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ERGODICITY AND LOCAL LIMITS FOR STOCHASTIC LOCAL AND NONLOCAL p-LAPLACE EQUATIONS*

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Abstract. Ergodicity for local and nonlocal stochastic singular p-Laplace equations is proven, without restriction on the spatial dimension and for all $p \in [1, 2)$. This generalizes previous results from [B. Gess and J. M. Tölle, *J. Math. Pures Appl.*, 101 (2014), pp. 789–827], [W. Liu and J. M. Tölle, *Electron. Commun. Probab.*, 16 (2011), pp. 447–457], [W. Liu, *J. Evol. Equations*, 9 (2009), pp. 747–770]. In particular, the results include the multivalued case of the stochastic (nonlocal) total variation flow, which solves an open problem raised in [V. Barbu, G. Da Prato, and M. Röckner, *SIAM J. Math. Anal.*, 41 (2009), pp. 1106–1120]. Moreover, under appropriate rescaling, the convergence of the unique invariant measure for the nonlocal stochastic p-Laplace equation to the unique invariant measure of the local stochastic p-Laplace equation is proven.

Key words. stochastic variational inequality, nonlocal stochastic partial differential equations, singular-degenerate SPDE, stochastic p-Laplace equation, ergodicity

AMS subject classifications. 35K55, 35K92, 60H15, 37L15, 45E10

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1. Introduction.

We consider stochastic nonlocal singular p-Laplace equations of the type

$$\begin{aligned} dX_t &\in \left(\int_{\mathcal{O}} J(\cdot - \xi) |X_t(\xi) - X_t(\cdot)|^{p-2} (X_t(\xi) - X_t(\cdot))\, d\xi \right) dt + B dW_t, \\ X_0 &= x_0, \end{aligned} \tag{1.1}$$

and stochastic (local) singular p-Laplace equations of the type

$$\begin{aligned} dX_t &\in \operatorname{div} \left(|\nabla X_t|^{p-2} \nabla X_t \right) dt + B dW_t, \\ X_0 &= x_0, \end{aligned} \tag{1.2}$$

with zero Neumann boundary conditions on bounded, convex domains $\mathcal{O} \subseteq \mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$, and mean zero initial conditions $x_0 \in H := L^2_{\mathrm{av}}(\mathcal{O})$ and $p \in [1, 2)$. Here, W is a cylindrical Wiener process on H, $B \in L_2(H)$ is a symmetric Hilbert–Schmidt operator, and $J : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative, continuous, radial kernel with compact support and $J(0) > 0$. In particular, this includes the multivalued case of the stochastic total variation flow ($p = 1$) recently studied in [10, 11]. We note that for $p = 1$, (1.1) and (1.2) become evolution inclusions.

Our results are twofold: First, we prove the existence and uniqueness of an invariant probability measure to (1.1) and (1.2). Second, the convergence of the respective invariant probability measures for (1.1) to the invariant probability measure for (1.2) is shown, under appropriate rescaling of the kernel J.

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Uniqueness of invariant probability measures to (1.2) has been previously considered in [31, 47, 49]. The difficulties arising in proving uniqueness of invariant probability measures for (1.2) are due to the singular nature of the drift and the resulting low regularity properties of the solutions. More precisely, the energy space associated to (1.2) is given by $W^{1,p}_{\mathrm{av}}$, which is compactly embedded into L^2_{av} only if

$$d < \frac{2p}{2-p}. \tag{1.3}$$

The validity of this embedding is crucial for previously established methods, and thus (1.3) had to be assumed in all of the works [31, 47, 49], which led to stringent restrictions on the spatial dimension d, e.g., $d \leqslant 2$ for $p \approx 1$. For the case of nonlocal stochastic p-Laplace equations, the situation is even worse, since the energy associated to (1.1) is given by[1]

$$\varphi(u) = \frac{1}{2p} \int_{\mathcal{O}} \int_{\mathcal{O}} J(\zeta - \xi)|u(\xi) - u(\zeta)|^p \, d\xi d\zeta,$$

which is equivalent to the L^p norm. Hence, based on this, compactness and thus tightness for the laws of the solutions in L^2_{av} cannot be expected. These obstacles are overcome in the present work by establishing a cascade of energy inequalities for L^m norms of the solutions to (1.1) and (1.2) for all $m \geqslant 2$. These new estimates are then used in order to prove concentration of mass of the solutions around zero, which in turn allows the application of results developed in [43] based on coupling techniques. In conclusion, we prove the existence and uniqueness of an invariant probability measure for (1.1) and (1.2) without any restriction on the dimension $d \in \mathbb{N}$ and for all $p \in [1, 2)$. In particular, this solves the open question raised in [10] of uniqueness of invariant measures for the stochastic total variation flow.

In the second part of this paper, we consider the convergence of invariant probability measures under rescaling of the kernel J in (1.1). More precisely, we consider

$$dX^\varepsilon_t = \left(\int_{\mathcal{O}} J^\varepsilon\left(\cdot - \xi\right) |X^\varepsilon_t(\xi) - X^\varepsilon_t(\cdot)|^{p-2} (X^\varepsilon_t(\xi) - X^\varepsilon_t(\cdot)) d\xi \right) dt + B dW_t, \tag{1.4}$$

where $p \in (1, 2)$ and (dropping normalization constants)

$$J^\varepsilon(\xi) = \frac{1}{\varepsilon^{d+p}} J\left(\frac{\xi}{\varepsilon}\right), \quad \xi \in \mathbb{R}^d,$$

and prove that the corresponding invariant measures μ^ε converge weakly* to the invariant measure μ corresponding to (1.2). Somewhat related questions of convergence of invariant measures of (1.2) with respect to perturbations in p have been considered in [16, 17] under stringent restrictions on the spatial dimension, i.e., assuming (1.3). Again, such dimensional restrictions are crucial to the approach developed in [16, 17], since the argument relies on tightness of the respective sequence of invariant probability measures μ^p, which in turn is verified using the compactness of the embedding $W^{1,p} \hookrightarrow L^2$.

In the setting of local limits for (1.4) for general dimension d, this leads to two fundamental problems: First, no concentration of the invariant probability measures on some uniform, compactly embedded space can be expected. Second, as observed

[1]For the sake of notational simplicity we drop normalization constants in the introduction.

in [32], only weak convergence of the solutions to (1.1) to the solution to (1.2) is available, that is, $X_t^\varepsilon \rightharpoonup X_t$ in H for $\varepsilon \to 0$. Hence, we do not have the convergence of the associated Markovian semigroups $P_t^\varepsilon F$ for all $F \in \mathrm{Lip}_b(H)$, a crucial ingredient in previously developed methods such as in [16, 17]. These problems are overcome in the present work, and we prove that μ^ε converges to μ in the topology of weak* convergence of measures on L^p_{av}, without any restriction on the spatial dimension d.

