This is an electronic reprint of the original article.
This reprint may differ from the original in pagination and typographic detail.

Tzarouchis, Dimitrios; Sihvola, Ari

Polarizability and Light Scattering by Subwavelength Graded-Index Plasmonic Spheres

Published in:

2018 IEEE AP-S Symposium on Antennas and Propagation and URSI CNC/USNC Joint Meeting (AP-S/URSI 2018) Proceedings, July 8 - 13, 2018, Boston, Massachusetts, U.S.A.

DOI:

[10.1109/APUSNCURSINRSM.2018.8608310](https://doi.org/10.1109/APUSNCURSINRSM.2018.8608310)

Published: 01/01/2018

Document Version

Peer reviewed version

Please cite the original version:

Tzarouchis, D., & Sihvola, A. (2018). Polarizability and Light Scattering by Subwavelength Graded-Index Plasmonic Spheres. In *2018 IEEE AP-S Symposium on Antennas and Propagation and URSI CNC/USNC Joint Meeting (AP-S/URSI 2018) Proceedings, July 8 - 13, 2018, Boston, Massachusetts, U.S.A.* (pp. 871-872). (Digest of the IEEE Antennas and Propagation Society International Symposium). IEEE.
<https://doi.org/10.1109/APUSNCURSINRSM.2018.8608310>

This material is protected by copyright and other intellectual property rights, and duplication or sale of all or part of any of the repository collections is not permitted, except that material may be duplicated by you for your research use or educational purposes in electronic or print form. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone who is not an authorised user.

Polarizability and Light Scattering by Subwavelength Graded-Index Plasmonic Spheres

Dimitrios C. Tzarouchis and Ari Sihvola

Department of Electronics and Nanoengineering, Aalto University, Finland; dimitrios.tzarouchis@aalto.fi

Abstract—Light scattering by a subwavelength sphere exhibiting radially inhomogeneous permittivity is here presented. The theoretical foundations describing the scattering response are given in a simple manner and the concept of polarizability is generalized incorporating the inhomogeneity effects. Two illustrative examples are briefly discussed, i.e., a power-law and a Drude-like inhomogeneous profile, exposing the physical mechanisms of their scattering response. The presented results can open the path towards the implementation of graded-index subwavelength particles in modern nanoantenna applications.

I. INTRODUCTION

Scattering by a subwavelength homogeneous sphere is a simple, yet fundamental, problem that can be found in the core of many modern power control/harvesting applications [1], especially at the terahertz-optics regime. Its universality emerges from the fact that the scattering response of a small dielectric sphere can be rigorously quantified by a (normalized) polarizability expression $\alpha = \frac{\varepsilon_1 - \varepsilon_h}{\varepsilon_1 + 2\varepsilon_h}$ [2]. This simple expression reveals a wealth of physical phenomena, such as the position of the localized surface plasmon (LSP) or plasmonic resonances ($\varepsilon_1 = -2\varepsilon_h$).

In this work the generalized concept of the polarizability is presented by assuming the case of a sphere with a radially inhomogeneous (graded-index) permittivity profile. This kind of profiles occur either naturally [3] or as a result of sophisticated engineering processes [4]. Previous studies expanded both Mie and electrostatic scattering theory for certain cases of inhomogeneous profiles (see for example [5], [6], [7]). Here we present some simple analytical formulas for several cases of profiles such as the power-law, exponential, and inhomogeneous Drude profile. Special attention to the LSP resonances is given, exposing the underlying scattering mechanisms of the aforementioned spheres.

II. THEORY & RESULTS

Let us assume a sphere (subscript 0 for external and 1 for internal domain) of radius r_1 (Fig. 2) subject to a uniform electrostatic potential $\Phi_0(r, \theta) = -E_0 r$, causing a scattering potential of dipolar character, $\Phi_s(r, \theta) = \frac{B_0}{r^2} \cos \theta$, and an internal potential $\Phi_1(\mathbf{r}) = f(r) \cos \theta$ [8]. Requiring that the electric flux density be divergence-less at the internal region we have, $\nabla \cdot \mathbf{D}_1 = \nabla \cdot (\varepsilon_r(r) \nabla \Phi_1(r, \theta)) = 0$, resulting to the following O.D.E

$$f''(r) + \left(\frac{2}{r} + \frac{\varepsilon_r'(r)}{\varepsilon_r(r)} \right) f'(r) - \frac{2}{r^2} f(r) = 0 \quad (1)$$

Fig. 1: A radially inhomogeneous sphere immersed in a host medium subject to a constant excitation field.

