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Locking-free variational formulations and isogeometric analysis for the Timoshenko beam models of strain gradient and classical elasticity

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Abstract

The Timoshenko beam bending problem is formulated in the context of strain gradient elasticity for both static and dynamic analysis. Two non-standard variational formulations in the Sobolev space framework are presented in order to avoid the numerical shear locking effect pronounced in the strain gradient context. Both formulations are shown to be reducible to their locking-free counterparts of classical elasticity. Conforming Galerkin discretizations for numerical results are obtained by an isogeometric C^{p-1} -continuous approach with B-spline basis functions of order $p \geq 2$. Convergence analyses cover both statics and free vibrations as well as both strain gradient and classical elasticity. Parameter studies for the thickness and gradient parameters, including micro-inertia terms, demonstrate the capability of the beam model in capturing size effects. Finally, a model comparison between the gradient-elastic Timoshenko and Euler-Bernoulli beam models justifies the relevance of the former, confirmed by experimental results on nano-beams from literature.

Keywords: Timoshenko beam, Variational formulation, Strain gradient elasticity, Isogeometric analysis, Size effect, Shear locking

1. Introduction

The relevance of the classical beam bending models – the Euler–Bernoulli and Timoshenko models from 1750s and 1920s, respectively – to the classical engineering fields such as (macro-scale) civil and mechanical engineering is undeniable. Moreover, these models have nowadays numerous applications in many other fields of science and engineering as well, from (meso-scale) biomechanics to micro- and nanotechnology. On the other hand, experimental results of micro- and nanosciences, indeed, have shown that the classical continuum mechanics is not capable of describing the size effects of

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small scale objects such as micro- and nano-beams [1, 2, 3, 4, 5, 6]: the underlying well-defined axioms of the homogenizing conception of Cauchy’s continuum are violated as the microstructural lengths of the material such as the crystal size become comparable to the dimensions of the structure itself such as the thickness of the beam. Besides, microarchitected metamaterials or structures of any scale have shown to share the same nature, typically meaning that coarse microstructures are relatively stiffer than the corresponding denser ones [7]. This experimental evidence have implied a number of generalized beam models relying still on the classical kinematic dimension reduction assumptions but within the framework of generalized continuum theories (see a very recent review in [8]).

Regarding the modeling framework, the present contribution considers perhaps the most common simplified variant of Mindlin’s strain gradient elasticity theory [9] introduced by Vardoulakis et al. [10], and Altan and Aifantis [11] and later adopted, further developed and mathematically analyzed by numerous scholars (see [12], for instance). In particular, differing from Mindlin’s theory which incorporates five static length scale parameters for isotropic materials, the simplified variant possesses only one static parameter and hence remains experimentally validatable. Regarding generalized beam models, reviews on the existing Euler–Bernoulli models including numerical treatments can be found in [8] and [13], whereas for Timoshenko beams we refer to [8] and only briefly review those ones of the earlier contributions which are very close to the model analyzed in the present contribution: [14] and [15], in particular, miss a crucial size effect term and the latter does not include micro-inertia terms; [16] includes the crucial size effect and micro-inertia terms, whereas the corresponding variational formulation is not provided; the model developed in [17] includes the size effect term but not the micro-inertia terms (see [18] as well) and is based on a simplified strain gradient theory involving three length scale parameters. Finally, we note that references for higher-order shear-deformable strain gradient beam models can be found in [19].

Literature on numerical methods and analysis for models of generalized continua is still very limited, even for beam models as reviewed in [8]:

“...most of existing size-dependent models focused on analytical solutions... limited to beam and plate structures subjected to certain loading and boundary conditions and geometries... Therefore, further efforts should be devoted to developing finite element models of size-dependent theories, especially the strain gradient-based models.”

One reason for the lack of methods and analysis might be the fact that strain gradient theories, especially, lead to higher-order differential equations requiring more complex numerical methods than the classical C^0 -continuous Lagrange finite elements [12, 13]. On the other hand, it is well known that numerical methods for Timoshenko beams and Reissner–Mindlin plates and shells, in particular, suffer from the so-called *numerical shear locking phenomena*, extreme for thin beams and low-order approximations. Within beam finite elements of classical elasticity, the so-called selective reduced integration, or an L^2 -projection of the shear term in mathematical terms [20], can be considered being a remarkable achievement in solving the locking problem. For plate finite elements, instead, it has been necessary to develop much more sophisticated tech-

niques in order to reach stability and optimal convergence rates, which has generated a vast literature by the scientific communities of engineering and mathematics—and milestones in the triumph of the finite element method. For shell finite elements, the locking problem is not restricted to the shear term and the problem has not been completely resolved until today. During the past few years, departing from low-order basis functions and standard C^0 -continuity with the aid of isogeometric analysis have given new ways to avoid the locking problem: for Timoshenko beams, a reformulation of the variational problem up to a one-variable problem [21] or simply departing from the Galerkin framework [22] have been shown to cure, or better avoid, the problem; for Reissner–Mindlin plates and shells, another type of reformulation [23, 24, 25] has shown to be a clever way to avoid locking *ab initio*.

For generalized Timoshenko beam models, in particular, there seems to exist only a couple of contributions focusing on numerical methods: [26] and [27] propose finite elements with basis functions based on analytical solutions. Therefore, there is an evident need for a general-purpose numerical method with optimal convergence properties, which is the substance of the present contribution.

In this paper, we first use Hamilton’s variational principle for deriving the governing equations and boundary conditions as well as variational formulations for the statics and dynamics of the gradient-elastic Timoshenko beam bending model. For dynamics, an additional material parameter related to the so-called micro-inertia terms is taken into account. Importantly, the micro-inertia terms in the kinetic energy provide physical dispersion curves (cf. [28, 13]). Second, we propose two variational formulations in order to avoid *shear locking*: the first one (not considered as fully locking-free from the theoretical point of view in the strain gradient framework) imitates the reformulation of the Reissner–Mindlin plate bending problem of classical elasticity by Beirão da Veiga et al. [23], having originally roots in the shell formulations of [24, 25] (see the earlier references therein as well); the second one can be considered as a non-trivial strain gradient extension of the hierarchic formulation for Timoshenko beams of classical elasticity by Kiendl et al. [21], having relatives among shell formulations in [29, 30]. It should be noticed, however, that in the strain gradient context the hierarchic beam formulation is not extendable to a single-variable formulation as in the context of classical elasticity in [21]. The resulting formulations still involve two variables requiring $H^2 \times H^3$ -regular Sobolev functions as trial and test functions and hence call for C^2 -continuity for the conformity of numerical methods, which will be naturally satisfied in this work by adopting the concept of isogeometric analysis [31] with B-spline basis functions of order $p \geq 3$. It is worth noting that according to numerical results both formulations are practically locking-free for wide ranges of thickness and gradient parameters—for both strain gradient and classical Timoshenko beams. Finally, it is demonstrated by experimental and numerical results that the strain gradient Timoshenko beam model vanquishes the corresponding Euler–Bernoulli model in certain circumstances, as happens within the classical elasticity theory as well.

This paper is organized as follows: In Section 2, we introduce our notation by recalling the theory of strain gradient elasticity applied to the Timoshenko beam bending

model with governing equations and boundary conditions. In Section 3, we propose the variational formulations of the problem, and present the corresponding numerical method. In Section 4, we analyze a series of numerical benchmarks and examples. Conclusions are finally drawn in Section 5.

2. Gradient-elastic Timoshenko beam model

This section starts by recalling a simplified version of Mindlin's strain gradient elasticity theory and then adopts the kinematical assumptions of the Timoshenko beam model for deriving the virtual strain and kinetic energies and the work done by external loadings as well as governing equations and boundary conditions of the problem.