We note that, in general, the invariant measures μ^ε to (1.4) will only be concentrated on the domains of the corresponding energy functionals (dropping normalization constants)

$$\varphi^\varepsilon(u) := \frac{1}{2p}\int_{\mathcal{O}}\int_{\mathcal{O}} J^\varepsilon(\zeta-\xi)\,|u(\xi)-u(\zeta)|^p\, d\xi d\zeta,$$

rather than on $W^{1,p}_{\mathrm{av}}$ as for (1.2). Roughly speaking, one has $p\varphi^\varepsilon(u) \uparrow \|u\|^p_{W^{1,p}}$. In this sense, at least *asymptotic* concentration on $W^{1,p}_{\mathrm{av}}$ is still satisfied. This is reflected in our proof by working with *asymptotic* tightness rather than tightness. Noncompactness of $W^{1,p}_{\mathrm{av}}$ in L^2_{av} is dealt with by considering weak* convergence of measures on L^p_{av} rather than on L^2_{av}. However, this leads to the further difficulty of working with two topologies: weak* convergence of μ^ε on L^p_{av} and weak convergence of X_t^ε on L^2_{av}. These issues are resolved by a careful treatment in section 5 below.

For simplicity, we restrict ourselves to the case of zero Neumann boundary conditions. In the nonlocal form (1.1) the choice of zero Neumann boundary conditions is reflected by the choice of the domain of integration as $\mathcal{O}$, rather than, for example, $\mathcal{O} + \operatorname{supp} J$. Under appropriate rescaling of J it is known (cf. [5]) that the solutions to the nonlocal deterministic equations, that is, (1.1) with $B \equiv 0$, converge to the solution of the local p-Laplace equation with zero Neumann boundary conditions, that is, to (1.2) with $B \equiv 0$. The nonlocal analogue to homogeneous Dirichlet boundary conditions involves a penalizing term (cf. [3, 5]) which can be viewed as a nonlocal analogue of the boundary trace. While we focus on Neumann boundary conditions, we expect that the case of Dirichlet boundary conditions can be treated by similar methods.

The case of a degenerate drift, that is, $p \geqslant 2$ in (1.1), (1.2), can be treated by rather different and somewhat more simple methods. More precisely, for $p \geqslant 2$ the dissipativity method (cf., e.g., [26, Theorem 3.7]) can be applied to obtain the ergodicity and strong mixing property for both (1.1) and (1.2). Concerning the convergence of the invariant measures, in contrast to the singular case $p < 2$, in the degenerate case the embedding $W^{1,p}_{\mathrm{av}} \hookrightarrow L^2_{\mathrm{av}}$ is always compact. Hence, this compactness may be used to deduce (asymptotic) tightness of the invariant measures without any restriction on the dimension, thus allowing for a rather direct argument similar to the one given in [16].

The problems of existence, uniqueness, and stability with respect to parameters of invariant measures for stochastic partial differential equations (SPDEs) are classical, and a review of the available results would exceed the scope of this paper. Thus, we refer the interested reader to some exemplary works in this direction and the references therein. A typical approach to the uniqueness of invariant measures is given by the Doob–Khasminskii theorem [22, 20, 36] and its more recent generalization [34, 35]. In both cases, this strategy requires smoothing properties of the associated Markov semigroup and its irreducibility, which have been successfully verified for many semilinear SPDEs with degenerate noise (see, e.g., [35, 25, 57, 1] and the references therein). The route followed in this paper is different and relies on a contractivity (e-property) of the Markov semigroup, rather than on a smoothing property (asymptotic strong Feller property), as suggested in the abstract framework of [43]. Some details on the relation of the asymptotic strong Feller property and the e-property can be found in [40, 41, 61]. We note that this type of argument bears a resemblance to arguments used to prove ergodicity of stochastic scalar conservation laws [14, 21, 59]. Concerning the stability of invariant measures for stochastic Navier–Stokes and stochastic Burgers equations with respect to parameters, we refer the reader to [44, 45, 58] and the references therein.

A detailed treatment of deterministic nonlocal p-Laplace equations may be found in [2, 3, 4, 5] and the references therein. Decay estimates and extinction results for solutions of deterministic nonlocal p-Laplace equations have been considered, e.g., in [12, 24, 37, 38, 39, 54, 55]. Relying on nondegeneracy assumptions on the noise, gradient estimates, Harnack inequalities, and exponential convergence rates for stochastic p-Laplace equations and stochastic porous media equations have been obtained in [47, 65, 66] and the references therein. A stochastic variational inequality (SVI) approach to stochastic fast diffusion equations was developed in [29], and their ergodicity has been considered in [9, 31, 48, 49, 50, 65, 66]. The case of stochastic degenerate p-Laplace equations, that is, for $p > 2$, was investigated in [8, 30, 46, 52, 56, 64, 65, 66], and ergodicity for stochastic porous media equations was obtained in [7, 8, 18, 19, 33, 42, 46, 53, 56, 63, 64].

1.1. Structure of the paper.

In section 2 ergodicity for the stochastic nonlocal p-Laplace equation is proven. The case of the stochastic local p-Laplace equation is treated in section 3. Convergence of the solutions of the nonlocal stochastic p-Laplace equation to its local version is shown in section 4. The respective convergence of invariant probability measures is shown in section 5. For notation, see Appendix A.

2. Ergodicity for stochastic nonlocal p-Laplace equations.

In this section we derive a stochastic variational inequality (SVI) formulation for stochastic singular nonlocal p-Laplace equations with homogeneous Neumann boundary condition of the type

$$\begin{aligned} dX_t &\in \left(\int_{\mathcal{O}} J(\cdot-\xi)|X_t(\xi)-X_t(\cdot)|^{p-2}(X_t(\xi)-X_t(\cdot))d\xi\right)dt + BdW_t, \\ X_0 &= x_0 \in L^2(\Omega,\mathcal{F}_0;L^2_{\mathrm{av}}(\mathcal{O})), \end{aligned} \tag{2.1}$$

where $p \in [1,2)$ and $\mathcal{O}$ is a bounded, smooth domain in $\mathbb{R}^d$. The kernel $J:\mathbb{R}^d \to \mathbb{R}$ is supposed to be a nonnegative, continuous, radial function with compact support, $J(0) > 0$ and $\int_{\mathbb{R}^d} J(z)\,dz = 1$. Furthermore, W is a cylindrical Wiener process on H and $B \in L_2(H)$ symmetric with $H = L^2_{\mathrm{av}}(\mathcal{O})$. Hence,

$$W_t^B := BW_t$$

is a trace-class Wiener process in H. We further assume that there is an orthonormal basis e_k of H such that

$$\sum_{k=1}^{\infty} \|Be_k\|_\infty^2 < \infty; \tag{2.2}$$

cf., e.g., [10], where similar conditions on B have been used in the case of the stochastic total variation flow. For $u \in L^p(\mathcal{O})$ we set

$$\varphi(u) := \frac{1}{2p}\int_{\mathcal{O}}\int_{\mathcal{O}} J(\zeta-\xi)|u(\xi)-u(\zeta)|^p\,d\xi d\zeta$$

