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Gradient-elastic stress analysis near cylindrical holes
in a plane under bi-axial tension fields

Sergei Khakalo∗ Jarkko Niiranen†

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Abstract

This article is devoted to a gradient-elastic stress analysis of an infinite plate weakened by a
cylindrical hole and subjected to two perpendicular and independent uni-axial tensions at infinity.
The problem setting can be considered as an extension and generalization of the well-known Kirsch
problem of the classical elasticity theory which is here extended with respect to the external
loadings and generalized with respect to the continuum framework. A closed-form solution in
terms of displacements is derived for the problem within the strain gradient elasticity theory on
plane stress/strain assumptions. The main characters of the total and Cauchy stress fields are
analyzed near the circumference of the hole for different combinations of bi-axial tensions and
for different parameter values. For the original Kirsch problem concerning a uni-axially stretched
plate, the analytical solution fields for stresses and strains are compared to numerical results.
These results are shown to be in a full agreement with each other and, in particular, they reveal
a set of new qualitative findings about the scale-dependence of the stresses and strains provided
by the gradient theory, not common to the classical theory. Based on these findings, we finally
consider the physicalness of the concepts total and Cauchy stress appearing in the strain gradient
model.

Strain gradient elasticity, Kirsch problem, Cauchy stress, total stress, plane stress/strain problem

1 Introduction

The fundamental knowledge of the mechanical behavior of solids has been successfully accumulated by
studying certain applied model problems of the classical theory of elasticity, e.g., the Lamé problem,
the Kirsch problem and the Eshelby’s inclusion problem. In particular, analytically derived stress and
strain fields allow one to investigate both the qualitative and the quantitative material behavior of
solids such as predicting locations of stress concentrations and further failure for brittle materials, or
plasticity initiation for ductile materials.

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On the macro scale, the classical solutions of these model problems, which have applicational relevance as well, are in a very good agreement with experimental results. On smaller scales, however, limitations of the classical theory of elasticity are evident since the solutions given by the classical theory are size-independent, i.e., the classical models are not able to take into account the relation between geometric dimensions of the body and inner micro structures of the material such as grain or crystal lattice sizes, for instance (already studied in the 1850’s by Cauchy who is considered as the father of the classical continuum models; see the references in [1]). In the Lamé problem, for instance, focusing on an annulus with radial boundary loads, the stress field depends on the ratio of the internal and external radii of the annulus and the applied boundary stress distributions only [2]. In the Kirsch problem, in turn, focusing on an infinite plate weakened by a hole in uni-axial tension (analyzed thoroughly in this work), the stress concentration factor is equal to 3 independently of the hole size, and the position of the stress peak remains fixed [3]. According to experimental results, however, the basic reactions of structures are essentially size-dependent. A couple of simple examples of size dependence can be found in [4, 5] showing a significant increase in the bending rigidity of a beam specimen thin enough, and in [6] reporting an increase in Young’s modulus for ZnO nano-wires with the diameter small enough. As demonstrated by these examples, the classical theory of elasticity fails to describe the fundamental mechanical behavior of solids on micro scales in which the dimensions of the body are comparable to the size of the micro structure of the material. The main reason for these deficiencies, which can be considered as a defence for the classical continuum theories as well, are the underlying well-defined axioms of the conception of the Cauchy continuum.

During the past century, so-called generalized continuum theories, such as micromorphic theories, strain gradient theories, micropolar theories, couple stress theories and non-local theories [7], have been developed for overcoming the single-scale shortcomings of the classical continuum theory – without losing the powerful homogenizing nature of the continuum approach [7, 8], however. Actually, one of the main motivations for further developments of single-scale continuum mechanics towards multi-scale capabilities has been the fact that theories and computational methods developed for studying small scale phenomena, such as molecular dynamics, are often inefficient in many applications eventually ruled by macro-scale conservation laws. The Cosserat brothers are considered to be the founders of the generalized continuum mechanics due to their monograph published in 1909 [9]. Unlike the classical continuum theory giving each material point three translational degrees of freedom, the Cosserat continuum theory introduces three additional rotational degrees of freedom. In such a media, a new material parameter with the dimension of length appears, however. In particular, according to the Cosserat theory the solution of the Kirsch problem is sensitive to this micro-length parameter, and for holes small enough the stress concentration factor tends to the value 1.8 [10] – but still not to unity being the desired limit for a structure without any holes, which have served as a motivation for further developments of the theory and analysis.

A more advanced continuum model was introduced by Mindlin in 1964 [11] which is considered as
the second milestone of the field. According to Mindlin’s model, each material point has nine additional micro-deformational degrees of freedom. As a low-frequency (long wave length) approximation to this theory, Mindlin proposed the strain gradient theory (similar to Toupin’s generalization of the couple-stress theory [12]) in which the strain energy density depends on both strains and strain gradients. In this theory, on a par with the Cauchy stress work conjugate to the strain, there appears a double stress work conjugate to the (first) strain gradient. In 1965, Mindlin proposed an extension to the strain gradient theory by including the second strain gradients into the expression of the strain energy density [13]. A simplified (first) strain gradient theory adopted in this work was originally proposed by Altan and Aifantis in 1997 [14] and later further developed, generalized and applied by many others; see especially the works by Georgiadis (2003) [15] and Lazar et al. (2005) [16] in the context of dislocations and cracks (for more detailed reviews, see [1, 17]). In particular, simplified models give a possibility to analytically solve some model problems in order to understand the fundamental implications of the underlying continuum theory. In general, one of main features of the non-classical continuum theories is that solving comparatively simple model problems becomes evidently more complicated than solving their classical counterparts, even within the simplified gradient theories of elasticity. Another complicating feature common to almost all extended continuum theories is the appearance of additional material parameters. Methods for finding appropriate values for the new material parameters for the simplified models, in particular, have been reported in [4, 18, 19, 5, 20], while methods for determining the additional material parameters of the original Mindlin model are reported in [21, 22].

Regardless of the more complex structure of analytical solutions, there are a number of model problems which have been solved within the strain gradient theory of elasticity by analytical means during the past decades. The strain gradient effects around spherical and cylindrical inclusions and cavities in the field of spherically and cylindrically symmetric tensions have been considered in [23] and [24], respectively. Although the Eshelby’s inclusion problem in the framework of the classical elasticity theory was solved by Eshelby already in 1957 [25], an extension to the strain gradient theory of elasticity was introduced comparatively recently, see [26, 27, 28, 29, 30, 31, 32, 33, 34]. The influence of the strain gradient on the stress concentration at spherical and cylindrical holes in a field of uniaxial tension has been analyzed in [35] and [36], respectively. Axisymmetric problems (e.g., the Lamé problem) for strain gradient elasticity have been discussed in [37], whereas a solution of the Lamé problem in the framework of a simplified strain gradient elasticity theory has been obtained in [38] and [39]. The main common result of these articles is that the values of the stress concentration factor related to the Cauchy stress obtained by the gradient theory are at large variance compared to the corresponding classical ones. Although this review is limited to problems of medium with cavities or inclusions, it is worth noting that there are studies on crack problems within the gradient elasticity with surface energy [40, 41], non-singular dislocations [42, 43] and disclinations [44] in the theory of gradient elasticity and dislocation based fracture mechanics [45, 46].
When moving away from the model problems solvable by analytical means only numerical approaches such as finite element methods can be considered as a general-purpose tool. For gradient-elastic problems and other non-classical theories, the related literature is very limited, however. The main reason for this is clearly the fact that the higher-order partial differential equations of the strain gradient theories, in particular, are not a straightforward application for the classical finite element methods due to the higher-order regularity requirements concerning the basis functions of the methods. For further discussion on the computational methods, we refer to [47, 48, 49, 50, 51, 52] and our recent contributions [53, 54, 55, 56] focusing on the variational formulation, implementation and verification benchmarks and hence providing a reference for the analytical solutions of the present contribution.