2.1. Simplified strain gradient elasticity theory

Mindlin's strain gradient elasticity theory of Form II [9] defines the strain energy density for the isotropic case in the form

$$\begin{aligned} \mathcal{W} = & \frac{1}{2} \lambda \varepsilon_{ii} \varepsilon_{jj} + \mu \varepsilon_{ij} \varepsilon_{ij} + g_1 \varepsilon_{ik,i} \varepsilon_{jj,k} + g_2 \varepsilon_{ii,k} \varepsilon_{jj,k} \\ & + g_3 \varepsilon_{ik,i} \varepsilon_{jk,j} + g_4 \varepsilon_{ij,k} \varepsilon_{ij,k} + g_5 \varepsilon_{ij,k} \varepsilon_{ik,j}, \end{aligned} \quad (2.1)$$

with five non-classical material parameters $g_1, \dots, g_5$ besides the classical Lamé parameters λ and μ of the generalized Hooke's law relating the classical Cauchy stresses, or better Cauchy-like stresses of strain gradient elasticity, as

$$\sigma_{ij} = \frac{\partial \mathcal{W}}{\partial \varepsilon_{ij}} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} = \sigma_{ji}. \quad (2.2)$$

In (2.2), the linear strains are defined by the partial derivatives of displacements u_i in the form

$$\varepsilon_{ij} = (u_{j,i} + u_{i,j})/2. \quad (2.3)$$

As usual, indices i, j, k take values from the set of Cartesian coordinates $\{x, y, z\}$. With identity tensor $\mathbf{I}$ of the second rank, the tensor form of the generalized Hooke's law reads as

$$\boldsymbol{\sigma} = \lambda \text{tr} \boldsymbol{\varepsilon} \mathbf{I} + 2\mu \boldsymbol{\varepsilon}. \quad (2.4)$$

A one-parameter simplification proposed by Altan and Aifantis [11], and later adopted and analyzed by numerous authors, considers from (2.1) only those non-classical terms which relate to parameters g_2 and g_4 ($g_2 = g^2 \lambda / 2$, $g_4 = g^2 \mu$) reducing the strain energy density to the form

$$\mathcal{W} = \frac{1}{2} \lambda \varepsilon_{ii} \varepsilon_{jj} + \mu \varepsilon_{ij} \varepsilon_{ij} + g^2 \left(\frac{1}{2} \lambda \varepsilon_{ii,k} \varepsilon_{jj,k} + \mu \varepsilon_{ij,k} \varepsilon_{ij,k} \right), \quad (2.5)$$

where the extra material coefficient, gradient parameter g , describes the length scale of the micro-structure of the material. Within this formulation, the so-called double-stresses [9]

$$\tau_{ijk} = \frac{\partial \mathcal{W}}{\partial \varepsilon_{jk,i}} = g^2(\lambda \varepsilon_{ll,i} \delta_{jk} + 2\mu \varepsilon_{jk,i}) = \tau_{ikj} \quad (2.6)$$

are related to the strain derivatives by the Lamé parameters and the gradient parameter. For constant Lamé parameters, expression (2.6) gives $\tau_{ijk} = g^2 \sigma_{jk,i}$ [32, 33], or in the tensor notations

$$\boldsymbol{\tau} = g^2 \nabla \boldsymbol{\sigma}, \quad (2.7)$$

with the tensor-valued gradient operator denoted by ∇ . Finally, stress tensor expressions (2.4) and (2.7) give the strain energy density (2.5) in the form

$$\mathcal{W} = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \frac{1}{2} g^2 \nabla \boldsymbol{\sigma} : \nabla \boldsymbol{\varepsilon}, \quad (2.8)$$

where $:$ and $\dot{}$ denote the scalar products defined for second- and third-order tensors with Einstein's summation convention, respectively, as

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sigma_{ij} \varepsilon_{ij}, \quad \boldsymbol{\kappa} \dot{\boldsymbol{\gamma}} = \kappa_{ijk} \gamma_{ijk}. \quad (2.9)$$

2.2. Dimensionally reduced strain energy

In order to dimensionally reduce the presented energy setting to beam structures, let us consider a three-dimensional prismatic domain

$$\mathcal{P} = A \times \Omega, \quad (2.10)$$

where $\Omega = (0, L)$ is the central axis of the structure of length L and $A \subset \mathbb{R}^2$ denotes the cross section of the beam, with $\text{diam}(A) \ll L$. For simplicity, the cross section is assumed to be constant. In general, throughout the paper, the beam is assumed to be in *bending* rather than in *stretching* state. It is further assumed that the applied loadings as well as the geometry and material parameters of the beam structure are selected such that the problem reduces to a planar bending problem (in the xz -plane) with the classical kinematical Timoshenko assumptions allowing a displacement field of the form

$$\mathbf{u}(x, z) = -z\beta(x)\mathbf{e}_x + w(x)\mathbf{e}_z, \quad (2.11)$$

where two scalar fields $w : \Omega \rightarrow \mathbb{R}$ and $\beta : \Omega \rightarrow \mathbb{R}$ representing the transverse deflection of the central axis and rotations of planar cross sections serve as the independent unknowns of the problem, as in the classical elasticity theory for Timoshenko beams. The displacement field implies two non-trivial strain tensor components

$$\varepsilon_{xx} = -z\beta', \quad \varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}(w' - \beta), \quad (2.12)$$

where prime means a derivative with respect to x . The corresponding stresses are given by the standard stress assumptions on (2.2) for beams [34] giving

$$\sigma_{xx} = E\varepsilon_{xx}, \quad \sigma_{xz} = \sigma_{zx} = 2\mu\varepsilon_{xz}, \quad (2.13)$$

where $E = 2\mu(1 + \nu)$ stands for Young's modulus with ν denoting Poisson's ratio.

With these assumptions, the variation of the strain energy for the Timoshenko beam model following the simplified strain elasticity theory of (2.8) is given as

$$\begin{aligned} \delta W_{\text{int}}^g &= \int_{\mathcal{P}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta \mathbf{u}) \, d\mathcal{P} + \int_{\mathcal{P}} g^2 \nabla \boldsymbol{\sigma} : \nabla \boldsymbol{\varepsilon}(\delta \mathbf{u}) \, d\mathcal{P} \\ &= - \int_0^L \int_A \left(\sigma_{xx} z \delta \beta' - \sigma_{xz} (\delta w' - \delta \beta) \right) \, dA \, dx \\ &\quad - \int_0^L \int_A g^2 \left(\sigma'_{xx} z \delta \beta'' + \frac{\partial \sigma_{xx}}{\partial z} \delta \beta' - \sigma'_{xz} (\delta w' - \delta \beta)' \right) \, dA \, dx \\ &= - \int_0^L (M + g^2 R) \delta \beta' \, dx + \int_0^L Q (\delta w' - \delta \beta) \, dx \\ &\quad - \int_0^L g^2 M' \delta \beta'' \, dx + \int_0^L Q' (\delta w' - \delta \beta)' \, dx. \end{aligned} \quad (2.14)$$

In the last line, we have assumed a constant cross-sectional area and a non-constant gradient parameter $g = g(x)$, and defined the classical resultants, bending moment and shear force, and the so-called couple bending moment, respectively, as

$$M(x) = \int_A \sigma_{xx} z \, dA, \quad Q(x) = \int_A \sigma_{xz} \, dA, \quad R(x) = \int_A \frac{\partial \sigma_{xx}}{\partial z} \, dA. \quad (2.15)$$

Integration by parts in (2.14) proposes defining the bending moment and shear force of the gradient-elastic model, or the total bending moment and total shear force, as

$$M^g = M + g^2 R - (g^2 M')', \quad Q^g = Q - (g^2 Q')' \quad (2.16)$$

which reduce to $M^g = M - g^2 M'' + g^2 R$ and $Q^g = Q - g^2 Q''$ for constant g . With constant Lamé parameters, which is our assumption in what follows, the energy expression takes the form

$$\begin{aligned} \delta W_{\text{int}}^g &= \int_0^L (EI \beta' \delta \beta' + KGA(w' - \beta)(\delta w' - \delta \beta)) \, dx \\ &\quad + \int_0^L g^2 (EI \beta'' \delta \beta'' + KGA(w'' - \beta')(\delta w'' - \delta \beta') + EA \beta' \delta \beta') \, dx, \end{aligned} \quad (2.17)$$

where $G = \mu$ denotes the shear moduli, I stands for the geometric moment of inertia of the cross section around the y -axis, and K is the so-called shear correction factor depending on the cross section shape (e.g. see [35]).

Remark 1. *The contribution of the partial derivatives with respect to z -coordinate of the ∇ -operator in (2.14), stemming from the three-dimensional continuum theory, is neglected in some papers devoted to gradient-elastic beam models. However, the results in [13] demonstrate its importance in producing a thickness-dependent stiffening effect.*

2.3. Dimensionally reduced kinetic energy

Regarding the kinetic energy for the gradient elasticity theory, an additional gradient parameter χ introducing a micro-inertia term has been already proposed by Mindlin [9] in order to achieve a physically satisfactory dispersion relation for a large range of wave numbers [28]. The kinetic energy density is then written in the form

$$\mathcal{W}_{\text{kin}}^\chi = \frac{1}{2}\rho\dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \frac{1}{2}\chi^2\rho\nabla\dot{\mathbf{u}} : \nabla\dot{\mathbf{u}}, \quad (2.18)$$

where ρ stands for mass density and the upper dot symbol for the time derivative. By substituting the kinematical assumptions of (2.11) in (2.18), taking a variation and integrating over time interval (t_0, t_1) , one can obtain an expression for the virtual kinetic energy (with time variable τ) in the form

$$\begin{aligned} \delta \int_{t_0}^{t_1} \mathcal{W}_{\text{kin}}^\chi d\tau = & - \int_{t_0}^{t_1} \int_0^L \rho (I\ddot{\beta}\delta\beta + A\ddot{w}\delta w \\ & + \chi^2(I\ddot{\beta}'\delta\beta' + A\ddot{\beta}\delta\beta + A\ddot{w}'\delta w')) dx d\tau. \end{aligned} \quad (2.19)$$

It should be mentioned that in the right-hand side above, integration by parts with respect to the time variable together with zero initial boundary conditions have been applied, as usual.