and obtain, if $p > 1$,

$$A(u) := -\partial_{L^2}\varphi(u) = \int_{\mathcal{O}} J(\cdot - \xi)|u(\xi) - u(\cdot)|^{p-2}(u(\xi) - u(\cdot))\, d\xi$$

and, if $p = 1$,

$$\begin{aligned} A(u) &:= -\partial_{L^2}\varphi(u) \\ &= \Big\{ \int_{\mathcal{O}} J(\cdot - \xi)\eta(\xi, \cdot)\, d\xi : \|\eta\|_{L^\infty} \leqslant 1,\, \eta(\xi,\zeta) = -\eta(\zeta,\xi) \text{ and} \\ &\qquad J(\zeta - \xi)\eta(\xi,\zeta) \in J(\zeta - \xi)\operatorname{sgn}(u(\xi) - u(\zeta)) \text{ for a.e. } (\xi,\zeta) \in \mathcal{O}\times\mathcal{O} \Big\}, \end{aligned}$$

where $\partial_{L^2}\varphi$ denotes the L^2 subgradient of φ restricted to L^2. We note that A defines a continuous, monotone operator on H, satisfying

$$\|A(u)\|_H^2 \lesssim 1 + \|u\|_H^2 \quad \forall u \in H. \tag{2.3}$$

Hence, we can write (2.1) in its relaxed form,

$$dX_t \in -\partial\varphi(X_t)dt + BdW_t.$$

Existence and uniqueness of an SVI solution $X = X^{x_0} \in L^2(\Omega; C([0,T]; L^2(\mathcal{O})))$ to (2.1) has been proven in [32, section 4] and

$$\mathbb{E}\|X_t^x - X_t^y\|_{L^2}^2 \lesssim \|x - y\|_{L^2}^2. \tag{2.4}$$

Since $\int_{\mathcal{O}} \eta d\zeta = 0$ for all $\eta \in A(u)$, $u \in H$, from the construction of SVI solutions (see [32, Definition 2.1] for the definition) presented in [32, section 4] it easily follows that the average value is preserved, that is,

$$X_t \in L^2_{\text{av}}(\mathcal{O}) \quad \forall t \geqslant 0, \mathbb{P}\text{-a.s.} \tag{2.5}$$

if $x_0 \in L^2(\Omega; L^2_{\text{av}}(\mathcal{O}))$. Furthermore, analogously to [31, proof of Proposition 5.2], it follows that

$$P_t F(x) := \mathbb{E}F(X_t^x) \quad \text{for } F \in \mathcal{B}_b(H)$$

defines a Feller semigroup on $\mathcal{B}_b(H)$. As a main result in this section we obtain the following.

Theorem 1. *There is a unique invariant measure μ for P_t satisfying*[2]

$$0 \in \operatorname{supp}(\mu) \subseteq \mathcal{T}(\mu)$$

and

$$\int_H \varphi(x)d\mu(x) \lesssim \|B\|^2_{L_2(H)}. \tag{2.6}$$

We first need to derive suitable a priori bounds on general L^m norms of the solutions.

[2]See Appendix A for notation.

LEMMA 2. *Let $x_0 \in L^m(\Omega, \mathcal{F}_0; L^m_{\mathrm{av}}(\mathcal{O}))$, $m \in [2,\infty)$, and let X be the corresponding SVI solution to* (2.1). *Then there are $c = c(m), C = C(m) > 0$ such that*

$$\frac{1}{m}\mathbb{E}\|X_t\|_m^m + c\mathbb{E}\int_0^t \|X_r\|_{p+m-2}^{p+m-2} dr \leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m + tC \quad \forall t \geqslant 0. \tag{2.7}$$

If $B \equiv 0$, then we can choose $C = 0$.

Proof. For notational convenience let

$$\psi(\xi) := \frac{1}{p}|\xi|^p,$$
$$\phi(\xi) := \partial\psi(\xi) = |\xi|^{p-2}\xi, \quad \xi \in \mathbb{R}^d.$$

Step 1. We start by proving that for $x \in L^m(\Omega, \mathcal{F}_0; L^m_{\mathrm{av}}(\mathcal{O}))$, we have $\mathbb{E}\|X_t\|_{L^m}^m < \infty$ for all $t \geqslant 0$.

We aim to apply Itô's formula for $\frac{1}{m}\|\cdot\|_m^m$. To do so, we need to consider appropriate approximations. Let

$$\iota^\alpha(r) := \frac{1}{m}\begin{cases} |r|^m & \text{if } |r| \leqslant \frac{1}{\alpha}, \\ |\frac{1}{\alpha}|^m + \frac{m}{\alpha^{m-1}}(r - \frac{1}{\alpha}) + \frac{m(m-1)}{2\alpha^{m-2}}(r-\frac{1}{\alpha})^2 & \text{if } r \geqslant \frac{1}{\alpha}, \\ |\frac{1}{\alpha}|^m + \frac{m}{\alpha^{m-1}}(r + \frac{1}{\alpha}) + \frac{m(m-1)}{2\alpha^{m-2}}(r+\frac{1}{\alpha})^2 & \text{if } r \leqslant -\frac{1}{\alpha} \end{cases}$$

and observe, for α small enough, that

$$\begin{aligned}(\iota^\alpha)''(r) &:= \begin{cases} (m-1)|r|^{m-2} & \text{if } |r| \leqslant \frac{1}{\alpha}, \\ \frac{m-1}{|\alpha|^{m-2}} & \text{otherwise} \end{cases} \\ &\lesssim 1 + \iota^\alpha(r).\end{aligned} \tag{2.8}$$

Let θ^β be a standard Dirac sequence on $\mathbb{R}^d$. For $v \in L^2(\mathcal{O})$ we set (for notation, see the appendix)

$$\eta^\alpha(v) := \int_{\mathcal{O}} \iota^\alpha(v)\, d\zeta,$$
$$\eta^{\alpha,\beta}(v) := \int_{\mathcal{O}} \iota^\alpha\left(\theta^\beta * \bar{v}\right) d\zeta$$

and observe that $\eta^{\alpha,\beta} \in C^2(L^2)$ with uniformly continuous derivatives on bounded sets. We recall that the SVI solution X to (2.1) has been constructed in [32, section 4] as a limit in $L^2(\Omega; C([0,T]; H))$ of (strong) solutions X^δ corresponding to the approximating SPDE

$$dX_t^\delta = \left(\int_{\mathcal{O}} J(\cdot - \xi)\phi^\delta(X_t^\delta(\xi) - X_t^\delta(\cdot))d\xi\right)dt + BdW_t,$$

where ψ^δ is the Moreau–Yosida approximation (cf., e.g., [6]) of $\psi(\cdot) = \frac{1}{p}|\cdot|^p$ and $\phi^\delta := \partial\psi^\delta$. Hence, by Itô's formula

$$\begin{aligned}\mathbb{E}\eta^{\alpha,\beta}(X_t^\delta) &= \mathbb{E}\eta^{\alpha,\beta}(x_0) + \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)'(\theta^\beta * X_r^\delta)(\theta^\beta * A^\delta(X_r^\delta))d\zeta dr \\ &\quad + \sum_{k=1}^\infty \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)''(\theta^\beta * X_r^\delta)(\theta^\beta * Be_k)^2 d\zeta dr.\end{aligned} \tag{2.9}$$

