(((1 - \delta) \sum_{i=0}^{N-1} \delta^i(1, 1) + \delta^N(3.5, 3.5), N = 0, \ldots, \infty)\); see the black dots in Figure 3. The equilibrium paths are of the form \(d^N a^\infty\), where \(d^k\) denotes
Table 2: Equilibrium payoffs with different $\delta$ for different strategies.

<table>
<thead>
<tr>
<th>Discount factor</th>
<th>Pure</th>
<th>Correlated pure</th>
<th>Mixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq \delta &lt; 1/6$</td>
<td>$u(d)$</td>
<td>$u(d)$</td>
<td>$u(d)$</td>
</tr>
<tr>
<td>$1/6 \leq \delta &lt; 2/7$</td>
<td>Points between $u(a)$ and $u(d)$</td>
<td>Line $u(a) - u(d)$</td>
<td>$\delta = 0.25$ in Fig. 4</td>
</tr>
<tr>
<td>$2/7 \leq \delta &lt; 8/13$</td>
<td>Points in FIR</td>
<td>FIR</td>
<td>$\delta = 0.4$ in Fig. 4</td>
</tr>
<tr>
<td>$\delta \geq 8/13$</td>
<td>FIR</td>
<td>FIR</td>
<td>FIR</td>
</tr>
</tbody>
</table>

that the action profile $d$ is repeated $k$ times. In mixed strategies, it is possible that the pure-action profiles $b$ and $c$ are realized. See the pure and mixed-strategy payoffs for $\delta = 0.25$ in the left of Figure 4.

When $2/7 \leq \delta < 8/13 \approx 0.615$, it is possible to play all the action profiles in pure strategies. The set of correlated pure payoffs is the set of FIR payoffs. The set of pure-strategy payoffs contains only some points in the FIR set; more specifically, the points have to lie inside the corresponding $K_2$ sets, $x \in A$. See the pure and mixed-strategy payoffs for $\delta = 0.4$ on the right of Figure 4. Finally, when $\delta \geq 8/13$, the set of equilibria is the FIR set for all the strategies. Note that the set of equilibria need not be inside the FIR set for the unequal discount factors.

6. Conclusions

This paper presents the first method to solve the set of mixed-strategy equilibria in infinitely repeated $2 \times 2$ games. It is now possible to analyze the games in more detail and compare what difference does it make if the players have different types of strategies available. The set of mixed-strategy equilibria can be much larger compared to the pure strategies. The additional payoffs are not only interior payoffs as one would expect, but there may be more Pareto efficient outcomes and the maximum payoff may be higher. Thus, the mixed strategies are vital when the players consider the reasonable equilibria that can be played in the game.

It is not easy to solve all the mixed-strategy equilibria in repeated games. We have presented the framework of set-valued games to illustrate what the problem is about. The main idea is to decompose the problem into smaller parts that can be easily solved. This involves using the classification of stage games, splitting the space into X-Y convex parts and finding the extreme
points of the set using the monotonicity results. It should be noted that it may not be a simple task to find the classes that are needed even in the most simple $2 \times 2$ games. The problem is even more difficult to solve in multiplayer games due to the nonlinearities and the loss of monotonicity. However, the ideas presented here can be used in finding a subset of equilibria in the more general games beyond the $2 \times 2$ setting.

There are many ways to enhance the computation. The method is based on the outer approach where we iterate the set-valued mapping. The iteration starts from a superset that contains the equilibrium payoffs, and the payoffs that cannot be obtained in the game are removed on each iteration. There is no guarantee how many iterations are needed and when it has converged. We have observed that the number of corner points keeps increasing fast, and this is one issue that can be improved. It is possible to use outer approximations to reduce the number of corner points, since it is not necessary to keep all the details during the iteration when the payoff set has not converged to its correct shape. The exact details should only be computed in the final iterations. Of course, if it is acceptable to compute approximate equilibria, it is not necessary to compute all the corner points at any point.

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References