Remark 2. *The rotational inertia terms $\rho I\ddot{\beta}\delta\beta$ and $\rho\chi^2 I\ddot{\beta}'\delta\beta'$ are often omitted in the derivations of the governing equations of gradient-elastic beams. Nevertheless, the importance of this terms for the dynamics of Euler–Bernoulli beams has been recently shown in [13]. In the present work, the rotational inertia terms are involved and, therefore, the present model can be considered as a gradient-elastic analogue of the classical Timoshenko-Rayleigh model.*

2.4. Dimensionally reduced work done by external forces

The classical expression of the virtual work done by external forces is augmented by one additional gradient term:

$$\delta W_{\text{ext}}^g = \int_{\mathcal{P}} \mathbf{f} \cdot \delta \mathbf{u} d\mathcal{P} + \int_{\partial\mathcal{P}} \mathbf{t} \cdot \delta \mathbf{u} d\partial\mathcal{P} + \int_{\partial\mathcal{P}} \mathbf{r} \cdot (\mathbf{n} \cdot \nabla \delta \mathbf{u}) d\partial\mathcal{P}, \quad (2.20)$$

where $\mathbf{f}$ and $\mathbf{t}$ stand for the body force and surface traction force, respectively, and $\mathbf{r}$ denotes the double traction force with $\mathbf{n}$ denoting an outward unit normal to the boundary surface $\partial\mathcal{P}$. In the expression above, it has been assumed that only classical body forces are present and there are no forces on the sharp edges or corners of the structure (cf. [16, 36, 37, 33]).

Following the dimension reduction assumptions of the Timoshenko beam model, one can reduce expression (2.20) into the form

$$\delta W_{\text{ext}}^g = \int_0^L (m\delta\beta + q\delta w) dx + (\mathcal{M}\delta\beta + \mathcal{Q}\delta w)|_0^L + (\mathfrak{M}\delta\beta' + \mathfrak{R}\delta w')|_0^L, \quad (2.21)$$

with distributed loadings resulting from body loads,

$$m(x) = - \int_A f_x z \, dA, \quad q(x) = \int_A f_z \, dA,$$

and end point loadings resulting from tractions,

$$\mathcal{M} = - \int_A t_x z \, dA, \quad \mathcal{Q} = \int_A t_z \, dA, \quad \mathfrak{M} = - \int_A r_x z \, dA, \quad \mathfrak{R} = \int_A r_z \, dA.$$

2.5. Governing equations and boundary conditions

With Hamilton's principle for energy expressions (2.17) and (2.19) with (2.21) in the form

$$\int_{t_0}^{t_1} (\delta W_{\text{kin}}^x - \delta W_{\text{int}}^g + \delta W_{\text{ext}}^g) = 0 \quad (2.22)$$

together with standard derivations, one can obtain the governing equations

$$\begin{aligned} KGA((w'' - \beta') - g^2(w'''' - \beta''')) \\ - \rho A(\ddot{w} - \chi^2 \ddot{w}'') = -q, \\ EI(\beta'' - g^2 \beta''') + KGA((w' - \beta) - g^2(w''' - \beta'')) + EA g^2 \beta'' \\ - \rho A \chi^2 \ddot{\beta} - \rho I(\ddot{\beta} - \chi^2 \ddot{\beta}') = -m \end{aligned} \quad (2.23)$$

along with boundary conditions

$$\begin{aligned} KGA(w' - \beta) - KGA g^2(w''' - \beta'') + \rho A \chi^2 \ddot{w}' &= \overline{\mathcal{Q}} \text{ or } w = \overline{w}, \\ EI(\beta' - g^2 \beta''') - KGA g^2(w'' - \beta') + EA g^2 \beta' + \rho I \chi^2 \ddot{\beta}' &= \overline{\mathcal{M}} \text{ or } \beta = \overline{\beta}, \\ EI g^2 \beta'' &= \overline{\mathfrak{M}} \text{ or } \beta' = \overline{\kappa}, \\ KGA g^2(w'' - \beta') &= \overline{\mathfrak{R}} \text{ or } w' = \overline{\phi}, \end{aligned} \quad (2.24)$$

where the overlined symbols denote the appropriate given boundary data. For statics, the governing equations reduce to the form

$$\begin{aligned} KGA((w'' - \beta') - g^2(w'''' - \beta''')) &= -q, \\ EI(\beta'' - g^2 \beta''') + KGA((w' - \beta) - g^2(w''' - \beta'')) + EA g^2 \beta'' &= -m, \end{aligned} \quad (2.25)$$

with the corresponding boundary conditions

$$\begin{aligned} KGA(w' - \beta) - g^2 KGA(w''' - \beta'') &= \overline{\mathcal{Q}} \text{ or } w = \overline{w}, \\ EI(\beta' - g^2 \beta''') - g^2 KGA(w'' - \beta') + EA g^2 \beta' &= \overline{\mathcal{M}} \text{ or } \beta = \overline{\beta}, \\ g^2 EI \beta'' &= \overline{\mathfrak{M}} \text{ or } \beta' = \overline{\kappa}, \\ g^2 KGA(w'' - \beta') &= \overline{\mathfrak{R}} \text{ or } w' = \overline{\phi}. \end{aligned} \quad (2.26)$$

From the physical point of view, the boundary conditions above should be now grouped such that they describe at least the three standard types: clamped, simply

supported and free. Let us follow [12, 33, 13] and distinguish the clamped and simply supported boundaries into two different types according to the curvature: *singly* and *doubly* referring to unprescribed and prescribed curvature, accordingly – with the subscripts s and d. In this way, five different boundary condition types can be defined: *doubly clamped* and *singly clamped* boundaries, respectively,

$$\begin{aligned} w &= \bar{w} \text{ and } \beta = \bar{\beta} \text{ and} \\ \beta' &= \bar{\kappa} \text{ and } g^2 KGA(w'' - \beta') = \bar{\mathfrak{R}} \text{ on } \Gamma_{\text{Cd}}, \end{aligned} \quad (2.27)$$

$$\begin{aligned} w &= \bar{w} \text{ and } \beta = \bar{\beta} \text{ and} \\ g^2 EI\beta'' &= \bar{\mathfrak{M}} \text{ and } g^2 KGA(w'' - \beta') = \bar{\mathfrak{R}} \text{ on } \Gamma_{\text{Cs}}, \end{aligned} \quad (2.28)$$

doubly simply supported boundaries

$$\begin{aligned} w &= \bar{w} \text{ and} \\ EI(\beta' - g^2 \beta''') - g^2 KGA(w'' - \beta') + EA g^2 \beta' &= \bar{\mathcal{M}} \text{ and} \\ \beta' &= \bar{\kappa} \text{ and} \\ g^2 KGA(w'' - \beta') &= \bar{\mathfrak{R}} \text{ on } \Gamma_{\text{Sd}}, \end{aligned} \quad (2.29)$$

singly simply supported boundaries

$$\begin{aligned} w &= \bar{w} \text{ and} \\ EI(\beta' - g^2 \beta''') - g^2 KGA(w'' - \beta') + EA g^2 \beta' &= \bar{\mathcal{M}} \text{ and} \\ g^2 EI\beta'' &= \bar{\mathfrak{M}} \text{ and} \\ g^2 KGA(w'' - \beta') &= \bar{\mathfrak{R}} \text{ on } \Gamma_{\text{Ss}}, \end{aligned} \quad (2.30)$$

and *free* boundaries,

$$\begin{aligned} KGA(w' - \beta) - g^2 KGA(w''' - \beta'') &= \bar{\mathcal{Q}} \text{ and} \\ EI(\beta' - g^2 \beta''') - g^2 KGA(w'' - \beta') + EA g^2 \beta' &= \bar{\mathcal{M}} \text{ and} \\ g^2 EI\beta'' &= \bar{\mathfrak{M}} \text{ and} \\ g^2 KGA(w'' - \beta') &= \bar{\mathfrak{R}} \text{ on } \Gamma_{\text{F}}, \end{aligned} \quad (2.31)$$

It should be noticed that fixing w' independently (see (2.26)) would give even more boundary condition types.

3. Variational formulations

This section aims at presenting the standard and two locking-free variational formulations of the problem and then reduces them to the case of classical elasticity theory, including finally conforming Galerkin discretizations.