Using (for $\alpha > 0$ fixed)

$$(\iota^\alpha)'(r) \lesssim 1 + |r|,$$
$$(\iota^\alpha)''(r) \lesssim 1 \quad \forall r \in \mathbb{R},$$

and dominated convergence, we may let $\beta \to 0$ in (2.9) to obtain that

$$\begin{aligned}\mathbb{E}\eta^\alpha(X_t^\delta) \leqslant \mathbb{E}\eta^\alpha(x_0) + \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)'(X_r^\delta) A^\delta(X_r^\delta) d\zeta dr \\ + \sum_{k=1}^\infty \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)''(X_r^\delta)(Be_k)^2 d\zeta dr.\end{aligned} \tag{2.10}$$

We note that, using [5, Lemma 6.5], monotonicity of $(\iota^\alpha)'$, and $\operatorname{sgn}(\phi^\delta(a-b)) = \operatorname{sgn}(a-b)$, we have that

$$\begin{aligned}&\int_{\mathcal{O}} (\iota^\alpha)'(v) A^\delta(v) d\zeta \\ &= \int_{\mathcal{O}} \int_{\mathcal{O}} J(\zeta - \xi)\phi^\delta(v(\xi) - v(\zeta))(\iota^\alpha)'(v(\zeta)) d\xi d\zeta \\ &= -\frac{1}{2}\int_{\mathcal{O}} \int_{\mathcal{O}} J(\zeta - \xi)\phi^\delta(v(\xi) - v(\zeta))((\iota^\alpha)'(v(\xi)) - (\iota^\alpha)'(v(\zeta))) d\xi d\zeta \\ &\leqslant 0\end{aligned}$$

for all $v \in H$. Hence, using (2.8), we observe that

$$\mathbb{E}\eta^\alpha(X_t^\delta) \leqslant \mathbb{E}\eta^\alpha(x_0) + C\mathbb{E}\int_0^t (1 + \eta^\alpha(X_r^\delta)) dr.$$

Gronwall's lemma then implies that

$$\begin{aligned}\mathbb{E}\eta^\alpha(X_t^\delta) &\lesssim \mathbb{E}\eta^\alpha(x_0) + 1 \\ &\leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m + 1.\end{aligned}$$

Hence, taking $\alpha \to 0$ and using Fatou's lemma we obtain that

$$\frac{1}{m}\mathbb{E}\|X_t^\delta\|_m^m \lesssim \frac{1}{m}\mathbb{E}\|x_0\|_m^m + 1. \tag{2.11}$$

Taking $\delta \to 0$ finishes the proof.

Step 2. We first note that it is enough to prove (2.7) for $x \in L^\infty(\Omega, \mathcal{F}_0; L^\infty_{\mathrm{av}}(\mathcal{O}))$. Due to (2.4) the case of $x \in L^m(\Omega, \mathcal{F}_0; L^m_{\mathrm{av}}(\mathcal{O}))$ can then be concluded by approximation and Fatou's lemma. Hence, assume $x \in L^\infty(\Omega, \mathcal{F}_0; L^\infty_{\mathrm{av}}(\mathcal{O}))$ from now on. By step one we have $\mathbb{E}\|X_t\|_m^m < \infty$ for all $t \geqslant 0$, $m \in \mathbb{N}$.

Letting $\alpha \to 0$ in (2.10), using dominated convergence and (2.3), we obtain

$$\begin{aligned}\frac{1}{m}\mathbb{E}\|X_t^\delta\|_m^m \leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m + \mathbb{E}\int_0^t \int_{\mathcal{O}} (X_r^\delta)^{[m-1]} A^\delta(X_r^\delta) d\zeta dr \\ + (m-1)\sum_{k=1}^\infty \mathbb{E}\int_0^t \int_{\mathcal{O}} |X_r^\delta|^{m-2}(Be_k)^2 d\zeta dr.\end{aligned}$$

Using [5, Lemma 6.5] we observe that

$$\begin{aligned}
&\int_{\mathcal{O}} (X_r^\delta)^{[m-1]} A^\delta(X_r^\delta) d\zeta \\
&= \int_{\mathcal{O}} \int_{\mathcal{O}} J(\zeta-\xi)\phi^\delta(X_r^\delta(\xi)-X_r^\delta(\zeta))(X_r^\delta)^{[m-1]}(\zeta) d\xi d\zeta \\
&= -\frac{1}{2}\int_{\mathcal{O}} \int_{\mathcal{O}} J(\zeta-\xi)\phi^\delta(X_r^\delta(\xi)-X_r^\delta(\zeta))((X_r^\delta)^{[m-1]}(\xi)-(X_r^\delta)^{[m-1]}(\zeta)) d\xi d\zeta.
\end{aligned}$$

From [19], for every $m \in [2,\infty)$ there is a $c>0$ such that

$$(a-b)(a^{[m-1]}-b^{[m-1]}) \geqslant c|a-b|^m \quad \forall a,b \in \mathbb{R}.$$

Since $\mathrm{sgn}(\phi^\delta(a-b)) = \mathrm{sgn}(a-b)$ and (cf. [32, Appendix A])

$$\begin{aligned}
\phi^\delta(a)a &\geqslant \psi^\delta(a) \\
&\geqslant c\psi(a) - C\delta
\end{aligned}$$

for some $c>0$, $C>0$, and $\delta>0$ small enough, this yields

$$\begin{aligned}
\phi^\delta(a-b)(a^{[m-1]}-b^{[m-1]}) &\geqslant c\phi^\delta(a-b)(a-b)|a-b|^{m-2} \\
&\geqslant c\psi^\delta(a-b)|a-b|^{m-2} \\
&\geqslant c|a-b|^{p+m-2} - C\delta|a-b|^{m-2} \quad \forall a,b\in\mathbb{R}.
\end{aligned}$$