As indicated above, the Kirsch problem can be considered as one of the most illustrative model problems for analyzing the fundamental features of the generalized continuum theories. In particular, the stress concentration factor and the location of the stress peak(s) have direct implications for further applications, e.g., in porous materials, in perforated micro-structures or even in metamaterials with holes of different scales. For instance, in fatigue life estimation the stress concentration factor $K_t$ (size-independent in the classical elasticity) is typically modified such that the so-called fatigue notch factor $K_f$ will be size-dependent since fatigue is known to be a size-dependent phenomena [57]. More precisely, the fatigue notch factor can be calculated by Peterson’s equation $K_f = 1 + (K_t - 1)/(1 + a/r)$, with $r$ denoting the notch radius and $a$ denoting a material constant. It can be seen that $K_f \to K_t$ for large $r$, whereas for small notches $K_f \to$ tends to unity. Within the gradient elasticity theory, the Kirsch problem has been studied by Eshel and Rosenfeld in [36] providing a quasi-analytical solution by adopting Mindlin’s strain gradient elasticity theory of form II [58] with two independent micro structural parameters. However, the solution corresponds to a modified version of the original Kirsch problem with the pressure applied at the hole surface (not at infinity), and the constants present in the solution are determined numerically.

In the current article, we generalize the Kirsch problem to cover an infinite plate weakened by a hole in the field of two independent and perpendicular uni-axial tensions (applied at infinity). We adopt the one parameter modification of Mindlin’s strain gradient elasticity theory of form II which will be briefly recalled in Section 2. An analytical solution of the generalized Kirsch problem is derived in terms of displacements in Section 3. In particular, all of the coefficients appearing in the solution are derived symbolically. In Section 4 (and Appendix), the total and Cauchy stresses and the related stress concentration factors are analyzed near the hole for different loading cases and parameter values. Perhaps the most significant finding of the analysis is that for hole radii less than a certain critical value the total stress demonstrates a qualitatively and quantitatively different behavior in comparison with the Cauchy stress which, in turn, essentially differs from the one of the classical elasticity theory. In Section 5, for the case of a uni-axially stretched plate (the original Kirsch problem) the analytical solutions fields are compared to numerical results including a detailed analysis of different strain
components in both Cartesian and cylindrical coordinate systems and for different parameter values.

Finally, in Section 6, we discuss about the significance of the findings and, accordingly, consider the physicalness of the concepts Cauchy and total stress.

2 One-parameter strain gradient elasticity theory

According to the strain gradient elasticity theory of Mindlin [11] (see the works of Toupin [12] or Mindlin and Eshel [58] as well) the variation of the strain energy in a volume \( \Omega \) and, correspondingly, the variation of the work done by external forces can be defined, respectively, as

\[
\delta W^{\text{int}} = \int_{\Omega} \delta \mathcal{W}^{\text{int}} dV = \int_{\Omega} (\tau : \varepsilon(\delta u) + \mu : \nabla \varepsilon(\delta u)) dV, \tag{1}
\]

\[
\delta W^{\text{ext}} = \int_{\Omega} F \cdot \delta u dV + \int_{\partial \Omega} P \cdot \delta u dS + \int_{\partial \Omega} R \cdot (n \cdot \nabla \delta u) dS, \tag{2}
\]

where \( \tau \) denotes the Cauchy, or better Cauchy-like, stress tensor with the strain tensor \( \varepsilon \) as its work conjugate acting on the displacement variation \( \delta u \), whereas \( \mu \) stands for the double stress tensor of the third rank. The density of the body forces is denoted by \( F \), whereas \( P \) and \( R \) denote the traction force and double traction force, respectively, with \( n \) denoting the unit vector normal to the boundary surface of the volume. For simplicity, we have omitted traction forces acting on possible wedge lines of the body. Here and in what follows, the tensor multiplication sign \( \otimes \) is omitted (for example, \( ab = a \otimes b \)), other notation follows [59]. We remark that a slightly different expression for (2), including a double body force, in particular, has been proposed by Bleustein [60] (see [61] as well).

The principle of virtual work, in the form \( \delta W^{\text{int}} = \delta W^{\text{ext}} \) valid for all kinematically admissible displacement variations \( \delta u \), implies the equilibrium equation and boundary conditions as

\[
\nabla \cdot \sigma + F = 0 \quad \text{in } \Omega, \tag{3}
\]

\[
\n \cdot \sigma - \nabla_s \cdot (n \cdot \mu) + (\nabla_s \cdot n) mn : \mu = P \quad \text{or} \quad u = u_P \quad \text{on } \partial \Omega_P, \tag{4}
\]

\[
\nnn : \mu = R \quad \text{or} \quad n \cdot \nabla u = u_R \quad \text{on } \partial \Omega_R, \tag{5}
\]

where \( \sigma = \tau - \nabla \cdot \mu \) introduces the so-called total stress tensor [40] and \( \nabla_s = (I - nn) \cdot \nabla \) stands for the surface part of the gradient operator \( \nabla \). The boundary surface is composed of disjoint sets as \( \partial \Omega = \partial \Omega_P \cup \partial \Omega_R \).

In case of linear isotropic gradient-elastic continuum, the strain energy density can be written as a quadratic form of \( \varepsilon \) and \( \nabla \varepsilon \) in the form ([14], equation (5))

\[
\mathcal{W}^{\text{int}} = \frac{1}{2} \varepsilon : C : \varepsilon + \frac{1}{2} l^2 C :: ((\nabla \varepsilon)^T \cdot \nabla \varepsilon), \tag{6}
\]

with \( C = 2\mu I + \lambda II \). Here, \( C \) stands for the fourth rank tensor of elastic modulii, \( \mu \) and \( \lambda \) denote the Lamé parameters, \( ^4I \) and \( I \) are the symmetric forth and second rank unit tensors, respectively. This energy expression with only one micro-structural parameter \( l \) represents the simplest variant of
Mindlin’s strain gradient elasticity theory of form II [58] (see the energy postulates and discussion in [14, 15, 16]). For simplicity, \( l \) is assumed to be constant. Accordingly, the Cauchy stress \( \tau \), work-conjugate to \( \varepsilon \), and the double stress \( \mu \), work-conjugate to \( \nabla \varepsilon \), are defined as partial derivatives of the strain energy density (6) with respect to \( \varepsilon \) and \( \nabla \varepsilon \), respectively, as

\[
\tau = \frac{\partial W^{\text{int}}}{\partial \varepsilon}, \quad \mu = \frac{\partial W^{\text{int}}}{\partial \nabla \varepsilon},
\]

(7)

leading for constant Lamé parameters to the constitutive laws of the form (cf. [16])

\[
\tau = 2\mu \varepsilon + \lambda \text{tr}(\varepsilon)I, \quad \mu = l^2 \nabla \tau.
\]

(8)

The Cauchy stress tensor \( \tau \) and the strain tensor \( \varepsilon \) are coupled by the Hooke’s law as usual, whereas the double stress tensor \( \mu \) is written in terms of the Cauchy stress tensor giving the total stress tensor in the form

\[
\sigma = \tau - l^2 \Delta \tau.
\]

(9)

It can be seen that in the classical limit, \( l \to 0 \), the total stress is equal to the Cauchy stress.

Within this framework, the stress equation of equilibrium as well as the traction and double traction forces are rewritten below in terms of the Cauchy stress:

\[
\nabla \cdot (\tau - l^2 \Delta \tau) + F = 0 \quad \text{in } \Omega,
\]

(10)

\[
\n \cdot (\tau - l^2 \Delta \tau) - l^2 \nabla \cdot (\n \cdot \nabla \tau) + l^2 (\n \cdot \n) \n : \nabla \tau = P \quad \text{on } \partial \Omega_P,
\]

(11)

\[
\n : \nabla \tau = R \quad \text{on } \partial \Omega_R.
\]

(12)

Finally, by assuming small deformations, the strain tensor \( \varepsilon \) takes its standard form

\[
\varepsilon = \frac{1}{2} (\nabla u + (\nabla u)^T),
\]

(13)

and after substituting (8) and (13) into (10), the final displacement equation of equilibrium can be written as

\[
(1 - l^2 \Delta) [\mu \Delta u + (\lambda + \mu) \nabla \nabla : u] + F = 0 \quad \text{in } \Omega.
\]

(14)

### 3 Formulation and analytical solution of the plane problem

In this section, we first describe the problem setting for the plane problem and then derive the analytical solution of the problem in terms of displacements within the strain gradient theory of elasticity reviewed in the previous section.

#### 3.1 Problem description

A thin infinite plate weakened by a circular hole of radius \( a \) is schematically shown in Figure 1. At infinity, the plate is uni-axially stretched or compressed by a uniformly distributed force \( \sigma_\infty \) in one
direction and uni-axially stretched or compressed by a uniformly distributed force $\xi \sigma_\infty$ in the other direction, with a loading parameter $\xi \in \mathbb{R}$.