3.1. Standard formulation and shear locking effect

The weak form of the boundary value problem consisting of (2.25) and the chosen boundary conditions among (2.27)–(2.31) can be formulated in the following form: For $q, m \in L^2(\Omega)$, $\Omega = (0, L)$, find $(w, \beta) \in \mathcal{W} \times \mathcal{V}$ such that

$$a(w, \beta; \hat{w}, \hat{\beta}) = l(\hat{w}, \hat{\beta}) \quad \forall (\hat{w}, \hat{\beta}) \in \widehat{\mathcal{W}} \times \widehat{\mathcal{V}}, \quad (3.1)$$

where the bilinear form $a : (\mathcal{W} \times \mathcal{V}) \times (\widehat{\mathcal{W}} \times \widehat{\mathcal{V}}) \rightarrow \mathbb{R}$, with $a(w, \beta; \hat{w}, \hat{\beta}) = a^c(w, \beta; \hat{w}, \hat{\beta}) + a^g(w, \beta; \hat{w}, \hat{\beta})$, and load functional $l : \widehat{\mathcal{W}} \times \widehat{\mathcal{V}} \rightarrow \mathbb{R}$ are defined as

$$a^c(w, \beta; \hat{w}, \hat{\beta}) = \int_0^L (EI\beta'\hat{\beta}' + KGA(w' - \beta)(\hat{w}' - \hat{\beta})) dx, \quad (3.2)$$

$$a^g(w, \beta; \hat{w}, \hat{\beta}) = \int_0^L g^2(EI\beta''\hat{\beta}'' + KGA(w'' - \beta')(\hat{w}'' - \hat{\beta}') + EA\beta'\hat{\beta}') dx, \quad (3.3)$$

$$l(\hat{w}, \hat{\beta}) = \int_0^L (q\hat{w} + m\hat{\beta}) dx. \quad (3.4)$$

The trial function spaces are defined as

$$\mathcal{W} = \{v \in H^2(\Omega) \mid v|_{\Gamma_S \cup \Gamma_C} = \bar{w}\} \quad (3.5)$$

$$\mathcal{V} = \{\eta \in H^2(\Omega) \mid \eta|_{\Gamma_C} = \bar{\beta}, \quad \eta'|_{\Gamma_{C_d} \cup \Gamma_{S_d}} = \bar{\kappa}\}, \quad (3.6)$$

with given Dirichlet data $\bar{w}, \bar{\beta}, \bar{\kappa}$ and with $\Gamma_C = \Gamma_{C_s} \cup \Gamma_{C_d}$ denoting (singly and doubly) clamped ends, $\Gamma_S = \Gamma_{S_s} \cup \Gamma_{S_d}$ denoting (singly and doubly) simply supported ends and Γ_F standing for free ends, whereas test function spaces $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{V}}$ consist of H^2 functions satisfying the corresponding homogeneous Dirichlet boundary conditions.

It is a well known fact that Galerkin methods for Timoshenko beams and Reissner–Mindlin plates and shells suffer from the *numerical shear locking effect* for small values of the thickness parameter. In order to demonstrate the origin of the *shear locking effect*, let us divide the both sides of equation (3.1) by bending rigidity EI giving scaled counterparts for the bilinear forms and load functional:

$$\tilde{a}^c(w, \beta; \hat{w}, \hat{\beta}) = \int_0^L (\beta'\hat{\beta}' + \frac{KGA}{EI}(w' - \beta)(\hat{w}' - \hat{\beta})) dx, \quad (3.7)$$

$$\tilde{a}^g(w, \beta; \hat{w}, \hat{\beta}) = \int_0^L g^2(\beta''\hat{\beta}'' + \frac{KGA}{EI}(w'' - \beta')(\hat{w}'' - \hat{\beta}') + \frac{EA}{EI}\beta'\hat{\beta}') dx, \quad (3.8)$$

$$\tilde{l}(\hat{w}, \hat{\beta}) = \int_0^L (\frac{q}{EI}\hat{w} + \frac{m}{EI}\hat{\beta}) dx. \quad (3.9)$$

For simplicity and without loss of generality, let us consider a rectangular beam cross section satisfying $b = t$, with b and t denoting the width and thickness of the beam, respectively, and giving the cross-sectional area and the moment of inertia in terms of thickness as $A = bt = t^2, I = bt^3/12 = t^4/12$, respectively (or more generally

$A = \alpha_A t^2, I = \alpha_I t^4$ with cross-sectional constants α_A and α_I). Accordingly, the scaled version of the energy balance is written as

$$\tilde{a}(w, \beta; \hat{w}, \hat{\beta}) = \tilde{l}(\hat{w}, \hat{\beta}) \quad (3.10)$$

where the load functional is now defined in the form

$$\tilde{l}(\hat{w}, \hat{\beta}) = \int_0^L (\tilde{q}\hat{w} + \tilde{m}\hat{\beta}) dx, \quad (3.11)$$

with $\tilde{q}$ and $\tilde{m}$ being independent of t . The classical locking problem stems from the scaled shear energy term

$$\frac{KGA}{EI}(w' - \beta)(\hat{w}' - \hat{\beta}) = \frac{6K}{(1 + \nu)t^2}(w' - \beta)(\hat{w}' - \hat{\beta}) \quad (3.12)$$

which, for bounded shear energies in the limit $t \rightarrow 0$, imposes the Euler–Bernoulli limit condition $w' - \beta \rightarrow 0$ forcing constraint $w' = \beta$ (satisfied by Euler–Bernoulli beams). In general, this condition cannot be satisfied in numerical methods by equal order shape function approximations for w and β , which causes the locking effect (not only in limit $t \rightarrow 0$ but for small values of t as well).

Within strain gradient elasticity, another locking term is present in the formulation:

$$\frac{g^2 KGA}{EI}(w'' - \beta')(\hat{w}'' - \hat{\beta}') = \frac{6g^2 K}{(1 + \nu)t^2}(w'' - \beta')(\hat{w}'' - \hat{\beta}'). \quad (3.13)$$

Accordingly, an extra condition is imposed in limit $t/g \rightarrow 0$, i.e., $w'' - \beta' \rightarrow 0$. Both locking effects are demonstrated by numerical examples in Section 4 for different parameter values. It should be noticed that as long as g is considered to represent a material length scale, it typically holds that $g < t$.

3.2. Locking-free formulations for strain gradient elasticity

In what follows, we propose two formulations for avoiding the locking effect by following the central idea of [24, 25, 23] proposing locking-free Reissner–Mindlin shell and plate formulations within the classical elasticity theory. As clarified below, for strain gradient elasticity one needs to introduce another change of variable, having roots in [21], in order to reach a fully, or better intrinsically, locking-free formulation.

Let us first introduce a change of variable in the following fashion:

$$\beta \hookrightarrow w' - \gamma \quad (3.14)$$

with a new independent variable, (engineering) shear strain $\gamma = w' - \beta$, substituting rotation β . Accordingly, Problem (3.1) is reformulated as follows:

Problem 1. For $q, m \in L^2(\Omega)$, $\Omega = (0, L)$, find $(w, \gamma) \in \mathcal{W} \times \mathcal{V}$ such that

$$a_1(w, \gamma; \hat{w}, \hat{\gamma}) = l_1(\hat{w}, \hat{\gamma}) \quad \forall (\hat{w}, \hat{\gamma}) \in \widehat{\mathcal{W}} \times \widehat{\mathcal{V}}, \quad (3.15)$$

where the bilinear form $a_1 : (\mathcal{W} \times \mathcal{V}) \times (\widehat{\mathcal{W}} \times \widehat{\mathcal{V}}) \rightarrow \mathbb{R}$, with $a_1(w, \gamma; \hat{w}, \hat{\gamma}) = a_1^c(w, \gamma; \hat{w}, \hat{\gamma}) + a_1^g(w, \gamma; \hat{w}, \hat{\gamma})$, load functional $l_1 : \widehat{\mathcal{W}} \times \widehat{\mathcal{V}} \rightarrow \mathbb{R}$ and the trial function spaces are defined as

$$a_1^c(w, \gamma; \hat{w}, \hat{\gamma}) = \int_0^L (EI(w'' - \gamma')(\hat{w}'' - \hat{\gamma}') + KGA\gamma\hat{\gamma}) \, dx, \quad (3.16)$$

$$a_1^g(w, \gamma; \hat{w}, \hat{\gamma}) = \int_0^L g^2(EI(w''' - \gamma'')(\hat{w}''' - \hat{\gamma}'') + KGA\gamma'\hat{\gamma}' + EA(w'' - \gamma')(\hat{w}'' - \hat{\gamma}')) \, dx, \quad (3.17)$$

$$l_1(\hat{w}, \hat{\gamma}) = \int_0^L (q\hat{w} + m(\hat{w}' - \hat{\gamma})) \, dx, \quad (3.18)$$

$$\mathcal{W} \times \mathcal{V} = \{(v, \eta) \in H^3(\Omega) \times H^2(\Omega) \mid v|_{\Gamma_S \cup \Gamma_C} = \bar{w}, (v' - \eta)|_{\Gamma_C} = \bar{\beta}, (v'' - \eta')|_{\Gamma_{Cd} \cup \Gamma_{Sd}} = \bar{\kappa}\}. \quad (3.19)$$

We call this *first locking-free (LF) formulation*—which according to numerical results is shown to be practically locking-free although it actually (after scaling) includes term

$$\frac{g^2 EA}{EI}(w'' - \gamma')(\hat{w}'' - \hat{\gamma}') = \frac{g^2}{t^2}(w'' - \gamma')(\hat{w}'' - \hat{\gamma}') \quad (3.20)$$

which can still be a source of locking for small values of ratio t/g .