Using this and the Poincaré-type inequality [5, Proposition 6.19] in combination with (2.5) we get

$$\begin{aligned}
&\mathbb{E}\int_{\mathcal{O}} (X_r^\delta)^{[m-1]} A^\delta(X_r^\delta) d\zeta \\
&\leqslant -\mathbb{E}\int_{\mathcal{O}}\int_{\mathcal{O}} J(\zeta-\xi)(c|X_r^\delta(\xi)-X_r^\delta(\zeta)|^{p+m-2} - C\delta|X_r^\delta(\xi)-X_r^\delta(\zeta)|^{m-2}) d\zeta d\xi \\
&\leqslant -c\mathbb{E}\|X_r^\delta\|_{p+m-2}^{p+m-2} + C\delta\mathbb{E}\int_{\mathcal{O}}\int_{\mathcal{O}} J(\zeta-\xi)|X_r^\delta(\xi)-X_r^\delta(\zeta)|^{m-2} d\zeta d\xi \\
&\leqslant -c\mathbb{E}\|X_r^\delta\|_{p+m-2}^{p+m-2} + C\delta(\mathbb{E}\|X_r^\delta\|_m^m + 1)
\end{aligned}$$

for some $c>0$. Now, using (2.2) we obtain that

$$\sum_{k=1}^\infty \int_{\mathcal{O}} |X_t^\delta|^{m-2}(Be_k)^2 d\zeta \leqslant \varepsilon\|X_t^\delta\|_{p+m-2}^{p+m-2} + C_\varepsilon$$

for all $\varepsilon>0$ and some $C_\varepsilon \geqslant 0$. Choosing $\varepsilon,\delta>0$ small enough, we conclude that

$$\frac{1}{m}\mathbb{E}\|X_t^\delta\|_m^m \leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m - c\mathbb{E}\int_0^t \|X_r^\delta\|_{p+m-2}^{p+m-2} dr + C\delta\mathbb{E}\int_0^t \|X_r^\delta\|_m^m dr + tC.$$

Letting $\delta \to 0$ concludes the proof. ∎

We next analyze the deterministic situation, i.e., $B=0$ in (2.1). Let u be the unique SVI solution to

$$\begin{aligned}
du_t &\in \left(\int_{\mathcal{O}} J(\cdot-\xi)|u_t(\xi)-u_t(\cdot)|^{p-2}(u_t(\xi)-u_t(\cdot))d\xi\right) dt, \qquad (2.12)\\
u_0 &= x_0 \in H.
\end{aligned}$$

LEMMA 3. *Let $m_0 = 4 - p \in (2,3]$, let $x_0 \in L^{m_0}_{av}(\mathcal{O}) \subseteq H$, and let u be the corresponding SVI solution to* (2.12). *Then there is a $C > 0$ such that*

$$\|u_t\|_2^2 \leqslant \frac{C}{t}\|x_0\|_{m_0}^{m_0}.$$

Proof. From Lemma 2 we know that

$$\begin{aligned}\frac{1}{m_0}\|u_t\|_{m_0}^{m_0} &\leqslant \frac{1}{m_0}\|x_0\|_{m_0}^{m_0} - c\int_0^t \|u_r\|_{p+m_0-2}^{p+m_0-2}dr\\ &= \frac{1}{m_0}\|x_0\|_{m_0}^{m_0} - c\int_0^t \|u_r\|_2^2 dr.\end{aligned}$$

In particular, $t \mapsto \|u_t\|_{m_0}^{m_0}$ is nonincreasing. Using that also $t \mapsto \|u_t\|_2^2$ is nonincreasing yields

$$\frac{1}{m_0}\|u_t\|_{m_0}^{m_0} \leqslant \frac{1}{m_0}\|x_0\|_{m_0}^{m_0} - ct\|u_t\|_2^2.$$

Hence,

$$\|u_t\|_2^2 \leqslant \frac{C}{t}\|x_0\|_{m_0}^{m_0}.$$

∎

Next we prove concentration on bounded L^{m_0} sets for sufficiently regular initial conditions.

LEMMA 4. *Let $\varepsilon > 0$ and $x \in L^{m_1}_{\mathrm{av}}(\mathcal{O}) \subseteq H$ with $m_1 = m_0 + 2 - p \in (2,4]$, $m_0 = 4 - p \in (2,3]$. Then there is an $R = R(\varepsilon) > 0$ such that*

$$Q_T(x, B_R^{m_0}(0)) \geqslant 1 - \varepsilon$$

for all $T \geqslant 1$.

Proof. By Lemma 2 we have

$$\frac{1}{t}\mathbb{E}\|X_t^x\|_{m_1}^{m_1} + c\mathbb{E}\frac{1}{t}\int_0^t \|X_r^x\|_{m_0}^{m_0}dr \leqslant \frac{1}{t}\mathbb{E}\|x\|_{m_1}^{m_1} + C.$$

Thus, for $T \geqslant 1$,

$$\begin{aligned}Q_T(x, B_R^{m_0}(0)) &= \frac{1}{T}\int_0^T P_r(x, B_R^{m_0}(0))dr\\ &\geqslant \frac{1}{T}\int_0^T \left(1 - \frac{\mathbb{E}\|X_r^x\|_{m_0}^{m_0}}{R}\right)dr\\ &= 1 - \frac{1}{R}\frac{1}{T}\int_0^T \mathbb{E}\|X_r^x\|_{m_0}^{m_0}dr\\ &\geqslant 1 - \frac{C}{RT}\mathbb{E}\|x\|_{m_1}^{m_1} - \frac{C}{R}.\end{aligned}$$

Choosing R large enough yields the claim. ∎

LEMMA 5. *For each $T \geqslant 0$, $\eta > 0$ we have*

$$\inf_{x\in B}\mathbb{P}\Big(\sup_{t\in[0,T]}\|X_t^x - u_t^x\|_H^2 \leqslant \eta\Big) > 0$$

for all bounded sets $B \subseteq H$.

Proof. We consider $Y_t^\delta := X_t^\delta - W_t^B$, which satisfies

$$\begin{aligned}\frac{d}{dt}Y_t^\delta &= A^\delta(Y_t^\delta + W_t^B)dt,\\ Y_0^\delta &= x_0.\end{aligned}$$

Accordingly, let u^δ be the unique solution to (2.12) with $\phi(z) = |z|^{p-2}z$ replaced by ϕ^δ. Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Y_t^\delta\|_H^2 &= (Y_t^\delta, A^\delta(Y_t^\delta + W_t^B))_H\\ &\leqslant \|Y_t^\delta\|_H^2 + C\|W_t^B\|_H^2.\end{aligned}$$

Thus,

$$\sup_{t\in[0,T]}\|Y_t^\delta\|_H^2 \lesssim 1 + \|x_0\|_H^2.$$

Similarly,

$$\sup_{t\in[0,T]}\|u_t^\delta\|_H^2 \lesssim 1 + \|x_0\|_H^2.$$

Moreover,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Y_t^\delta - u_t^\delta\|_H^2 &= (Y_t^\delta - u_t^\delta, A^\delta(Y_t^\delta + W_t^B) - A^\delta(u_t^\delta))_H\\ &\leqslant -(W_t^B, A^\delta(Y_t^\delta + W_t^B) - A^\delta(u_t^\delta))_H\\ &\leqslant \|W_t^B\|_H\|A^\delta(Y_t^\delta + W_t^B) - A^\delta(u_t^\delta)\|_H\\ &\leqslant C\|W_t^B\|_H(\|Y_t^\delta\|_H + \|W_t^B\|_H + \|u_t^\delta\|_H)\\ &\leqslant C\|W_t^B\|_H(\|x_0\|_H + \|W_t^B\|_H + 1).\end{aligned}$$