The problem will be written as a plane stress/strain problem of gradient elasticity, i.e., the equilibrium equation (10), or simply (14), and the boundary conditions (11)–(12) will be reduced to their planar counterparts.

![Infinite plate weakened by a hole under distributed uni-axial forces at infinity.](image)

It should be mentioned that the case of applied shear forces $\tau_\infty$ (at infinity) can be transformed to an equivalent problem of Figure 1 by applying uni-axial tension and compression ($\sigma_\infty = \tau_\infty$, $\xi = -1$).

### 3.2 Derivation of the gradient-elastic solution

As a first step, it is assumed that the problem can be solved in the plane stress or plane strain state with the in-plane parts of the solution field taken into account only. The classical assumptions for displacements, strains and stresses are adopted (see [53] for details and references). Accordingly, the plane displacement field can be written in the polar coordinate system $(r, \theta)$ as

$$u = u_r(r, \theta)e_r + u_\theta(r, \theta)e_\theta. \quad (15)$$

The corresponding strain tensor field is of the form

$$\varepsilon = \varepsilon_{rr}e_r \otimes e_r + \varepsilon_{r\theta}e_r \otimes e_\theta + \varepsilon_{\theta\theta}(e_\theta \otimes e_\theta + e_r \otimes e_r), \quad (16)$$

whereas the Cauchy stress tensor is written as

$$\tau = \tau_{rr}e_r \otimes e_r + \tau_{r\theta}e_r \otimes e_\theta + \tau_{\theta\theta}(e_\theta \otimes e_\theta + e_r \otimes e_r). \quad (17)$$

In general, analytical solutions of plane strain or plane stress problems can be derived in two ways. The first way is to solve the problem in terms of displacements and then calculate the strains and stresses from the obtained solution. The other way is to solve the problem in terms of stresses by
introducing the so-called Airy stress function \[62\]. This approach has been adopted for the gradient-elastic case of the Lamé problem by Aravas in \[39\]. However, it is not so straightforward to properly extend the Airy stress function approach to the strain gradient case since there appear additional stress measures, i.e., double and total stress tensors. Moreover, this approach implies the derivation of displacements by integrating the strains in contrast to the direct approach of the first way where the strains are obtained as derivatives of the displacements. In this article, the first option is used.

In view of (15), we first recall that in the polar coordinate system operators \( \nabla \) and \( \Delta \) take the forms
\[
\nabla = e_r \frac{\partial}{\partial r} + e_\theta \frac{\partial}{\partial \theta}, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \tag{18}
\]
Next, we set \( F = 0 \) in (14). For the boundary conditions, in turn, we notice that in a plane problem the surface gradient reduces to \( \nabla_s = s \frac{\partial}{\partial s} \) and hence (11) and (12), respectively, take the forms \[53\]
\[
\mathbf{n} \cdot (\mathbf{\tau} - l^2 \Delta \mathbf{\tau}) - l^2 s \cdot \frac{\partial}{\partial s} (\mathbf{n} \cdot \nabla \mathbf{\tau}) = \mathbf{P} \text{ on } \partial \Omega_P, \tag{19}
\]
\[
l^2 \mathbf{n} : \nabla \mathbf{\tau} = \mathbf{R} \text{ on } \partial \Omega_R, \tag{20}
\]
where \( s \) denotes the unit vector tangential to the boundary.

We note that by following \[36\] focusing on the modified version of the original Kirsch problem the solution of the displacement equilibrium equation (14) could be expressed in terms of a vector function \( \mathbf{B} \) and a scalar function \( B_0 \) as follows (simplification of Mindlin’s solution in \[11\]):
\[
\mathbf{u} = \mathbf{B} - l^2 \nabla \nabla \cdot \mathbf{B} - \frac{1}{2} (\kappa - l^2 \nabla^2) \nabla [r \cdot (1 - l^2 \nabla^2) \mathbf{B} + B_0], \tag{21}
\]
where \( \kappa = (\mu + \lambda)/(2\mu + \lambda) \), and \( r \) denotes the position vector, whereas \( \mathbf{B} \) and \( B_0 \) are the solutions of the problems
\[
(1 - l^2 \nabla^2) \nabla^2 \mathbf{B} = 0, \quad (1 - l^2 \nabla^2) \nabla^2 B_0 = 0, \tag{22}
\]
respectively. However, in \[36\] the integration constants has been determined for the problem with loading applied to the hole surface, whereas in our case the inner surface \( r = a \) is free of loading implying the boundary conditions
\[
\mathbf{P} = 0, \quad \mathbf{R} = 0 \text{ at } r = a. \tag{23}
\]

As the second step, we focus on the details of the boundary conditions. With the conditions (23) valid at the free inner boundary, the boundary conditions (19)–(20) written in terms of the Cauchy stress take the form
\[
\tau_{rr} - l^2 \left( \frac{\partial^2}{\partial r^2} \tau_{rr} + \frac{1}{a^2} \left( \frac{\partial^2}{\partial \theta^2} \tau_{rr} - 2 \tau_{r\theta} + 2 \tau_{\theta\theta} - 4 \frac{\partial}{\partial \theta} \tau_{\theta\theta} - 4 \frac{\partial}{\partial \theta} \tau_{r\theta} \right) \right) = 0,
\]
\[
\tau_{r\theta} - l^2 \left( \frac{\partial^2}{\partial r^2} \tau_{r\theta} + \frac{1}{a^2} \left( \frac{\partial^2}{\partial \theta^2} \tau_{r\theta} - 4 \tau_{r\theta} - 2 \frac{\partial}{\partial \theta} \tau_{\theta\theta} + 2 \frac{\partial}{\partial \theta} \tau_{r\theta} \right) + \frac{1}{a^2} \left( \frac{\partial^2}{\partial \theta^2} \tau_{\theta\theta} - \frac{\partial}{\partial \theta} \tau_{\theta\theta} \right) \right) = 0,
\]
\[
\frac{\partial}{\partial r} \tau_{rr} = 0, \quad \frac{\partial}{\partial r} \tau_{r\theta} = 0, \tag{24}
\]
at $r = a$. For the boundary conditions at infinity, it is assumed (as in [36]) that far away from the hole, the size effects are negligible. Thus, the double traction force can be excluded and the traction force can be rewritten in its classical form

$$ P = n \cdot \boldsymbol{\tau} \quad \text{at} \quad r \to +\infty. \quad (25) $$

In the polar coordinate system, the components of the far field Cauchy stress tensor have the form

$$ \tau_{rr}(r, \theta) \xrightarrow{r \to +\infty} \frac{\sigma_{\infty}}{2} (1 + \xi) + \frac{\sigma_{\infty}}{2} (1 - \xi) \cos 2\theta, \quad (26) $$

$$ \tau_{r\theta}(r, \theta) \xrightarrow{r \to +\infty} \frac{\sigma_{\infty}}{2} (1 + \xi) - \frac{\sigma_{\infty}}{2} (1 - \xi) \cos 2\theta, \quad (27) $$

$$ \tau_{\theta\theta}(r, \theta) \xrightarrow{r \to +\infty} \frac{\gamma}{2} (1 - \xi) \sin 2\theta. \quad (28) $$

Accordingly, it is next assumed that the method of separation of variables can be utilized and the components of the stress tensor $\boldsymbol{\tau}$ can be presented as

$$ \tau_{rr}(r, \theta) = \tilde{\tau}_{rr}(r) + \tilde{\tau}_{rr}(r) \cos 2\theta, \quad (29) $$

$$ \tau_{r\theta}(r, \theta) = \tilde{\tau}_{r\theta}(r) + \tilde{\tau}_{r\theta}(r) \cos 2\theta, \quad (30) $$

$$ \tau_{\theta\theta}(r, \theta) = \tilde{\tau}_{\theta\theta}(r) + \tilde{\tau}_{\theta\theta}(r) \sin 2\theta. \quad (31) $$

As a result, the solution of the governing equation (14) can be found in the simplified form

$$ \mathbf{u} = (\hat{u}_1(r) + \hat{u}_1(r) \cos 2\theta) \mathbf{e}_r + (\hat{u}_2(r) + \hat{u}_2(r) \sin 2\theta) \mathbf{e}_\theta. \quad (32) $$