Nevertheless, in order to avoid locking terms of any kind we introduce the *second locking-free (LF) formulation* with another change of variables:

$$w \hookrightarrow v + \vartheta, \beta \hookrightarrow \vartheta', \quad (3.21)$$

for which the weak formulation is written as follows:

Problem 2. For $q, m \in L^2(\Omega)$, $\Omega = (0, L)$, find $(v, \vartheta) \in \mathcal{W} \times \mathcal{V}$ such that

$$a_2(v, \vartheta; \hat{v}, \hat{\vartheta}) = l_2(\hat{v}, \hat{\vartheta}) \quad \forall (\hat{v}, \hat{\vartheta}) \in \widehat{\mathcal{W}} \times \widehat{\mathcal{V}}, \quad (3.22)$$

where the bilinear form $a_2 : (\mathcal{W} \times \mathcal{V}) \times (\widehat{\mathcal{W}} \times \widehat{\mathcal{V}}) \rightarrow \mathbb{R}$, with $a_2(v, \vartheta; \hat{v}, \hat{\vartheta}) = a_2^c(v, \vartheta; \hat{v}, \hat{\vartheta}) + a_2^g(v, \vartheta; \hat{v}, \hat{\vartheta})$, load functional $l_2 : \widehat{\mathcal{W}} \times \widehat{\mathcal{V}} \rightarrow \mathbb{R}$ and the trial function spaces are defined

as

$$a_2^c(v, \vartheta; \hat{v}, \hat{\vartheta}) = \int_0^L (EI\vartheta''\hat{\vartheta}'' + KGA v'\hat{v}') \, dx, \quad (3.23)$$

$$a_2^g(v, \vartheta; \hat{v}, \hat{\vartheta}) = \int_0^L g^2 (EI\vartheta'''\hat{\vartheta}''' + KGA v''\hat{v}'' + EA\vartheta''\hat{\vartheta}'') \, dx, \quad (3.24)$$

$$l_2(\hat{v}, \hat{\vartheta}) = \int_0^L (q(\hat{v} + \hat{\vartheta}) + m\hat{\vartheta}') \, dx, \quad (3.25)$$

$$\mathcal{W} \times \mathcal{V} = \{(v, \eta) \in H^2(\Omega) \times H^3(\Omega) \mid (v + \eta)|_{\Gamma_S \cup \Gamma_C} = \bar{w}, \\ \eta'|_{\Gamma_C} = \bar{\beta}, \eta''|_{\Gamma_{Cd} \cup \Gamma_{Sd}} = \bar{\kappa}\}. \quad (3.26)$$

This formulation does not contain any sources of locking. Moreover, the bilinear forms do not contain any cross terms meaning that the stiffness matrix consists of two decoupled square blocks. The problem variables, however, are coupled by the boundary conditions fixing the deflection of the standard formulation: $w = v + \vartheta = \bar{w}$ on $\Gamma_S \cup \Gamma_C$. The energy norm induced by the bilinear form of Problem 2 is defined as

$$\|(\hat{v}, \hat{\vartheta})\|_{a_2}^2 = \int_0^L (EI + g^2 EA)(\hat{\vartheta}'')^2 \, d\Omega + \int_0^L g^2 EI(\hat{\vartheta}''')^2 \, d\Omega \\ + \int_0^L KGA(\hat{v}')^2 \, d\Omega + \int_0^L g^2 KGA(\hat{v}'')^2 \, d\Omega. \quad (3.27)$$

The symmetry of the bilinear form is clearly guaranteed whenever $\mathcal{W} \times \mathcal{V} = \widehat{\mathcal{W}} \times \widehat{\mathcal{V}}$, i.e., $a_2(v, \vartheta; \hat{v}, \hat{\vartheta}) = a_2(\hat{v}, \hat{\vartheta}; v, \vartheta) \, \forall (v, \vartheta), (\hat{v}, \hat{\vartheta}) \in \mathcal{W} \times \mathcal{V}$.

Remark 3. *This formulation with $w = \vartheta + v$ and $\beta = \vartheta'$ turns out to be using the same change of variables as introduced in [21] for Timoshenko beams of classical elasticity splitting the deflection in bending and shear parts as $w = w_b + w_s$ giving rotation $\varphi = -w'_b$ (note the sign convention in [21], $\phi = -\beta$) and shear strain $\gamma = w'_s$. However, the strain gradient formulation cannot be formulated as a single variable problem as the formulation in [21] based on solving (integrating) the deflection in terms of rotation from the classical moment balance equation*

$$EI\beta'' + KGA(w' - \beta) = 0 \quad (3.28)$$

obtained from (2.25) with $g = 0$ and $m = 0$.

Remark 4. *Compared to the corresponding Euler–Bernoulli formulation*

$$a^c(w, \hat{w}) = \int_0^L EI w'' \hat{w}'' \, d\Omega, \quad (3.29)$$

$$a^g(w, \hat{w}) = \int_0^L g^2 EA w'' \hat{w}'' \, d\Omega + \int_0^L g^2 EI w''' \hat{w}''' \, d\Omega \quad (3.30)$$

analyzed in [13], the present Timoshenko formulation simply includes the additional shear energy terms (in (3.23)–(3.24) with factor KGA). However, proving continuity and coercivity of the bilinear form in the present case is not as straightforward as in [13] due to the essential boundary conditions in the function space $\mathcal{W} \times \mathcal{V}$ coupling the problem variables (cf. [23, 21]). Nevertheless, the numerical results in Section 4 adopting the Nitsche method [38] for enforcing the boundary conditions in a weak sense demonstrate the stability of the formulation.

Finally, the eigenvalue problem for free vibrations (see Appendix B) corresponding to the standard formulation (3.1) and locking-free formulations of Problems 1 and 2 can be formulated as follows:

Problem 3. Find eigenpairs $(w, \beta), \Lambda$, with $(w, \beta) \in \mathcal{W} \times \mathcal{V}, \Lambda = \omega^2 \in \mathbb{R}$, such that

$$a(w, \beta; \hat{w}, \hat{\beta}) - \omega^2 b(w, \beta; \hat{w}, \hat{\beta}) = 0 \quad \forall (\hat{w}, \hat{\beta}) \in \widehat{\mathcal{W}} \times \widehat{\mathcal{V}}, \quad (3.31)$$

where bilinear form $a(w, \beta; \hat{w}, \hat{\beta})$ follows the definition of formulation (3.1) and $b : (\mathcal{W} \times \mathcal{V}) \times (\widehat{\mathcal{W}} \times \widehat{\mathcal{V}}) \rightarrow \mathbb{R}$, $b(w, \beta; \hat{w}, \hat{\beta}) = b^c(w, \beta; \hat{w}, \hat{\beta}) + b^x(w, \beta; \hat{w}, \hat{\beta})$, forming the generalized mass matrix, is defined by forms

$$b^c(w, \beta; \hat{w}, \hat{\beta}) = \int_0^L \rho (Aw\hat{w} + I\beta\hat{\beta}) \, dx, \quad (3.32)$$

$$b^x(w, \beta; \hat{w}, \hat{\beta}) = \int_0^L \rho \chi^2 (A(w'\hat{w}' + \beta\hat{\beta}) + I\beta'\hat{\beta}') \, dx. \quad (3.33)$$

For Problems 1 and 2, a is replaced by a_1 and a_2 (or by the scaled versions $\bar{a}_1$ and $\bar{a}_2$), respectively, and b is modified accordingly:

$$b_1^c(w, \gamma; \hat{w}, \hat{\gamma}) = \int_0^L \rho (Aw\hat{w} + I(w' - \gamma)(\hat{w}' - \hat{\gamma})) \, dx, \quad (3.34)$$

$$b_1^x(w, \gamma; \hat{w}, \hat{\gamma}) = \int_0^L \rho \chi^2 (A(w'\hat{w}' + (w' - \gamma)(\hat{w}' - \hat{\gamma})) + I(w'' - \gamma')(\hat{w}'' - \hat{\gamma}')) \, dx, \quad (3.35)$$

$$b_2^c(v, \vartheta; \hat{v}, \hat{\vartheta}) = \int_0^L \rho (A(v + \vartheta)(\hat{v} + \hat{\vartheta}) + I\vartheta'\hat{\vartheta}') \, dx, \quad (3.36)$$

$$b_2^x(v, \vartheta; \hat{v}, \hat{\vartheta}) = \int_0^L \rho \chi^2 (A((v + \vartheta)'(\hat{v} + \hat{\vartheta})' + \vartheta'\hat{\vartheta}') + I\vartheta''\hat{\vartheta}'') \, dx. \quad (3.37)$$

It should be noticed that in b_1^x , one locking term is still present (cf. (3.20)):

$$\frac{\rho \chi^2 A}{EI} (w' - \gamma)(\hat{w}' - \hat{\gamma}). \quad (3.38)$$

3.3. Locking-free formulations for classical elasticity

In consideration of the foregoing, one can reduce the locking-free strain gradient formulations to the corresponding ones of the classical elasticity:

Problem 4. For $q, m \in L^2(\Omega)$, $\Omega = (0, L)$, find $(w, \gamma) \in \mathcal{W} \times \mathcal{V}$ such that

$$a_1^c(w, \gamma; \hat{w}, \hat{\gamma}) = l_1(\hat{w}, \hat{\gamma}) \quad \forall (\hat{w}, \hat{\gamma}) \in \widehat{\mathcal{W}} \times \widehat{\mathcal{V}}, \quad (3.39)$$

with function space

$$\mathcal{W} \times \mathcal{V} = \{(v, \eta) \in H^2(\Omega) \times H^1(\Omega) \mid v|_{\Gamma_S \cup \Gamma_C} = \bar{w}, (v' - \eta)|_{\Gamma_C} = \bar{\beta}\}. \quad (3.40)$$

Remark 5. The corresponding Reissner–Mindlin plate formulation has been proposed and analyzed in [23].