Since W^B is a trace-class Wiener process in H, for each $\eta \in (0,1], T > 0$ we can find a subset $\Omega_\eta \subseteq \Omega$ of positive mass such that $\sup_{t\in[0,T]}\|W_t^B(\omega)\|_H < \eta$ for all $\omega \in \Omega_\eta$. For $\omega \in \Omega_\eta$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|Y_t^\delta - u_t^\delta\|_H^2 \leqslant C\eta(\|x_0\|_H + 1).$$

Choosing $\eta > 0$ small enough and letting $\delta \to 0$ yields the claim. ∎

LEMMA 6. *Let $\varepsilon > 0$, $x \in L^{m_1}_{\mathrm{av}}(\mathcal{O})$ with m_1 as before and $\delta > 0$. Then*

$$\liminf_{T\to\infty} Q_T(x, B_\delta(0)) > 0.$$

Proof. By Lemma 4 there is an $R > 0$ such that

$$Q_T(x, B_R^{m_0}(0)) \geqslant \frac{1}{2}$$

for all $T \geqslant 1$. Moreover, by Lemma 3 we have

$$\|u_t^x\|_2^2 \leqslant \frac{C}{t}\|x\|_{m_0}^{m_0} \leqslant \frac{C}{t}R$$

for all $x \in B_R^{m_0}(0)$, and thus there is a $T_0 = T_0(R,\delta)$ such that

$$\|u_t^x\|_2^2 \leqslant \frac{\delta}{2}$$

for all $t \geqslant T_0$. Using Lemma 5 we observe

$$P_{T_0}(x, B_\delta(0)) = P(\|X^x_{T_0}\|_H \leqslant \delta) \geqslant P\left(\|X^x_{T_0} - u^x_{T_0}\|_H \leqslant \frac{\delta}{2}\right) \geqslant \gamma > 0$$

for some $\gamma = \gamma(\delta, T_0) > 0$ and all $x \in B^{m_0}_R(0)$. Thus, following an idea from [23], we conclude that

$$\begin{aligned}
\liminf_{T\to\infty} Q_T(x, B_\delta(0)) &= \liminf_{T\to\infty} \frac{1}{T}\int_0^T P_s(x, B_\delta(0))ds \\
&= \liminf_{T\to\infty} \frac{1}{T}\int_0^T P_{s+T_0}(x, B_\delta(0))ds \\
&= \liminf_{T\to\infty} \frac{1}{T}\int_0^T \int_H P_s(x, dz)P_{T_0}(z, B_\delta(0))ds \\
&\geqslant \liminf_{T\to\infty} \frac{1}{T}\int_0^T \int_{B^{m_0}_R(0)} P_s(x, dz)P_{T_0}(z, B_\delta(0))ds \\
&\geqslant \gamma \liminf_{T\to\infty} Q_T(x, B^{m_0}_R(0)) \\
&\geqslant \frac{\gamma}{2} > 0.
\end{aligned}$$

∎

LEMMA 7. *Let X, Y be two SVI solutions to* (2.1) *with initial conditions $x_0, y_0 \in L^2(\Omega, \mathcal{F}_0; L^m_{\mathrm{av}}(\mathcal{O}))$, respectively. Then, for all $m \geqslant 1$,*

$$\|X_t - Y_t\|_{L^m(\mathcal{O})} \leqslant \|x_0 - y_0\|_{L^m(\mathcal{O})} \quad \mathbb{P}\text{-a.s. } \forall t \geqslant 0.$$

In particular, the semigroup P_t satisfies the e-property on H.

Proof. We have that

$$d(X^\delta_t - Y^\delta_t) = (A^\delta(X^\delta_t) - A^\delta(Y^\delta_t))dt,$$

and thus $t \mapsto (X^\delta_t - Y^\delta_t) \in W^{1,2}([0,T]; H)$. Let ι^α be the Moreau–Yosida approximation of $\frac{1}{m}|\cdot|^m$ and

$$\eta^\alpha(v) := \int_{\mathcal{O}} \iota^\alpha(v)d\zeta$$

for $v \in H$. By [60, Lemma IV.4.3] we obtain that

$$\frac{d}{dt}\eta^\alpha(X^\delta_t - Y^\delta_t) = (\gamma^\alpha_t, A^\delta(X^\delta_t) - A^\delta(Y^\delta_t))_H \quad \text{for a.e. } t \in [0,T],$$

where $\gamma^\alpha_t := (\iota^\alpha)'(X^\delta_t - Y^\delta_t) \in L^2([0,T]; H)$. Using [5, Lemma 6.6] we conclude that, $\mathbb{P}$-a.s.,

$$\frac{d}{dt}\eta^\alpha(X^\delta_t - Y^\delta_t) \leqslant 0.$$

Letting $\alpha \to 0$, then $\delta \to 0$ concludes the proof. ∎

Proof of Theorem 1.

Step 1: *Existence and uniqueness of invariant measures.* The proof relies on an application of [43, Theorem 1]. Let $x \in H$ and $\delta > 0$. Then we may choose $y \in L^{m_1}_{\mathrm{av}}(\mathcal{O})$, with m_1 as in Lemma 4, such that

$$\|x - y\|^2_H \leqslant \frac{\delta}{2}.$$

By Lemma 7 we then have

$$\|X_t^x - X_t^y\|_H^2 \leqslant \frac{\delta}{2} \quad \forall t \geqslant 0. \tag{2.13}$$

Lemma 6 yields

$$\liminf_{T\to\infty} Q_T(y, B_{\frac{\delta}{2}}(0)) > 0.$$

Due to (2.13) we conclude

$$\liminf_{T\to\infty} Q_T(x, B_\delta(0)) \geqslant \liminf_{T\to\infty} Q_T(y, B_{\frac{\delta}{2}}(0)) > 0. \tag{2.14}$$

An application of [43, Theorem 1] implies that P_t has a unique invariant probability measure μ.