As the third step, the trial solution (32) is substituted into the governing equation (14) resulting in a system of equations split below into two parts. The first part is an uncoupled system of differential equations written in terms of $\hat{u}_i(r)$, $i = 1, 2$, as

$$ \hat{u}_i'''' + \frac{\hat{u}_i'}{r} - \frac{\hat{u}_i''}{r^2} \bigg\{ \gamma (\hat{u}_i'' + \frac{\hat{u}_i'}{r} - \frac{\hat{u}_i''}{r^2} + 2\frac{\hat{u}_i'''}{r^2} - \frac{\hat{u}_i'''}{r^3}) \bigg\} = 0, \quad i = 1, 2. \quad (33) $$

The second part is a coupled system of two differential equations written in terms of $\hat{u}_i(r)$, $i = 1, 2$, as

$$ \gamma (\hat{u}_1'''' + \frac{\hat{u}_1'}{r} - \frac{\hat{u}_1''}{r^2}) - (1 + \gamma)(\hat{u}_1'' + \frac{\hat{u}_1'}{r} - \frac{\hat{u}_1''}{r^2} + 2\frac{\hat{u}_1'''}{r^2} - \frac{\hat{u}_1'''}{r^3}) = 0, $n$$

$$ \gamma (\hat{u}_2'''' + \frac{\hat{u}_2'}{r} - \frac{\hat{u}_2''}{r^2}) - (1 + \gamma)(\hat{u}_2'' + \frac{\hat{u}_2'}{r} - \frac{\hat{u}_2''}{r^2} + 2\frac{\hat{u}_2'''}{r^2} - \frac{\hat{u}_2'''}{r^3}) = 0. \quad (34) $$
where $\gamma = \mu / \lambda$, and prime indicates a derivative with respect to $r$. The analytical solution of the two parts system can be found in the following general form:

$$
\tilde{u}_1(r) = A_1 r + \frac{A_2}{r} + A_3 I_1\left(\frac{r}{l}\right) + A_4 K_1\left(\frac{r}{l}\right),
$$

$$
\tilde{u}_2(r) = B_1 r + \frac{B_2}{r} + B_3 I_1\left(\frac{r}{l}\right) + B_4 K_1\left(\frac{r}{l}\right),
$$

$$
\tilde{u}_1(r) = - C_1 r - C_2 \frac{1 + 2\gamma}{r} + C_3 \frac{1}{r^3} - C_4 \frac{1}{2 + 3\gamma}
$$

$$
= C_5 I_1\left(\frac{r}{l}\right) + C_6 \left[I_1\left(\frac{r}{l}\right) \left(1 + 8\frac{l^2}{r^2}\right) - 4 \frac{l}{r} I_0\left(\frac{r}{l}\right)\right]
$$

$$
- C_7 K_1\left(\frac{r}{l}\right) + C_8 \left[K_1\left(\frac{r}{l}\right) \left(1 + 8\frac{l^2}{r^2}\right) + 4 \frac{l}{r} K_0\left(\frac{r}{l}\right)\right],
$$

$$
\tilde{u}_2(r) = C_{12} \frac{1}{r} + C_{13} \frac{1}{r^3} + C_{14} r^3
$$

$$
+ C_5 I_1\left(\frac{r}{l}\right) + C_6 I_3\left(\frac{r}{l}\right) + C_7 K_1\left(\frac{r}{l}\right) + C_8 K_3\left(\frac{r}{l}\right),
$$

where $I_i$, and $K_i$, $i = 0, 1, 3$, are the modified Bessel functions of the first and second kind, respectively, of order 0, 1 and 3.

As the fourth step, constants $A_i$, $B_i$ and $C_i$ above are determined by applying the boundary conditions. Substituting (32) into (13) and further into (8) gives the components of the Cauchy stress tensor in terms of displacements $\tilde{u}_1(r)$ and $\tilde{u}_2(r)$, $i = 1, 2$, as

$$
\tilde{\tau}_{rr} = (2\mu + \lambda) \tilde{u}_1' + \lambda \tilde{u}_1',
$$

$$
\tilde{\tau}_{r\theta} = (2\mu + \lambda) \tilde{u}_1' + \lambda \tilde{u}_1',
$$

$$
\tilde{\tau}_{\theta\theta} = \mu \left(\tilde{u}_2' - \frac{\tilde{u}_1}{r}\right),
$$

$$
\tilde{\tau}_{rr} = (2\mu + \lambda) \left(\frac{\tilde{u}_1}{r} + 2 \frac{\tilde{u}_2}{r}\right),
$$

$$
\tilde{\tau}_{\theta\theta} = (2\mu + \lambda) \left(\frac{\tilde{u}_1}{r} + 2 \frac{\tilde{u}_2}{r}\right) + \lambda \tilde{u}_1',
$$

$$
\tilde{\tau}_{\theta\theta} = \mu \left(\frac{\tilde{u}_1}{r} + \tilde{u}_2' - \frac{\tilde{u}_2}{r}\right).
$$

These expressions for the stresses of the form (29)–(31) are substituted in the boundary conditions (24) and (26) with (26)–(28), which splits them into two parts. The first one, composed of the conditions

$$
\tilde{\tau}_{rr} - l^2 \left(\tilde{\tau}_{rr}'' + \frac{2}{a^2} (\tilde{\tau}_{r\theta} - \tilde{\tau}_{rr}) - \frac{1}{a} \tilde{\tau}_{r\theta}'\right) = 0,
$$

$$
\tilde{\tau}_{r\theta} - l^2 \left(\tilde{\tau}_{r\theta}'' - \frac{4}{a^2} \tilde{\tau}_{r\theta}\right) = 0,
$$

$$
\tilde{\tau}_{rr}' = 0,
$$

$$
\tilde{\tau}_{r\theta}' = 0, \text{ at } r = a,
$$

$$
\tilde{\tau}_{rr}(r) \rightarrow \frac{1}{2} \sigma_{\infty} (1 + \xi),
$$

$$
\tilde{\tau}_{r\theta}(r) \rightarrow \frac{1}{2} \sigma_{\infty} (1 + \xi),
$$

$$
\tilde{\tau}_{\theta\theta}(r) \rightarrow 0,
$$

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corresponds to the tilde part of the stress tensor, whereas the second one, formed by the conditions
\[ \hat{\tau}_{rr} - l^2 \left( \hat{\tau}_{rr}'' + \frac{2}{a^2} (\hat{\tau}_{\theta\theta} - 3\hat{\tau}_{rr} - 4\hat{\tau}_{r\theta}) - \frac{1}{a} \hat{\tau}_{\theta\theta}' \right) = 0, \]
\[ \hat{\tau}_{r\theta} - l^2 \left( \hat{\tau}_{r\theta}'' + \frac{4}{a^2} (\hat{\tau}_{\theta\theta} - \hat{\tau}_{rr} - 2\hat{\tau}_{r\theta}) - \frac{2}{a} \hat{\tau}_{\theta\theta}' \right) = 0, \]
\[ \hat{\tau}_{rr}' = 0, \]
\[ \hat{\tau}_{r\theta}' = 0, \quad \text{at } r = a, \]
\[ \hat{\tau}_{r\theta}(r) \xrightarrow{r \to +\infty} \frac{1}{2} \sigma_\infty (1 - \xi), \]
\[ \hat{\tau}_{\theta\theta}(r) \xrightarrow{r \to +\infty} \frac{1}{2} \sigma_\infty (1 - \xi), \]
\[ \hat{\tau}_{\theta\theta}(r) \xrightarrow{r \to +\infty} - \frac{1}{2} \sigma_\infty (1 - \xi). \]