The main focus here is to study the cases where Eq. (1) obtains a closed form solution, hence extracting an analytical expression of the unknown scattering amplitudes. For these cases one can express the radial function as $f(r) = A_1 A(r) + B_1 B(r)$, where $A(r)$ is a constant-type and $B(r)$ is a dipole-type solutions of the radial function. It is tacitly assumed that function $A(r)$ is well-behaving at the origin, while $B(r)$ contains a singularity. This is, however, not always true and a special mathematical treatment is required, e.g., introduction of a singularity subtracting region at the center. Assuming $A(r)$ being a regular, non-singular, function the potential in the internal region is $\Phi_1(r, \theta) = A_1 A(r) \cos \theta$.

The unknown scattering coefficients, B_0 and A_1 , can be calculated by enforcing the required boundary conditions, i.e., $\partial_\theta \Phi_0(r, \theta) = \partial_\theta \Phi_1(r, \theta)$, and $\varepsilon_h \partial_r \Phi_0(r, \theta) = \varepsilon_r(r) \partial_r \Phi_1(r, \theta)$, at $r = r_1$. After some simple algebra, the polarizability (external coefficient) reads

$$B_0 = \frac{C \varepsilon_r(r_1) - \varepsilon_h}{C \varepsilon_r(r_1) + 2\varepsilon_h} r_1^3 E_0 \quad (2)$$

with $C = r_1 \frac{A'(r_1)}{A(r_1)}$ being the inhomogeneity parameter. Interestingly, this parameter is a function of $A(r)$ evaluated at the surface of the sphere, $r = r_1$, hence an effective permittivity of this inhomogeneous inclusion can be extracted *à la* Maxwell Garnett [9]. Inspecting Eq. (2) we observe that the inhomogeneity elegantly modifies the polarizability expression, generalizing the plasmonic resonance condition in a simple manner, i.e.,

$$\varepsilon_r(r_1) = -\frac{2\varepsilon_h}{C} \quad (3)$$

Apparently, the homogeneous case is when $A(r) = r$ and $C = 1$.

III. DISCUSSION & CONCLUSIONS

As an illustrative example we present the results for two different cases: the power-law profile and the inhomogeneous Drude-like profile. Note that the O.D.E. in Eq. (1) obtains a closed-form analytical solution for certain permittivity profiles such as power-law [7], polynomial [10], and exponential profiles. In our analysis the results are validated by implementing an iterative code of a multilayer sphere exhibiting the same profile [6]. For the case of the power-law profile, i.e., $\varepsilon_r(r) = \varepsilon_1 r^n$, the radial function reads

$$f(r) = A_1 r^{p_1} + B_1 r^{p_2}, \quad (4)$$

with $p_{1,2} = -\frac{1}{2} \left(n + 1 \mp \sqrt{(n+1)^2 + 8} \right)$, where n is a the power factor [7]. The inhomogeneity parameter in this case is $C = p_1$. The plot of the polarizability as a function of both permittivity and power factor can be found in Fig. 2 reveals a shift in the polarization enhancement as a function of n .

The second case of inhomogeneous Drude-like profile [11], $\varepsilon_r(\omega, r) = \varepsilon_\infty - \frac{\omega_p^2}{\omega^2} (1 + br^n)$, where ω_p is the plasma frequency, n is the power factor, and b is the inhomogeneity factor. For this case function $A(r)$ can be expressed as a hypergeometric function ${}_2F_1(\alpha, \beta; c; z)$ [12, Ch. 15] with the arguments

$$A(r) = r a_r {}_2F_1 \left(\nu_1, \nu_2; 1 + \frac{3}{n}; -\frac{ab}{a - \varepsilon_\infty} r^n \right) \quad (5)$$

where $a = \frac{\omega_p^2}{\omega^2}$, $a_r = \left(\frac{ab}{n^2(a - \varepsilon_\infty)} \right)^{1/n} n^{2/n}$, and $\nu_{1,2} = \frac{1}{2n} \left(n + 3 \mp \sqrt{(n+1)^2 + 8} \right)$. The inhomogeneity parameter is a function of ω , ω_p , n , and the inhomogeneity factor b . This compact description enables the study of inhomogeneous structures with dispersive characteristics, such as inhomogeneities in metals or other materials [13] at the optical-IR frequencies.