Problem 5. For $q, m \in L^2(\Omega)$, $\Omega = (0, L)$, find $(v, \vartheta) \in \mathcal{W} \times \mathcal{V}$ such that

$$a_2^c(v, \vartheta; \hat{v}, \hat{\vartheta}) = l_2(\hat{v}, \hat{\vartheta}) \quad \forall (\hat{v}, \hat{\vartheta}) \in \widehat{\mathcal{W}} \times \widehat{\mathcal{V}}, \quad (3.41)$$

with function space

$$\mathcal{W} \times \mathcal{V} = \{(v, \eta) \in H^1(\Omega) \times H^2(\Omega) \mid (v + \eta)|_{\Gamma_S \cup \Gamma_C} = \bar{w}, \eta'|_{\Gamma_C} = \bar{\beta}\}. \quad (3.42)$$

Remark 6. The corresponding Timoshenko beam formulation of classical elasticity with $m = 0$ has been reformulated as a single-variable formulation in [21].

3.4. Conforming isogeometric Galerkin implementations

A discrete counterpart of problem (3.1) for finding approximate numerical solutions reads as follows: For $q, m \in L^2(\Omega)$, $\Omega = (0, L)$, find $(w_h, \beta_h) \in \mathcal{W}_h \times \mathcal{V}_h \subset \mathcal{W} \times \mathcal{V}$ such that

$$a(w_h, \beta_h; \hat{w}, \hat{\beta}) = l(\hat{w}, \hat{\beta}) \quad \forall (\hat{w}, \hat{\beta}) \in \widehat{\mathcal{W}}_h \times \widehat{\mathcal{V}}_h \subset \widehat{\mathcal{W}} \times \widehat{\mathcal{V}}. \quad (3.43)$$

Regarding the corresponding locking-free formulations of Problems 1, 2 and 3, it should be noticed that H^3 -regular functions are required for one of the variables: $\mathcal{W}_h \subset H^3(\Omega)$ or $\mathcal{V}_h \subset H^3(\Omega)$, respectively. Instead, in the standard formulation (3.1) H^2 regularity is enough for both variables: $\mathcal{W}_h, \mathcal{V}_h \subset H^2(\Omega)$.

The discrete formulations of (3.43) have been implemented by using the principles of isogeometric analysis [31] (cf. [23, 21, 12, 39, 40, 13]). Associated to an open knot vector (allowing knot repetitions) $\{0 = x_1, \dots, x_{m+p+1} = 1\}$, with m denoting the number of basis functions, B-spline basis functions of order $p \geq 1$ are defined recursively by Cox–de Boor recursion (by definition $0/0 = 0$) as

$$N_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} N_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} N_{i+1,p-1}(x), \quad (3.44)$$

where the zeroth order ones are defined as follows: $N_{i,0}(x) = 1$ for $x_i \leq x < x_{i+1}$, $N_{i,0}(x) = 0$ elsewhere.

For the corresponding approximation $w_h(x) = \sum_{i=1}^m N_{i,p}(x) \hat{w}_i$ with (unknown) control variables $\hat{w}_i$, the B-spline basis functions provide regularity C^{p-1} over the mesh. In particular, since $C^{p-1}(\Omega) \subset H^3(\Omega)$ with $p \geq 3$ this approach provides $H^3(\Omega)$ -conforming discretizations for the locking-free formulations (whereas $p \geq 2$ is enough for the standard formulation).

4. Numerical results and model comparisons

In the present section, unless otherwise stated, the following problem parameters are used: we consider a beam with length $L = 10$, rectangular cross-section (width $b = 0.5$, thickness $t = 0.01$), Young's modulus $E = 210000$, Poisson's ratio $\nu = 0.3$, shear modulus $G = E/(1 + \nu)/2 = 80769$, shear correction factor $K = 0.85$, and gradient parameter $g = 0.001$. It is perceived that the value of parameter g cannot exceed the thickness value t . Therefore, it is assumed below that $g = t/10$, with some exclusions explicitly mentioned in context. In order to have a simple analytical reference solution for the problem, the *doubly simply supported* boundary conditions ((2.29) with zero tractions) are applied on both ends of the beam. A sinusoidal external loading (see (A.1) in Appendix A) with $\tilde{q}EI = 0.0001$ affects the beam, $m = 0$. The analytical solution for this model problem is presented in Appendix A. For the problems of free vibrations, in addition, the mass density and micro-inertial gradient parameter are chosen to be $\rho = 7.85\text{e-}9$, $\chi = 0.0005$, respectively.

The numerical results have been obtained by extending the package GeoPDEs [41] to the strain gradient context.

4.1. Convergence studies for statics

4.1.1. Locking-free formulations – strain gradient elasticity

In order to study the convergence properties of the *locking-free formulations*, first, we solve the problem with B-spline basis functions of orders $p = 3, 4, 5$. In Fig. 4.1, we plot the relative error of the numerical solution in different norms versus the mesh size, in logarithmic scales. Solid lines refer to the *first locking-free formulation* of Problem 1, dashed lines refer to the *second locking-free formulation* of Problem 2. It can be observed that the convergence curves follow the expected theoretical order: the error in the deflection measured in the H^3 norm follows the convergence order close to order $\mathcal{O}(h^{p-2})$, whereas the H^2 norm follows order $\mathcal{O}(h^{p-1})$, and the L^2 norm, as expected, converges as rapidly as order $\mathcal{O}(h^{p+1})$.

Fig. 4.1: Convergence curves for the deflection in the H^3 , H^2 and L^2 norms: B-splines of order $p = 3, 4, 5$ with $t/L = 0.001$ and $g/t = 0.1$.

In Fig.4.2, we depict relative errors in the L^2 norm for the stress resultants, shear force and bending moment: $Q = KGA(w' - \beta) = KGA\gamma = KGA v'$, $M = -EI\beta' = -EI(w'' - \gamma') = -EI\vartheta''$. Concerning the convergence curves for shear force (Fig. 4.2a), the following is observed: for the *first LF formulation* (solid lines), the error curves approximately follow order $\mathcal{O}(h^{p-1})$ for $p = 4, 5$ and $\mathcal{O}(h^p)$ for $p = 3$, whereas for the *second LF formulation* (dashed lines) the shear force error quite accurately follows order $\mathcal{O}(h^p)$ for every p . The error curves for moment (Fig. 4.2b) follow order $\mathcal{O}(h^{p-1})$ for both formulations.

Altogether, the convergence curves of Fig. 4.1 and 4.2 show that both of the new formulations are practically locking-free with respect to both deflection and moment. For shear force, however, the *first LF formulation* exhibits a minor deficiency in the convergence rates. This can be explained by the presence of the higher-order term (3.20) having a disposition for locking. The *second LF formulation*, by contrast, provides practically perfect convergence results for shear force as well.

Fig. 4.2: Convergence curves for the stress resultants in the L^2 norm: B-splines of order $p = 3, 4, 5$ with $t/L = 0.001$ and $g/t = 0.1$.

Second, for comparing the convergence properties of the *standard formulation* and *locking-free formulations*, we solve the same model problem for different thickness values: $t/L = 0.1, 0.01, 0.001, 0.0001$. The gradient parameter g is decreased alongside with thickness such that $g/t = 0.1$. The basis function order is $p = 3$. In Fig. 4.3, one can see the relative errors in the H^2 (for all formulations) and H^3 norms (only for the *locking-free (LF) formulations*). As expected, the locking effect is pronounced for the *standard formulation* with the decreasing beam thickness, whilst for the *locking-free formulations* the convergence curves are not sensitive to changes in the thickness.

Fig. 4.3: Convergence curves for the deflection in the H^3 and H^2 norms: B-splines of order $p = 3$ with $t/L = 0.1, 0.01, 0.001, 0.0001$ and $g/t = 0.1$.