corresponds to the hat part. According to these conditions, coefficients \( A_1, B_1, C_4, C_5 \) and \( C_6 \) in (35) must be set to zero, otherwise stresses tend to infinity as \( r \) goes to infinity. Utilizing the boundary conditions (37)–(38) and introducing coefficients \( \alpha_j, j = 1, 2, 3, 4 \), the displacement components can be written in the form
\[ \tilde{u}_1(r) = A_1 r + \frac{A_2}{r} + A_4 K_1 \left( \frac{r}{l} \right), \]
\[ \tilde{u}_2(r) = 0, \]
\[ \tilde{u}_1(r) = -C_1 r + C_7 \left[ -\alpha_1 \frac{1 + 2\gamma}{r^2} + \frac{\alpha_3 a^3}{r^3} - K_1 \left( \frac{r}{l} \right) \right] \]
\[ + C_8 \left[ -\alpha_2 \frac{1 + 2\gamma}{r^2} + \frac{\alpha_4 a^3}{r^3} + K_1 \left( \frac{r}{l} \right) (1 + 8 \frac{l^2}{r^2}) + 4 \frac{l}{r} K_0 \left( \frac{r}{l} \right) \right] \]
\[ \tilde{u}_2(r) = C_1 r + C_7 \left[ \alpha_1 \frac{a}{r} + \frac{\alpha_3 a^3}{r^3} + K_1 \left( \frac{r}{l} \right) \right] + C_8 \left[ \alpha_2 \frac{a}{r} + \frac{\alpha_4 a^3}{r^3} + K_3 \left( \frac{r}{l} \right) \right], \]

where coefficients \( A_i, C_j, \) and \( \alpha_i \) are determined and collected in Appendix A. It is worth noting that the derived solution (39) coincides with the corresponding expression given by Eshel and Rosenfeld [36] up to a linear with respect to \( r \) term (expression (3.4) in [36] should be reduced to a special case \( l_1 = l_2 = l \) which corresponds to the one-parameter simplified formulation used in the present contribution).

Altogether, we have derived the displacement solution of the form (15) in the simplified form of (32), with the components (39), giving the corresponding stresses of the form (17) with the components (29)–(31) and (36).

4 Stress analysis based on the analytical solution

For the first time, stresses around a hole placed in an infinite material under uni-axial tension were analysed by G. Kirsch [3], based on the classical elasticity theory. According to the solution obtained by applying the classical continuum approach, the stress concentration factor \( K_t \) is equal to \( 3 \) – independently of the size of the hole. Furthermore, the maximum value for the hoop stress is achieved always at the point \( r = a \) with \( \theta = \pi/2 \) or \( \theta = 3\pi/2 \) (see Figure 1).
In this section, we generalize the stress analysis of the Kirsch problem to other loading types (see Figure 1): the total and Cauchy stresses and the related stress concentration factors are analyzed near the hole for different loading cases by varying the loading parameter $\xi$. In particular, the influence of the micro-structural parameter $l$ on the stress components is clarified in detail. For a uni-axially stretched plate with a hole, i.e., for the original Kirsch problem (now studied within both the classical and gradient-elastic frameworks, however), analytical results based on the derivation of the previous section are compared to numerical results based on the Galerkin method proposed in [53].

For all of the computations of this section, we adopt the constitutive relations of the plane stress state.

### 4.1 Stresses around the hole circumference for different loadings

According to (17), the total stress tensor (9) can be written in the component form as

$$\sigma = \sigma_{rr} e_r e_r + \sigma_{\theta \theta} e_\theta e_\theta + \sigma_{r \theta} (e_r e_\theta + e_\theta e_r),$$

(40)

where

$$\sigma_{rr}(r, \theta) = \hat{\tau}_{rr} + \tau_{rr} \cos 2\theta - \frac{l^2}{r^2} \left[ (\hat{\tau}_{rr}'' + \frac{1}{r} \hat{\tau}_{rr}' + \frac{2}{r^2} (\hat{\tau}_{\theta \theta} - \hat{\tau}_{rr} - \hat{\tau}_{rr}')) \right],$$

(41)

$$\sigma_{\theta \theta}(r, \theta) = \hat{\tau}_{\theta \theta} + \tau_{\theta \theta} \cos 2\theta - \frac{l^2}{r^2} \left[ (\hat{\tau}_{\theta \theta}'' + \frac{1}{r} \hat{\tau}_{\theta \theta}' + \frac{2}{r^2} (\hat{\tau}_{rr} - \hat{\tau}_{\theta \theta} - 3\hat{\tau}_{rr} - 4\hat{\tau}_{r \theta})) \right],$$

(42)

$$\sigma_{r \theta}(r, \theta) = \hat{\tau}_{r \theta} - \frac{l^2}{r^2} \left[ \hat{\tau}_{r \theta}'' + \frac{1}{r} \hat{\tau}_{r \theta}' + \frac{4}{r^2} (\hat{\tau}_{\theta \theta} - \hat{\tau}_{rr} - 2\hat{\tau}_{r \theta})) \right] \sin 2\theta,$$

(43)

with the components given in terms of displacements in (36).

The classical material properties used in further calculations are taken to be $E = 210$ GPa and $\nu = 0.3$, whereas the gradient parameter $l$ varies from 0 to $+\infty$.

First, the normalized hoop stresses $\tau_{\theta \theta}$ and $\sigma_{\theta \theta}$ around the circumference of the hole are shown in Figures 2a–2c (angle $\theta$ varies from 0° to 180°, see Figure 1). The normalization is done with respect to the absolute value of the applied loading $\sigma_\infty$. The stresses in Figures 2a and 2b correspond to the biaxially ($\xi = 1$) and uni-axially ($\xi = 0$) stretched plates, respectively. Stresses in Figure 2c correspond to the plate stretched in one direction and compressed in the other direction ($\xi = -1$). Cauchy stress $\tau_{\theta \theta}$ and total stress $\sigma_{\theta \theta}$ are depicted in solid and dot lines, respectively. Curves corresponding to the classical theory are shown in black, whereas the gradient-elastic solutions for the cases $l/a = 0.2$ and $l/a = 0.7$ are plotted in red and blue, respectively.
(a) Biaxial tension, $\xi = 1$.

(b) Uniaxial tension, $\xi = 0$.

(c) Biaxial tension and compression, $\xi = -1$.

Figure 2: Stresses around the circumference of a hole.
In these graphs, it can be seen that the absolute values of all stresses decrease as the ratio $l/a$ increases. For the values of $l/a$ small enough, the total and Cauchy stresses almost coincide, while increasing $l/a$ makes the difference between the stress levels significant. It can be observed as well that as $l/a$ increases the Cauchy stress decreases faster than the total stress in the biaxially stretched plate, whereas for the two other loading cases the absolute value of the total stress decreases faster.

Second, the dependence of the stress concentration factor on the ratio $l/a$ is presented in Figures 3a–3c for all loading cases. The stress concentration factors for the total and Cauchy stresses are calculated by the formulae

$$K_t = \sigma_{\theta\theta}(a, \pi/2)/\sigma_{\infty} \quad \text{and} \quad K_t = \tau_{\theta\theta}(a, \pi/2)/\sigma_{\infty},$$

respectively. It can be seen that the normalized Cauchy stress tends to unity as the hole radius tends to zero, in all loading cases. It should be mentioned that in the case of biaxially loaded plate the level of the normalized Cauchy stress exceeds the value 2 when $l/a$ is in the range from 0 to 0.1, see Figure 3a (cf. [36]). In contrast, the normalized total stress tends to a higher value ($\approx 1.35$ till $l/a \approx 1000$; the range $l/a > 10$ is excluded from the picture).
Figure 3: Size-dependent stress concentration factor.
In the loading cases $\xi = 0$ and $\xi = -1$ (Figures 3b and 3c), it can be observed that both solid and dot lines are close to each other till the point $l/a \approx 0.5$, while after this point the curves differ significantly. This point, marked with a bold blue dot, is remarkable in another sense as well. Namely, it turns out, and will be demonstrated in the next subsection, that after this point the total stress $\sigma_{\theta\theta}$ (unlike the corresponding Cauchy stress) is not maximal at the "classical" material point $r = a$, $\theta = \pi/2$. It is considered meaningless to continue the curves corresponding to the total stress after the point $l/a \approx 1$ (the reason for this is explained in the next subsection).

### 4.2 Stresses around the hole circumference for uni-axial loadings

In this subsection, the stress fields for the original Kirsch problem are analyzed, i.e., the stresses around the hole circumference are considered for the loading case $\xi = 0$.

First, in Figures 4a–4c, the normalized hoop and radial stresses $\tau_{\theta\theta}$, $\sigma_{\theta\theta}$ and $\tau_{rr}$, $\sigma_{rr}$ for the Cauchy and total stresses, respectively, are plotted along the line $\theta = 3\pi/2$ for different values of the ratio $l/a$. The plots in Figures 4a and 4b are depicted with black, red and blue lines for $l/a = 0$, $l/a = 0.5$ and $l/a = 1$, respectively.
Figure 4: Distributions of normalized stresses for different values of $l/a$. 