Step 2. We first note that for all $x \in H$ such that $\{Q_T(x,\cdot)\}_{T\geqslant T_0}$ is tight for some $T_0 \geqslant 0$, we have that

$$Q_T(x,\cdot) \rightharpoonup^* \mu \quad \text{for } T \to \infty$$

by uniqueness of the invariant measure μ. Hence,

$$\mathcal{T}(\mu) = \left\{ x \in H : \{Q_T(x,\cdot)\}_{T\geqslant T_0} \text{ is tight for some } T_0 \geqslant 0 \right\}.$$

By [43, Proposition 1] we have that

$$\operatorname{supp} \mu \subseteq \mathcal{T}(\mu).$$

Moreover, using invariance of μ, Fatou's lemma, and (2.14) we note that

$$\begin{aligned}
\mu(B_\delta(0)) &= \liminf_{T\to\infty} Q_T\mu(B_\delta(0)) \\
&= \liminf_{T\to\infty} \int_H Q_T(x, B_\delta(0)) d\mu(x) \\
&\geqslant \int_H \liminf_{T\to\infty} Q_T(x, B_\delta(0)) d\mu(x) \\
&> 0
\end{aligned}$$

for all $\delta > 0$. Hence,

$$0 \in \operatorname{supp} \mu.$$

Step 3. An application of Itô's formula yields

$$\mathbb{E}\|X_t^\delta\|_H^2 \leqslant 2\mathbb{E}\int_0^t (A^\delta(X_r^\delta), X_r^\delta)_H dr + t\|B\|_{L_2(H)}^2.$$

By [5, Lemma 6.5] we have

$$2(A^\delta(v), v)_H = -p\varphi^\delta(v)$$

and thus

$$\frac{c}{t}\mathbb{E}\int_0^t \varphi^\delta(X_r^\delta) dr \leqslant \|B\|_{L_2(H)}^2$$

for some $c > 0$. Since, by [32, Appendix A],

$$|\varphi^\delta(v) - \varphi(v)| \leqslant C\delta(1 + \|v\|_H^2) \quad \forall v \in H$$

we obtain that

$$\frac{c}{t}\mathbb{E}\int_0^t \varphi(X_r^\delta)dr \leqslant \|B\|_{L_2(H)}^2 + \frac{C\delta}{t}\mathbb{E}\int_0^t (\|X_r^\delta\|_H^2 + 1)dr.$$

Letting $\delta \to 0$ yields

$$\frac{c}{t}\mathbb{E}\int_0^t \varphi(X_r)dr \leqslant \|B\|_{L_2(H)}^2.$$

Since $0 \in \mathcal{T}(\mu)$ this is easily seen to imply (2.6). ∎

3. Ergodicity for stochastic local p-Laplace equations.

In this section we consider stochastic singular p-Laplace equations with additive noise, that is,

$$\begin{aligned} dX_t &\in \operatorname{div}\left(|\nabla X_t|^{p-2}\nabla X_t\right) dt + BdW_t, \\ |\nabla X_t|^{p-2}\nabla X_t \cdot \nu &\ni 0 \quad \text{on } \partial\mathcal{O},\ t > 0, \\ X_0 &= x_0 \in L^2(\Omega, \mathcal{F}_0; L^2_{\mathrm{av}}(\mathcal{O})), \end{aligned} \tag{3.1}$$

with $p \in [1, 2)$ on a bounded, smooth domain $\mathcal{O} \subseteq \mathbb{R}^d$ with convex boundary $\partial\mathcal{O}$. In the following we set $H := L^2_{\mathrm{av}}(\mathcal{O})$ and $S := H^1_{\mathrm{av}}(\mathcal{O})$. Here, W is a cylindrical Wiener process on H and $B \in L_2(H)$ symmetric with $B \in L_2(H, H^3_{\mathrm{av}})$. Hence,

$$W_t^B := BW_t$$

is a trace-class Wiener process in $H^3_{\mathrm{av}} \subseteq H$. As in section 2, we further assume that there is an orthonormal basis e_k of H such that

$$\sum_{k=1}^\infty \|Be_k\|_\infty^2 < \infty; \tag{3.2}$$

cf. [10], where similar conditions on B have been used in the case $p = 1$. We define, for $p \in (1, 2)$,

$$\varphi(v) := \begin{cases} \frac{1}{p}\int_{\mathcal{O}}|\nabla u|^p\, d\xi & \text{if } v \in W^{1,p}(\mathcal{O}), \\ +\infty & \text{if } v \in L^p(\mathcal{O}) \setminus W^{1,p}(\mathcal{O}) \end{cases}$$

and, for $p = 1$,

$$\varphi(v) := \begin{cases} \|v\|_{TV} & \text{if } v \in BV(\mathcal{O}), \\ +\infty & \text{if } v \in L^1(\mathcal{O}) \setminus BV(\mathcal{O}). \end{cases}$$

Then (3.1) can be recast in its relaxed form,

$$dX_t \in -\partial_{L^2}\varphi(X_t)dt + BdW_t,$$

where $\partial_{L^2}\varphi$ denotes the L^2 subgradient of φ restricted to L^2. In [31, section 7.2.2] the existence and uniqueness of a (limit) solution $X = X^{x_0}$ to (3.1) has been proven and

$$\|X_t^x - X_t^y\|_H^2 \leqslant \|x - y\|_H^2 \quad \forall t \geqslant 0,\ \mathbb{P}\text{-a.s.} \tag{3.3}$$

Following [31, Appendix C] it is easy to see that X also is an SVI solution to (3.1), which by [32, section 3] is unique. From the construction of X it is easy to see that the average value is preserved, that is,

$$X_t \in L^2_{\mathrm{av}}(\mathcal{O}) \quad \forall t \geqslant 0,\ \mathbb{P}\text{-a.s.} \tag{3.4}$$

if $x_0 \in L^2(\Omega, \mathcal{F}_0; L^2_{\mathrm{av}}(\mathcal{O}))$. Moreover, by [31, Proposition 5.2],

$$P_t F(x) := \mathbb{E} F(X_t^x) \quad \text{for } F \in \mathcal{B}_b(H)$$

defines a Feller semigroup on $\mathcal{B}_b(H)$. By (3.3), P_t satisfies the e-property on H. As a main result in this section we obtain the following.

THEOREM 8. *There is a unique invariant measure μ for P_t, which satisfies*

$$0 \in \operatorname{supp}(\mu) \subseteq \mathcal{T}(\mu)$$

and

$$\int_H \varphi(x) d\mu(x) \lesssim \|B\|^2_{L_2(H)}.$$

The proof of Theorem 8 proceeds along the same principal ideas as that of Theorem 1. However, due to the local nature of (3.1) different arguments have to be used in order to deduce the cascade of L^m inequalities (cf. Lemma 9 below). Once these inequalities have been shown for (3.1), the proof can be concluded essentially as in section 2.

LEMMA 9. *Let $x_0 \in L^m(\Omega, \mathcal{F}_0; L^m_{\mathrm{av}}(\mathcal{O}))$, $m \in [2, \infty)$, and let X be the corresponding SVI solution to* (2.1)*. Then there is a constant $c = c(p, m) > 0$ such that*

$$\frac{1}{m}\mathbb{E}\|X_t\|_m^m + c\mathbb{E}\int_0^t \|X_r\|_{p+m-2}^{p+m-2} dr \leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m + tC \quad \forall t \geqslant 0. \tag{3.5}$$

Proof.

Step 1. We start by proving that for $x_0 \in L^m(\Omega, \mathcal{F}_0; L^m_{\mathrm{av}}(\mathcal{O}))$ we have $\mathbb{E}\|X_t\|^m_{L^m} < \infty$ for all $t \geqslant 0$.