Summarizing, the expression of Eq. (2) generalizes the concept of a homogeneous polarizability, allowing us to rigorously explore the non-trivial physical mechanisms for a whole new family of graded-index particles. Additionally, this simplified description can be used to reverse-engineer the inhomogeneity coefficient C fitting the experimental data, suggesting an alternative explanation to the experimentally observed deviations of plasmonic resonances on deeply subwavelength spheres [14].

The introduced description can be also implemented for a wide range of practical cases such as the accurate modeling of inhomogeneous structures (stratified spheres, transformation optics [13]), the implementation of temperature gradients (via a varying permittivity profile), the modeling of diffusive effects especially between interfaces or for extremely small particles [14], where interfaces are not hard but rather follow a radially dependent distribution. It is envisioned that the presented study will stimulate novel power control/harvesting

Fig. 2: Polarizability (in logarithmic scale) of a sphere with a power-law profile as a function of the power factor and the permittivity parameter.

ideas for nanophotonic applications, such as the implementation of subwavelength plasmonic particles exhibiting Luneburg, Eaton, or more exotic graded-index profiles.

REFERENCES

- [1] H. A. Atwater and A. Polman, "Plasmonics for improved photovoltaic devices," *Nature Materials*, vol. 9, no. 3, pp. 205–213, 2010.
- [2] C. F. Bohren and D. R. Huffman, *Absorption and scattering of light by small particles*. John Wiley & Sons, 2008.
- [3] J. Cai, J. P. Townsend, T. C. Dodson, P. A. Heiney, and A. M. Sweeney, "Eye patches: Protein assembly of index-gradient squid lenses," *Science* (80-.), vol. 357, no. 6351, pp. 564 LP – 569, 2017.
- [4] T. Zentgraf, Y. Liu, M. H. Mikkelsen, J. Valentine, and X. Zhang, "Plasmonic Luneburg and Eaton lenses," *Nat Nano*, vol. 6, no. 3, pp. 151–155, 2011.
- [5] M. Kerker, *The Scattering of Light and Other Electromagnetic Radiation: Physical Chemistry: A Series of Monographs*. Academic press, 2013, vol. 16.
- [6] A. Sihvola and I. V. Lindell, "Transmission Line Analogy for Calculating the Effective Permittivity of Mixtures with Spherical Multilayer Scatterers," *J. Electromagn. Waves Appl.*, vol. 2, no. 8, pp. 741–756, 1988.
- [7] L. Dong, G. Q. Gu, and K. W. Yu, "First-principles approach to dielectric response of graded spherical particles," *Phys. Rev. B*, vol. 67, no. 22, p. 224205, 2003.
- [8] J. D. Jackson, *Classical Electrodynamics*. Wiley Online Library, 1975.
- [9] A. H. Sihvola, *Electromagnetic mixing formulas and applications*. IET, 1999, no. 47.
- [10] V. Laquerbe, R. Pascaud, T. Callegari, L. Liard, and O. Pascal, "Analytical Model to Study the Electrostatic Resonance of Subwavelength Radially Inhomogeneous Negative Permittivity Spheres," *IEEE Antennas Wirel. Propag. Lett.*, vol. 16, pp. 2894–2897, 2017.
- [11] L. Sun and K. W. Yu, "Broadband electromagnetic transparency by graded metamaterials: scattering cancellation scheme," *J. Opt. Soc. Am. B*, vol. 28, no. 5, p. 994, 2011.
- [12] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions: with formulas, graphs, and mathematical tables, 9th printing*. Dover, 1972.
- [13] A. Vakil and N. Engheta, "Transformation Optics Using Graphene," *Science* (80-.), vol. 332, no. 6035, pp. 1291 LP – 1294, 2011.
- [14] T. Christensen, W. Yan, S. Raza, A.-P. Jauho, N. A. Mortensen, and M. Wubs, "Nonlocal Response of Metallic Nanospheres Probed by Light, Electrons, and Atoms," *ACS Nano*, vol. 8, no. 2, pp. 1745–1758, 2014.