(a) Normalized $\theta\theta$-stress component for $l/a = 0, 0.5, 1$. 

(b) Normalized $rr$-stress component for $l/a = 0, 0.5, 1$. 

(c) Normalized $\theta\theta$-stress component for $l/a = 1.2$. 

Figure 4: Distributions of normalized stresses for different values of $l/a$. 

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In Figure 4a, it can be seen that the hoop stresses reach their maximal values on the hole contour and these values decrease toward one as the hole radius decreases. It should be noticed that the stress concentration effect is confined to the region $1 \leq r/a \leq 3$ and the stress field becomes almost uniform beyond this area. In Figure 4b, the variation of the radial stresses is presented. Since the inner boundary is free of loading, the classical theory implies a zero radial stress at the periphery of the hole. In the gradient-elastic case, however, this holds true neither for the Cauchy stress nor for the total stress. Moreover, the total stress increases near the hole as $l/a$ increases. In addition, in Figure 4a and 4b it can be seen that the hoop Cauchy stresses fall to unity and the radial Cauchy stresses fall to zero, which corresponds to the stress distribution of a uni-axially stretched plate without any hole.

The first qualitatively unexpected fact can be observed in Figure 4c plotting the normalized hoop stresses for $l/a = 1.2$. The maximal value of the total stress is achieved inside the material, not on the hole contour. For this reason, in Figures 3b and 3c above the plots for the total stress concentration factor are cut short near $l/a = 1$.

The second unexpected fact can be seen in Figures 5a, 5b and 5c below illustrating the variation of the von Mises stresses, maximal shear stresses and maximal principle stresses, respectively, along the circumference of the hole for different values of $l/a$. The angular coordinate is redefined by the formula $\varphi = \theta - 3\pi/2$ as depicted in Figure 1 and the first quarter of the hole circumference with $0^\circ \leq \varphi \leq 90^\circ$ is considered. In all of the figures, the Cauchy stress fields tend to a uniform distribution as $l/a$ increases (the hole radius goes to zero), and the maximal value of the stresses is always achieved at $\varphi = 0$. In contrast, the total stresses behave differently. It can be observed that the location of the maximal stress value moves from its initial position ($\varphi = 0$ for the solid black line) to a limit value (the maximum of the dotted red and blue lines; see Figure 6 below as well) and the total stress level does not fall to a uniform value.
(a) Normalized von Mises stresses for $l/a = 0, 0.52, 1$.

(b) Normalized max shear stresses for $l/a = 0, 0.52, 1$.

(c) Normalized max principle stresses for $l/a = 0, 0.7, 1$.

Figure 5: Distributions of normalized von Mises, max shear and max principle stresses for different values of $l/a$. 

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In order to further investigate the unexpected situation observed above, it is shown in Figure 6 how the angular coordinate $\varphi$ of the maximal von Mises, maximal shear and maximal principle total stresses varies as $l/a$ increases. It can be seen that the maximal stress levels start to move along the circumference of the hole when $l/a$ exceeds its critical value ($a$ approximately equal to $2l$). Finally, the maximal stresses are achieved at the point with the coordinate value $\varphi \approx \pi/4$. This can be clearly seen in the surface plots of Section 5 as well.

![Figure 6: Angular coordinate of the maximal total stress levels.](image)

Finally, in Figures 7a–7c, it is shown how the maximum of the normalized von Mises, maximal shear and maximal principle stresses varies as the hole radius $a$ is decreased with respect to the parameter $l$. The Cauchy stresses start from the value 3 in Figures 7a and 7c and from the value 1.5 in Figure 7b, and then monotonically tend to the value which corresponds to a uniform stress distribution: the von Mises and maximal principle stresses tend to 1 (Figures 7a and 7c), whereas the maximal shear stress tends to 0.5 (Figure 7b). Once again, the total stresses behave differently. They start from the same values as the Cauchy stresses, and start to decrease as the Cauchy stresses, while after a critical value of $l/a$, they start to increase indefinitely (not shown in the plots).
Figure 7: Distributions of maximum levels of normalized von Mises, max shear and max principle stresses for different values of $l/a$. 
5 Numerical results for the original Kirsch problem

In this section, we further analyze the stress and strain fields of the original Kirsch problem according to numerical results obtained by applying an isogeometric Galerkin method for gradient-elastic models proposed and analyzed in [53]. The discretization have been implemented as a user subroutine in the commercial finite element software Abaqus [56]. The theoretical analysis and the numerical verification of the method and its implementation reported in [53] guarantee the reliability of the method. The weak form of the plane strain/stress gradient elasticity problem corresponds to (1) and (2), with (8) and (13).

The domain for the computational analysis of the problem and the mesh near the hole are shown in Figure 8. The symmetry conditions are utilized on the left and bottom edges and written as

\[ u \cdot n = 0 \quad \text{and} \quad \frac{\partial (u \cdot s)}{\partial n} = 0. \]  

(45)

where \( n \) and \( s \), respectively, denote the unit vectors normal and tangential to the boundary. A tensional loading \( \sigma_{\infty} = 100 \) MPa is applied to the upper edge. The length of the intact edge is denoted by \( L \) with \( L/a = 100 \). The classical elastic material parameters are taken to be \( E = 210 \) GPa and \( \nu = 0.3 \). The ratio of the micro-structural parameter \( l \) to the hole radius \( a \) is varied from 0 to 10 \((0 \leq l/a \leq 10)\). The computational domain is divided into 13312 elements. NURBS of the 5th order have been taken as basis functions for both the Galerkin displacement field basis functions and the geometry mappings. \( C^1 \)-continuity holds everywhere except on the diagonal line, see Figure 8. Across this line, the basis functions are \( C^1 \)-continuous satisfying the minimum regularity conditions.

First, the numerical and analytical values for the stress concentration factor related to the Cauchy stress are compared in Figure 3b. It can be seen that the red dots corresponding to the numerical values lie exactly on the black line representing the analytical solution.

Second, the distribution of the hoop Cauchy and total stress fields are shown in Figures 9–11 (left column) for \( l/a = 0 \) (classical elasticity theory) and \( l/a = 1 \). The maximum of the hoop stress

![Figure 8: Computational domain and the mesh near the hole.](image-url)
is achieved in the material point $r = a, \varphi = 0$ both for the classical and gradient cases. In the classical case ($l/a = 0$), the maximal value of the normalized stress is equal to 3. In the gradient case ($l/a = 1$), in turn, the maxima of the normalized Cauchy and total stresses are equal to 1.45 and 1.25, respectively. The numerical results coincide exactly with the analytical ones presented above in Figure 4a.

Figure 9: Distribution of the hoop (left) and von Mises (right) Cauchy stress for $l/a = 0$.

Figure 10: Distribution of the hoop (left) and von Mises (right) Cauchy stress for $l/a = 1$.

Figure 11: Distribution of the hoop (left) and von Mises (right) total stress for $l/a = 1$. 
Third, the distribution of the von Mises Cauchy and total stress fields are presented in Figures 9–11 (right column). The non-typical behavior of the total stress shown above in Figure 5a is confirmed by the numerical results presented in Figure 11: the maximum of the von Mises total stress has shifted along the circumference of the hole by the angle $\varphi$ approximately equal to $\pi/4$ (the maximum stress region indicated by red). The maximums of the normalized Cauchy and total stresses are equal to 1.41 and 1.93, respectively (cf. the corresponding values in Figures 5a and 7a).

More detailed information for the stress fields near the hole can be found in Appendix B. First, in Figure 16 for the classical case ($l/a = 0$) serving as a reference, the distributions of the Cauchy stress fields are shown for all components $xx, yy, xy$ and $rr, \theta\theta, r\theta$. Second, in Figure 17 dedicated to the $xx$-component, the distributions of the Cauchy and total stresses and their difference $l^2\Delta \tau$ are plotted for $l/a = 0.5$ and $l/a = 1$. It should be noticed, in particular, that the total stresses have peaks around both $\varphi = 0$ and $\varphi \approx \pi/4$. Third, in Figures 18–22, the same distributions are shown for components $yy, xy$ and $rr, \theta\theta, r\theta$. It should be noticed, in particular, how the component $r\theta$ having a clear maximum on the inner boundary at $\varphi = \pi/4$ (as for the analytical solution) deviates from the corresponding classical stress component in Figure 16.