Third, in order to study how changes in the gradient parameter affect the convergence order we consider another set of parameter values. We fix the thickness to $t = 0.01$ and change the gradient parameter such that $g/t = 1, 0.1, 0.01$. The case $g/t = 0.1$ has already been presented in Fig. 4.3(c), where the other two cases are depicted in Fig. 4.4. From these results, one can conclude that increasing parameter g tends to decrease the locking effect.

Fig. 4.4: Convergence curves for the deflection in the H^3 and H^2 norms: B-splines of order $p = 3$ with $t/L = 0.001$ and $g/t = 1, 0.01$.

4.1.2. Locking-free formulations – classical elasticity

It is worth noticing that the suggested *locking-free formulations* can be used for resolving the *locking problem* of the Timoshenko beam problem of classical elasticity as well, as indicated by formulating Problems 4 and 5. Accordingly, we next study the convergence properties of all the three models for a classical Timoshenko beam with *clamped ends*:

$$w(0) = 0 = w(L), \quad \beta(0) = 0 = \beta(L). \quad (4.1)$$

Fig. 4.5: Convergence curves for the deflection in the H^1 and H^2 norms: B-splines of order $p = 3$ with $t/L = 0.001$.

Fig. 4.5 demonstrates the desired convergence behavior for both *locking-free formulations* (red lines and blue markers). On the contrary, the convergence behavior of the

standard formulation does not follow any predictable rate. For the coarsest meshes, as expected, the accuracy is particularly poor.

4.2. Convergence studies for free vibrations

4.2.1. Shear locking effect – strain gradient elasticity

The quality of numerical solution for the eigenvalue problem of free vibrations is estimated by plotting the normalized eigen frequency ω_n^h/ω_n against the normalized mode number n/N . The analytical solution of the present eigenproblem can be found in equation (B.6) of Appendix B.

In order to demonstrate the locking effect, we solve the eigenvalue problem related to a *doubly simply supported* ((2.29) with boundary variables defined in (2.24) for dynamics) Timoshenko beam with two different thickness values: $t = L/100 = 0.01$ and $t = L/1000 = 0.001$.

Fig. 4.6: Normalized discrete spectra for the standard formulation for basis function orders $p = 2, 3, 4$ with the total number of degrees of freedom $N = 130$.

First, one can clearly see in Fig. 4.6 that raising the order of basis functions naturally increases the accuracy of the method apart from the so-called outliers, as already observed in [42, 12]. Second, the relative difference between the numerical and analytical solutions is much larger for the thinner beam (Fig. 4.6b) than for the thicker beam (Fig. 4.6a), which indicates that the numerical solution of the eigenvalue problem suffer from the *locking effect* in the same manner as the solution of the static problem. It should be noticed that the observations above mainly concern the first part of the spectra.

4.2.2. Convergence properties – strain gradient elasticity

First of all, we should mention that the results presented in this subsection for the *first LF formulation* are practically identical to the results for the *second LF formulation* not presented here. Only one beam thickness is considered now: $L/t = 0.001$ representing a case with a strongly pronounced *locking effect*.

A comparison of the results for normalized discrete spectrum for the *standard* and *first LF formulation* are presented in Fig. 4.7. The normalized discrete spectrum of standard formulation repeats the result plotted with a blue line in Fig. 4.6b. The analytical solution in (B.7) with $n_1 = 5$ serves as a reference solution.

Fig. 4.7: Normalized discrete spectra for a thin beam with $t/L = 0.001$: a comparison of standard and the first LF formulation with $p = 3$.

Fig. 4.8: Convergence curves for the 5th eigen mode of the deflection for a thin beam with $t/L = 0.001$: a comparison of standard and the first LF formulation with $p = 3$.

As one can see in the spectrum of Fig. 4.7, the *first LF formulation* works essentially better than the standard one for higher frequencies. The shape of the approximate spectrum is typical for isogeometric methods (for gradient-elastic rods and Euler–Bernoulli beams, see [12, 13]). Results presented in Fig. 4.8 for the eigen mode confirm the optimal convergence properties (order $\mathcal{O}(h^{p-2})$ for the H^3 norm and $\mathcal{O}(h^{p-1})$ for the H^2 norm) of the *first LF formulation* for the eigenvalue problem.

4.3. Timoshenko model versus Euler–Bernoulli model

4.3.1. Statics – model problem and comparison to experiments

Let us demonstrate the differences between Timoshenko and Euler–Bernoulli models within strain gradient elasticity by considering a beam of length L having a circular cross section with thickness (diameter) t and $K = 0.89$. Both ends of the beam are (singly) clamped and a concentrated force $P_{L/2}$ acts at the middle of the beam.

For the Euler–Bernoulli model (see [13]), we set the essential boundary conditions as

$$w(0) = 0 = w(L), \quad w'(0) = 0 = w'(L), \quad (4.2)$$

whilst for the Timoshenko model, the boundary conditions are expressed as

$$w(0) = 0 = w(L), \quad \beta(0) = 0 = \beta(L). \quad (4.3)$$

In the reformulations, i.e., Problems 1 and 2, weak enforcement for the given boundary values of the new problem variables have been adopted. In fact, enforcing boundary conditions in the strong sense might become a source of numerical deficiencies (cf. Remark 4 and [23]).

The problem is solved numerically with 32 elements and basis functions of order $p = 5$ for a set of thickness values from $t/L = 0.002$ to $t/L = 0.2$ with gradient parameter values $g/L = 0.002, 0.01, 0.02$. Note that the *first LF formulation* is used for the Timoshenko beam solution. The middle deflection $w_h(L/2)$ gives the beam bending rigidity as

$$D = \frac{P_{L/2}}{w_h(L/2)}. \quad (4.4)$$

The results are presented in Fig. 4.9, where the value of the relative (dimensionless) bending rigidity is defined as the ratio between the bending rigidity of (4.4) and the one for classical Euler–Bernoulli beams:

$$D_0 = \frac{192EI}{L^3}, \quad I = \frac{\pi t^4}{64}. \quad (4.5)$$

Consequently, the result does not depend on the problem scale.

Fig. 4.9: Bending rigidity versus thickness: a comparison between the beam models with different gradient parameter values.

Fig. 4.10: Bending rigidity versus thickness: a comparison between the beam models and experimental data.

The results in Fig. 4.9 show that, first, differences between the beam models, independently from gradient parameter g , are visible only for thick beams with $t/L \geq 0.05$, as typical within classic elasticity. Second (as a consequence from the first conclusion), one can see that depending on g value stiffening effect may appear already for quite large values of t/L —for which the behavior of the Timoshenko and Euler–Bernoulli beams is clearly different. Conversely, for certain ranges of g - and t -values no difference between the beam models can be distinguished. Third, the test results show (some of

them are not presented here) that values $t < 2g$ for a fixed g imply extremely (non-physically) large bending rigidity representing the underlying assumption of the strain gradient models: parameter g is related to the micro-structural dimensions which cannot be essentially larger than any of the macro-structural dimensions. Fourth, for a fixed thickness value, the difference between the beam models rises with the gradient parameter increasing.

Let us now appeal to the experimental results obtained by Jing et al. in [3] concerning nano-scale beams. Authors, similarly to Salvétat et al. in [43], state that the deflection has both bending and shear parts and that the shear part can be neglected for $t/L < 1/16$. Unfortunately, the authors do not report the length of the beam used in the experiment, or, more precisely, the diameter of the holes in the supporting silicon substrate. From the report, we only can get to know that they use "silicon substrates with holes of different diameters" in the capacity of the wafer plate for the silver nano-wires. We can, however, estimate from Fig. 1 of [3] that the diameter of the holes in the substrate is roughly equal to 900 nm , whilst the diameter of the nanowire is 79 nm . In accordance to results in Fig. 4.9, the influence of the shear component of the deflection is significant for the values of $t/L = 79/900 \approx 0.088$ meaning that the Timoshenko model should be used.

In Fig. 4.10, the result above is compared with the experimental data taken from Fig. 3 of [3] (by a recalculation for obtaining the relative bending rigidity). The parameters of the problem are taken from [3]: $L = 900\text{ nm}$, $E = 76\text{ GPa}$, $\nu = 0.37$.

In [3], using the Euler-Bernoulli beam model, the authors mention that the "effective Young's modulus" is smaller than the one for the bulk material. In our modeling approach, this corresponds to values $D/D_0 < 1$ (we recall that D_0 refers to the bending stiffness of the classical Euler-Bernoulli model). As it can be seen in Fig. 4.10, many of the experimental measurements (circles) situate under the line $D/D_0 = 1$, whilst the Euler-Bernoulli model naturally cannot give values less than 1. The Timoshenko model, instead, seems to capture the experimental measurements fairly well, which justifies the relevance of the model. Finally, for a macro-scale comparison, we refer to [7].

4.3.2. Free vibrations – model problem

Regarding free vibrations, the most crucial difference between the two models is the existence of the second branch in the spectrum of the Timoshenko model (ω_{n_2} in (B.6) of Appendix B). Discussions about the physicality of this branch are out of scope of the present contribution, we simply compare the eigen frequencies of the first branch only (ω_{n_1} in (B.6) of Appendix B).