Finally, for completing the picture about the differences between the classical and gradient elasticity theories the strain fields are plotted for different hole sizes with respect to both Cartesian and cylindrical coordinate systems. The distributions of the strain components $xx, yy, xy$ and $rr, \theta\theta, r\theta$ are compared in Figures 12–14 for $l/a = 0, l/a = 0.5$ and $l/a = 1$. One can see that the $xx$- and $rr$-components clearly change from $l/a = 0$ to $l/a = 0.5$ and $l/a = 1$, whereas the $yy$- or $\theta\theta$-components remain practically unchanged. The most radical change happens in the $xy$- and $r\theta$-components (in Figure 14): the position of the maximum moves from inside the medium to the circumference of the hole at $\varphi = \pi/4$ which seems to be the crucial territory from the qualitative point of view. This can be seen by comparing Figures 16 and 22 for the corresponding stress components as well. For further clarifying the behavior of the strain component $r\theta$, its distribution along the line $x = y$ has been plotted in Figure 15 for both the analytical and the numerical solutions (solid and dotted lines, respectively) with $l/a = 0, 0.1, 0.2, 0.3, 0.5$ and $l/a = 1$. This figure clearly shows how the maximum value of the strain moves from inside the material (at $r \approx 1.75a$ for $l/a = 0$, black line; at $r \approx 1.5a$ for $l/a = 0.3$, green line) towards the boundary of the hole ($r = a$ for $l/a = 0.5$ and $l/a = 1$; red and blue lines).
Figure 12: Distribution of the $xx$- and $rr$-components of the strain field for $l/a = 0, 0.5, 1$. 
Figure 13: Distribution of the $y y$- and $\theta \theta$-components of the strain field for $l/a = 0, 0.5, 1$. 

(a) Strain $\varepsilon_{yy}$, $l/a = 0$. 
(b) Strain $\varepsilon_{\theta\theta}$, $l/a = 0$. 
(c) Strain $\varepsilon_{yy}$, $l/a = 0.5$. 
(d) Strain $\varepsilon_{\theta\theta}$, $l/a = 0.5$. 
(e) Strain $\varepsilon_{yy}$, $l/a = 1$. 
(f) Strain $\varepsilon_{\theta\theta}$, $l/a = 1$. 

Figure 13: Distribution of the $y y$- and $\theta \theta$-components of the strain field for $l/a = 0, 0.5, 1$. 

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(a) Strain $\varepsilon_{xy}$, $l/a = 0$
(b) Strain $\varepsilon_{r\theta}$, $l/a = 0$
(c) Strain $\varepsilon_{xy}$, $l/a = 0.5$
(d) Strain $\varepsilon_{r\theta}$, $l/a = 0.5$
(e) Strain $\varepsilon_{xy}$, $l/a = 1$
(f) Strain $\varepsilon_{r\theta}$, $l/a = 1$

Figure 14: Distribution of the $xy$- and $r\theta$-components of the strain field for $l/a = 0$, 0.5, 1.

### 6 Discussions and conclusions

In this article, first an analytical solution of a generalized Kirsch problem, concerning an infinite plate weakened by a hole, has been derived in terms of displacements by adopting a one-parameter modification of Mindlin’s strain gradient elasticity theory. All of the coefficients appearing in the
solution have been derived symbolically. Second, the Cauchy and total stress distributions and the related stress concentration factors have been analysed in the vicinity of the hole for different parameter values and loading cases. For the case of uni-axial tension corresponding to the original Kirsch problem, the analytical stress and strain fields have been compared to numerical results provided by a Galerkin approximation. A detailed analysis of different stress and strain components in both Cartesian and cylindrical coordinate systems has been accomplished for different parameter values. The analytical and numerical results coincide with each other, which serves as a confirmation for the reliability of the analytical approach.

The main findings of the analysis are the following. First, the hoop total stress moves from the circumference of the hole inside the material with gradient parameter values greater than a critical value \( l \approx 1.2a \). Second, for hole radii less than its critical value \( (\approx 2l) \) the total stress demonstrates a qualitatively and quantitatively different behavior in comparison with the Cauchy stress which, in turn, essentially differs from the one of the classical elasticity theory for certain stress components. More precisely, for the total stress the maximum values for the typical stress measures for failure (von Mises, maximum principle stress, maximum shear stress and \( \sigma_{yy} \)) move from the classical hoop point of stress concentration \((r = a, \varphi = 0)\) to another hoop point \((r = a, \varphi \approx \pi/4 \) at the limit). Third, the total stresses become singular when the hole radius goes to zero, whereas the Cauchy stress field tends to the stress distribution of a plate without a hole. All of these findings are actually related to loadings which are unsymmetric with respect to the circular hole.

As a remark concerning the gradient parameter, we note that, for comprehensiveness, the ratio \( l/a \) between the gradient parameter \( l \) and the hole radius \( a \) is allowed to vary in this work between 0 and
+∞. However, as long as the gradient parameter, being a micro-structural parameter, is considered to represent the smallest geometrical problem dimension the admissible range for the ratio $l/a$ should be restricted to the range from 0 to 1, or from 0 to 2 when considering the hole diameter as the characteristic dimension.

Another remark can be given regarding the stress quantities of the gradient model. According to the present results, in the parameter limit the distributions of the Cauchy stress field tend to the stress field distributions of a plate without a hole emphasizing the physicalness of these stresses and serving as a reasoning for the assumption above regarding the parameter range. In contrast, the total stresses become singular when the hole radius goes to zero. Moreover, the total stresses demonstrate an unexpected and nonphysical behavior already for $l/a \approx 0.5$. In view of the aforesaid, in the current strain gradient elasticity theory of Mindlin type the Cauchy stress can be considered as a physical stress measure, whereas the role of the total stress should be clarified by further analysis based on both theoretical argumentation and experimental observations.

Experimental information regarding the influence of the hole size on the mechanical behavior of solids near cavities is very limited, perhaps due to the difficulties in the experimental realization. A related investigation has been accomplished by Weck et al. [63, 64] concerning crack initiation and further propagation in an aluminum sheet with micro-holes. Anyway, the results presented in this article, especially the strain fields, can be used as a reference and base for experimental analysis of the problem which necessitates experimental verification with micro-sized holes.

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Appendix A: Integration constants

Integration constants $A_1, A_2, A_4, C_1, C_7, C_8$ and $\alpha_i, i = 1, 2, 3, 4$, of the solution (39) are derived by considering the boundary conditions of the form (37)–(38):

\begin{equation}
A_1 = -\frac{\sigma_\infty}{4(\mu + \lambda)} (1 + \xi),
\end{equation}

\begin{equation}
A_2 = \frac{\sigma_\infty}{4\mu} \frac{a^2(1 + \xi)((1 + 2\gamma)h^2K_3(h) - 2(2 + 3\gamma)hK_2(h))}{4(1 + \gamma)K_1(h) - 2(3 + 5\gamma)hK_2(h) + (1 + 2\gamma)(h^2 - 12)K_3(h) + 2(1 + 2\gamma)hK_4(h)},
\end{equation}

\begin{equation}
A_4 = -\frac{\sigma_\infty}{\lambda} \frac{a(1 + \xi)}{4(1 + \gamma)K_1(h) - 2(3 + 5\gamma)hK_2(h) + (1 + 2\gamma)(h^2 - 12)K_3(h) + 2(1 + 2\gamma)hK_4(h)},
\end{equation}

\begin{equation}
C_1 = -\frac{\sigma_\infty}{4\mu}(1 - \xi),
\end{equation}

\begin{equation}
C_1 = -\frac{\sigma_\infty}{4\mu}(1 - \xi),
\end{equation}

\begin{equation}
C_1 = -\frac{\sigma_\infty}{4\mu}(1 - \xi),
\end{equation}

\begin{equation}
C_1 = -\frac{\sigma_\infty}{4\mu}(1 - \xi),
\end{equation}

\begin{equation}
C_1 = -\frac{\sigma_\infty}{4\mu}(1 - \xi),
\end{equation}

\begin{equation}
C_1 = -\frac{\sigma_\infty}{4\mu}(1 - \xi),
\end{equation}
\[ C_7 = -\frac{\sigma_\infty}{2\lambda} a(1 - \xi) \frac{\beta_2 + \beta_4}{\beta_1 \beta_4 - \beta_2 \beta_3}, \quad (50) \]