Once again, we consider a beam with *doubly simply supported* end points and solve the eigenvalue problem numerically for different ratios of t/L varying from $t/L = 0.002$ to $t/L = 0.2$ and for different gradient parameters g and χ . For the corresponding equations for the Euler–Bernoulli model, we refer to [13].

Fig. 4.11 presents the 1^{st} , 5^{th} and 10^{th} eigen frequencies as a function of the thickness parameter for different combinations of gradient parameters. The eigen frequencies have been nondimensionalized by the corresponding values of the classical Euler–Bernoulli

model

$$\omega_n^{0,0} = \sqrt{\frac{EI k^4}{\rho A + \rho I k^2}}, \quad k = \frac{\pi n}{L}. \quad (4.6)$$

Fig. 4.11: Eigen frequency versus thickness: different beam models and combinations of gradient parameters.

The results presented in Fig. 4.11 show that, first, for thick beams and higher frequencies the Timoshenko and Euler–Bernoulli models differ from each other, similarly to the static case considered above. Second, difference between the models for the lowest frequencies is small and distinguishable only for very thick beams, analogously to the static case. Third, parameter χ does not affect on the result for the lowest frequencies (in Fig. 4.11a, the green and purple curves overlap the red and blue ones accordingly), whereas for higher frequencies the inertial gradient parameter significantly affects on

the difference between the models. In contrast, parameter g has an effect even for low frequencies.

5. Conclusions

We have used Hamilton’s variational principle for deriving the governing equations, boundary conditions and variational formulations for the statics and dynamics of a strain gradient Timoshenko beam bending model relying on a simplified variant of Mindlin’s strain gradient elasticity theory. The model possesses only one static and one dynamic gradient parameter.

In order to avoid the shear locking effect typical for the numerical methods for Timoshenko beams, we have introduced two non-standard variational formulations. According to numerical results obtained by an isogeometric C^{p-1} -continuous approach with B-spline basis functions of order $p \geq 3$, both formulations are practically locking-free—for both strain gradient and classical elasticity. For the shear force, however, the first one of the new formulations exhibits a minor deficiency in the convergence rates. Convergence analyses cover both statics and free vibrations with a couple different boundary conditions types. Results not presented here have shown, however, that any combinations of boundary conditions can be handled in a similar fashion.

Parameter studies for the thickness and gradient parameters with a comparison to experimental results concerning static bending of nano-beams have been used for confirming that for relatively thick beams the gradient-elastic Timoshenko model is able to capture size effects more accurately than the corresponding Euler–Bernoulli model. Regarding free vibrations, it has been shown that already after the first eigen frequency the models start to differ from each other, which calls for experimental results for further validation. Finally, it should be mentioned that as in the case of Euler–Bernoulli beams in [13], our analysis can be extended to cover Timoshenko beam models relying on other generalized continuum theories such as in [17].

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Appendix A. Analytical solution for statics

Let us consider a beam governed by equations (2.25) with homogeneous *doubly simply supported* boundary conditions (2.29) applied at both ends. The beam is affected by the following external loadings:

$$q = q_1 \sin(\pi x/L), \quad m = 0, \tag{A.1}$$

where constant q_1 is chosen such that the distributed force causes deformations that can be considered small.

We assume that the deflection and rotation take the following forms:

$$w(x) = \sum_{n=1}^{\infty} W_n \sin \frac{\pi n x}{L}, \quad \beta(x) = \sum_{n=1}^{\infty} B_n \cos \frac{\pi n x}{L} \quad (\text{A.2})$$

fulfilling the essential boundary conditions, with W_n and B_n denoting unknown constants. Substituting (A.2) and (A.1) into (2.25) yields, after some simplifications, the following system of linear equations:

$$\begin{bmatrix} -KGA\phi^2\xi & KGA\phi\xi \\ KGA\phi\xi & -KGA\xi - EI\phi^2\xi - g^2EA\phi^2 \end{bmatrix} \begin{bmatrix} W_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} -q_1 \\ 0 \end{bmatrix}, \quad (\text{A.3})$$

where $\phi = \pi/L$, $\xi = (1 + g^2\phi^2)$. External loadings of (A.1) imply that $W_n = 0$ and $B_n = 0$ for all $n = 2, 3, \dots$. By solving system (A.3), expressions for W_1 and B_1 can be found and substituted into (A.2) yielding the solution of the problem in the form

$$\begin{aligned} w(x) &= \frac{KGA\xi + EI\phi^2\xi + g^2EA\phi^2}{KGA\phi^2\xi(KGA\xi + EI\phi^2\xi + g^2EA\phi^2) - (KGA\phi\xi)^2} q_1 \sin \phi x; \\ \beta(x) &= \frac{KGA\phi\xi}{KGA\phi^2\xi(KGA\xi + EI\phi^2\xi + g^2EA\phi^2) - (KGA\phi\xi)^2} q_1 \cos \phi x. \end{aligned} \quad (\text{A.4})$$

Appendix B. Analytical solution for free vibrations

Let us consider a free vibration problem with homogeneous *doubly simply supported* boundary conditions (the dynamic version of (2.29) following (2.24)) applied at both ends of the beam. A solution of the form

$$w(x, t) = w(x)e^{-i\omega t}, \quad \beta(x, t) = \beta(x)e^{-i\omega t}, \quad (\text{B.1})$$

are inserted into (2.23) giving eigenvalue problem

$$\begin{aligned} KGA((w'' - \beta') - g^2(w^{IV} - \beta''')) - \rho\omega^2 A(w - \chi^2 w'') &= 0, \\ EI(\beta'' - g^2\beta^{IV}) + KGA((w' - \beta) - g^2(w''' - \beta'')) & \\ + EA g^2\beta'' - \rho\omega^2(A\chi^2\beta - I(\beta - \chi^2\beta'')) &= 0. \end{aligned} \quad (\text{B.2})$$

Assuming that system (B.2) has the solutions in the form of exponential functions $w(x) = W \exp(ikx)$, $\beta(x) = B \exp(ikx)$, one can obtain the following system of linear equations:

$$\begin{bmatrix} c_1 - c_2\omega^2 & c_3 \\ c_3 & c_4 - c_5\omega^2 \end{bmatrix} \begin{bmatrix} W \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (\text{B.3})$$

where ω stands for the eigen frequency and the coefficients are defined as

$$\begin{aligned} c_1 &= k^2 KGA(1 + g^2 k^2); \\ c_2 &= \rho A(1 + \chi^2 k^2); \\ c_3 &= kKGA(1 + g^2 k^2); \\ c_4 &= (1 + g^2 k^2)(EI k^2 + KGA) + k^2 g^2 EA; \\ c_5 &= \rho I(1 + \chi^2 k^2) + \chi^2 \rho A; \end{aligned} \tag{B.4}$$

and k denotes the wave number. The standard condition on the existence of a non-trivial solution for system (B.3) engenders the biquadratic characteristic equation

$$c_2 c_5 \omega^4 - (c_1 c_5 + c_2 c_4) \omega^2 + (c_1 c_4 - c_3^2) = 0. \tag{B.5}$$

Equation (B.5) is bi-quartic with respect to k . Depending on ω , each of eight pair-conjugate roots $\pm k_1, \pm k_2, \pm k_3, \pm k_4$ can be real, imaginary or complex. Accordingly, functions $w(x)$ and $\beta(x)$, in general, consist of eight hyperbolic or trigonometric functions or their combinations.

By solving equation (B.5), one can obtain two spectra of the gradient-elastic Timoshenko beam model:

$$\omega_{n_{1,2}} = \sqrt{\frac{(c_1 c_5 + c_2 c_4) \pm \sqrt{(c_1 c_5 + c_2 c_4)^2 - 4 c_2 c_5 (c_1 c_4 - c_3^2)}}{2 c_2 c_5}}. \tag{B.6}$$

It can be shown also that for chosen boundary conditions and for $\omega_{n_1} < \tilde{\omega}$ (see below), displacement eigenmodes are of the form

$$w_{n_1}(x) = W_{n_1} \sin k_1 x; \quad k_1 = \frac{\pi n_1}{L}. \tag{B.7}$$

Even for Timoshenko beam model of classical elasticity there is no unique opinion about the physical meaning of the second spectrum (see [44] and the references therein). In the framework of gradient elasticity, the spectra of the Timoshenko beam model seem to have even more complex structure (the brunches in (B.6) can be divided on sub-parts). It is noteworthy that the first branch of the spectrum ω_{n_1} (minus sign in front of the internal square root in (B.6)) starts from $n_1 = 1$, whilst the second part starts from $n_2 = 0$. Moreover, similarly to the classical case, the first eigen frequency from the second branch of spectrum ω_{0_2} coincides with the so-called transition frequency $\tilde{\omega}$ which depends only on material parameters of the beam and geometry of its cross section (neither on length L nor on boundary conditions).

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