\[ C_8 = \frac{\sigma_\infty}{2\lambda} a(1 - \xi) \frac{\beta_1 + \beta_3}{\beta_1 \beta_4 - \beta_2 \beta_3}, \quad (51) \]

\[ \beta_1 = -2\gamma K_1(h) + \left[ (1 + 3\gamma) \frac{12}{h} + (5 + 17\gamma) \frac{h}{2} \right] K_2(h) - (3 + 8\gamma) \frac{h^2}{4} K_3(h) - (1 + 2\gamma) h K_4(h), \quad (52) \]

\[ \beta_2 = -4\gamma \left( \frac{1}{h} + \frac{24}{h^3} \right) K_0(h) - 2 \left[ 1 + 2\gamma + (1 + 6\gamma) \frac{4}{h^2} + \gamma \frac{96}{h^3} \right] K_1(h) \]
\[ - \left[ (2 + 3\gamma) \frac{h}{2} + (3 + 5\gamma) \frac{4}{h} + (1 + 2\gamma) \frac{32}{h^3} \right] K_2(h) \]
\[ + \left[ 3(1 + 2\gamma) \frac{h^2}{4} + 13 + 14\gamma + (7 + 8\gamma) \frac{8}{h^2} \right] K_3(h) \]
\[ + \left[ (2 + 7\gamma) \frac{4}{h} + (10\gamma - 1) \frac{h}{2} \right] K_4(h) - \left[ 2(1 + 2\gamma) + \gamma \frac{h^2}{2} \right] K_5(h), \quad (53) \]

\[ \beta_3 = 2\gamma K_1(h) + \left[ (1 + 3\gamma) \frac{12}{h} + (1 + \gamma) \frac{h}{2} \right] K_2(h) - \left[ 6(1 + 3\gamma) + (1 + 2\gamma) \frac{h^2}{4} \right] K_3(h) + \gamma h K_4(h), \quad (54) \]

\[ \beta_4 = -4\gamma \left( \frac{1}{h} + \frac{24}{h^3} \right) K_0(h) - 8\gamma \left( \frac{5}{h^2} + \frac{24}{h^3} \right) K_1(h) + \gamma \left( h - \frac{64}{h^3} \right) K_2(h) \]
\[ + \left[ 9 + 16\gamma + (1 + 2\gamma) \frac{h^2}{4} + (9 + 10\gamma) \frac{8}{h^2} \right] K_3(h) \]
\[ - \left[ (1 - 2\gamma) \frac{12}{h} + (1 + 2\gamma) \frac{h}{2} \right] K_4(h) - 14\gamma K_5(h) + \gamma h K_6(h), \quad (55) \]

\[ \alpha_1 = \frac{1 + 5\gamma}{2(1 + \gamma)} h K_2(h) - \frac{1 + 3\gamma}{4(1 + \gamma)} h^2 K_3(h), \quad (56) \]

\[ \alpha_2 = \left( \frac{1 + 2\gamma}{4(1 + \gamma)} h^3 + 3 \right) K_3(h) + \frac{3\gamma - h K_4(h)}{2(1 + \gamma)} - \frac{\gamma}{4(1 + \gamma)} h^2 K_5(h), \quad (57) \]

\[ \alpha_3 = -\frac{1 + 7\gamma}{12\gamma} h K_2(h) - \frac{1 + 4\gamma}{24\gamma} h^2 K_3(h), \quad (58) \]

\[ \alpha_4 = -\frac{2}{3\gamma} K_0(h) - \frac{1}{3} \left( \frac{1 + 4}{h^2} \right) K_1(h) - \frac{1}{12} \left( h + \frac{8}{h} \right) K_2(h) \]
\[ + \frac{1}{24} \left( \frac{1 + 2\gamma}{\gamma} h^2 + 4 + 2\gamma \right) K_3(h) + \frac{6\gamma - 1}{12\gamma} h K_4(h) - \frac{h^2}{12} K_5(h), \quad (59) \]

where \( \gamma = \mu/\lambda \) and \( h = a/l \).
Appendix B: Distributions of the stress fields

(a) Cauchy stress $\tau_{xx}$, $l/a = 0$.
(b) Cauchy stress $\tau_{rr}$, $l/a = 0$.
(c) Cauchy stress $\tau_{yy}$, $l/a = 0$.
(d) Cauchy stress $\tau_{\theta\theta}$, $l/a = 0$.
(e) Cauchy stress $\tau_{xy}$, $l/a = 0$.
(f) Cauchy stress $\tau_{r\theta}$, $l/a = 0$.

Figure 16: Distribution of the Cauchy stress field for $l/a = 0$. 

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Figure 17: Distribution of the $\sigma_{xx}$-component of the stress fields for $l/a = 0.5$ and $l/a = 1$. 

(a) Cauchy stress $\tau_{xx}$, $l/a = 0.5$. 
(b) Total stress $\sigma_{xx}$, $l/a = 0.5$. 
(c) Cauchy stress $\tau_{xx}$, $l/a = 1$. 
(d) Total stress $\sigma_{xx}$, $l/a = 1$. 
(e) Stress difference $(l^2 \Delta \tau : e_x e_x)$, $l/a = 0.5$. 
(f) Stress difference $(l^2 \Delta \tau : e_x e_x)$, $l/a = 1$. 

Figure 17: Distribution of the $xx$-component of the stress fields for $l/a = 0.5$ and $l/a = 1$. 

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Figure 18: Distribution of the $yy$-component of the stress fields for $l/a = 0.5$ and $l/a = 1$.
(a) Cauchy stress $\tau_{xy}$, $l/a = 0.5$.
(b) Total stress $\sigma_{xy}$, $l/a = 0.5$.
(c) Cauchy stress $\tau_{xy}$, $l/a = 1$.
(d) Total stress $\sigma_{xy}$, $l/a = 1$.
(e) Stress difference $(l^2 \Delta \tau : e_x e_y)$, $l/a = 0.5$.
(f) Stress difference $(l^2 \Delta \tau : e_x e_y)$, $l/a = 1$.

Figure 19: Distribution of the $xy$-component of the stress fields for $l/a = 0.5$ and $l/a = 1$. 

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(a) Cauchy stress $\tau_{rr}$, $l/a = 0.5$.
(b) Total stress $\sigma_{rr}$, $l/a = 0.5$.
(c) Cauchy stress $\tau_{rr}$, $l/a = 1$.
(d) Total stress $\sigma_{rr}$, $l/a = 1$.
(e) Stress difference $(l^2 \Delta \tau : e_r e_r)$, $l/a = 0.5$.
(f) Stress difference $(l^2 \Delta \tau : e_r e_r)$, $l/a = 1$.

Figure 20: Distribution of the $rr$-component of the stress fields for $l/a = 0.5$ and $l/a = 1$. 
(a) Cauchy stress $\tau_{\theta\theta}$, $l/a = 0.5$.
(b) Total stress $\sigma_{\theta\theta}$, $l/a = 0.5$.
(c) Cauchy stress $\tau_{\theta\theta}$, $l/a = 1$.
(d) Total stress $\sigma_{\theta\theta}$, $l/a = 1$.
(e) Stress difference $(l^2 \Delta \tau : e_{\theta\theta})$, $l/a = 0.5$.
(f) Stress difference $(l^2 \Delta \tau : e_{\theta\theta})$, $l/a = 1$.

Figure 21: Distribution of the $\theta\theta$-component of the stress fields for $l/a = 0.5$ and $l/a = 1$. 

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(a) Cauchy stress $\tau_{r\theta}$, $l/a = 0.5$.
(b) Total stress $\sigma_{r\theta}$, $l/a = 0.5$.
(c) Cauchy stress $\tau_{r\theta}$, $l/a = 1$.
(d) Total stress $\sigma_{r\theta}$, $l/a = 1$.
(e) Stress difference $(l^2\Delta \tau : e_r e_\theta)$, $l/a = 0.5$.
(f) Stress difference $(l^2\Delta \tau : e_r e_\theta)$, $l/a = 1$.

Figure 22: Distribution of the $r\theta$-component of the stress fields for $l/a = 0.5$ and $l/a = 1$. 

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References