In the following let $\psi(\cdot) = \frac{1}{p}|\cdot|^p$, let $\phi := \partial\psi$, let ψ^δ be the Moreau–Yosida approximation of ψ, and let $\phi^\delta := \partial\psi^\delta$. Recall that the unique SVI solution X to (2.1) has been constructed in [32, Theorem 4.1] as a limit of approximating solutions $X^{\varepsilon,\delta,n}$ to

$$\begin{aligned} dX_t^{\varepsilon,\delta,n} &= \varepsilon\Delta X_t^{\varepsilon,\delta,n}\, dt + \operatorname{div} \phi^\delta\left(\nabla X_t^{\varepsilon,\delta,n}\right) dt + B dW_t, \\ X_0^{\varepsilon,\delta,n} &= x_0^n, \end{aligned} \tag{3.6}$$

with zero Neumann boundary conditions, where $x_0^n \to x_0$ in $L^2(\Omega; H)$ with $x_0^n \in L^2(\Omega, \mathcal{F}_0; H^1_{\mathrm{av}})$ and $\varepsilon > 0$. From [32, Equation (3.6)] we recall the bound

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t^{\varepsilon,\delta,n}\|^2_{H^1} + 2\varepsilon\mathbb{E}\int_0^T \|\Delta X_r^{\varepsilon,\delta,n}\|_2^2 dr \leqslant C(\mathbb{E}\|x_0^n\|^2_{H^1} + 1), \tag{3.7}$$

with a constant $C > 0$ independent of ε, δ, and n. In [32, proof of Theorem 3.1] the following subsequent convergence has been shown in $L^2(\Omega; C([0,T]; H))$:

$$\begin{aligned} X^{\varepsilon,\delta,n} &\to X^{\varepsilon,n} \quad \text{for } \delta \to 0, \\ X^{\varepsilon,n} &\to X^n \quad \text{for } \varepsilon \to 0, \\ X^n &\to X \quad \text{for } n \to \infty. \end{aligned} \tag{3.8}$$

Let ι^α, θ^β, and $\eta^{\alpha,\beta}$ be as in the proof of Lemma 2. Then, since $X^{\varepsilon,\delta,n}$ is a strong solution to (3.6), by Itô's formula we obtain

$$
\begin{aligned}
\mathbb{E}\eta^{\alpha,\beta}(X_t^{\varepsilon,\delta,n}) &= \mathbb{E}\eta^{\alpha,\beta}(x_0)\\
&\quad + \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)'(\theta^\beta * X_r^{\varepsilon,\delta,n}) \left(\theta^\beta * \left(\varepsilon\Delta X_r^{\varepsilon,\delta,n} + \operatorname{div}\phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right)\right)\right) d\xi dr\\
&\quad + \sum_{k=1}^\infty \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)''(\theta^\beta * X_r^{\varepsilon,\delta,n})(\theta^\beta * Be_k)^2 d\xi dr.
\end{aligned} \tag{3.9}
$$

Using (3.2), (3.7), and dominated convergence, taking $\beta \to 0$ yields

$$
\begin{aligned}
\mathbb{E}\int_{\mathcal{O}} \iota^\alpha(X_t^{\varepsilon,\delta,n}) d\xi &= \mathbb{E}\int_{\mathcal{O}} \iota^\alpha(x_0) d\xi\\
&\quad + \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)'(X_r^{\varepsilon,\delta,n}) \left(\varepsilon\Delta X_r^{\varepsilon,\delta,n} + \operatorname{div}\phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right)\right) d\xi dr\\
&\quad + \sum_{k=1}^\infty \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)''(X_r^{\varepsilon,\delta,n})(Be_k)^2 d\xi dr.
\end{aligned} \tag{3.10}
$$

Since $\int_{\mathcal{O}} (\iota^\alpha)'(X_r^{\varepsilon,\delta,n})(\varepsilon\Delta X_r^{\varepsilon,\delta,n} + \operatorname{div}\phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right)) d\xi \leqslant 0$, using (2.8), this implies

$$
\mathbb{E}\int_{\mathcal{O}} \iota^\alpha(X_t^{\varepsilon,\delta,n}) d\xi \leqslant \mathbb{E}\int_{\mathcal{O}} \iota^\alpha(x_0) d\xi + C\mathbb{E}\int_0^t \int_{\mathcal{O}} 1 + \iota^\alpha(X_r^{\varepsilon,\delta,n}) dx dr.
$$

Hence, by Gronwall's lemma

$$
\mathbb{E}\int_{\mathcal{O}} \iota^\alpha(X_t^{\varepsilon,\delta,n}) d\xi \lesssim \mathbb{E}\int_{\mathcal{O}} \iota^\alpha(x_0) d\xi + 1 \lesssim \frac{1}{m}\mathbb{E}\|x_0\|_m^m + 1.
$$

Taking the limit $\alpha \to 0$ yields, by Fatou's lemma and dominated convergence,

$$
\frac{1}{m}\mathbb{E}\|X_t^{\varepsilon,\delta,n}\|_m^m \lesssim \frac{1}{m}\mathbb{E}\|x_0\|_m^m + 1. \tag{3.11}
$$

Taking the limits $\delta \to 0$, $\varepsilon \to 0$, $n \to \infty$ subsequently as in the proof of [32, Theorem 3.1] finishes the proof of this step by Fatou's lemma.

Step 2. As in the proof of Lemma 2 it is enough to prove (3.5) for $x_0 \in L^\infty(\Omega, \mathcal{F}_0; L^\infty_{\mathrm{av}}(\mathcal{O}))$. Hence, assume $x_0 \in L^\infty(\Omega, \mathcal{F}_0; L^\infty_{\mathrm{av}}(\mathcal{O}))$ from now on. By Step 1 we have $\mathbb{E}\|X_t\|_m^m < \infty$ for all $t \geqslant 0$, $m \in \mathbb{N}$.

Taking the limit $\alpha \to 0$ in (3.10) yields, using dominated convergence and (2.8), (3.11),

$$
\begin{aligned}
&\frac{1}{m}\mathbb{E}\int_{\mathcal{O}} \|X_t^{\varepsilon,\delta,n}\|_m^m d\xi\\
&\leqslant \frac{1}{m}\mathbb{E}\int_{\mathcal{O}} \|x_0\|_m^m + \mathbb{E}\int_0^t \int_{\mathcal{O}} (X_r^{\varepsilon,\delta,n})^{[m-1]} \left(\varepsilon\Delta X_r^{\varepsilon,\delta,n} + \operatorname{div}\phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right)\right) d\xi dr\\
&+ C\mathbb{E}\int_0^t \|X_r^{\varepsilon,\delta,n}\|_m^m dr + Ct.
\end{aligned}
$$

We observe that, for $v \in H^2$ with $\nabla v \cdot \nu = 0$ on $\partial\mathcal{O}$,

$$
\int_{\mathcal{O}} v^{[m-1]} \operatorname{div}\phi^\delta(\nabla v) d\zeta = -(m-1)\int_{\mathcal{O}} |v|^{m-2}\nabla v \cdot \phi^\delta(\nabla v) d\zeta.
$$

Further, note that (since $\psi^\delta(0)=0$)

$$\begin{aligned} a\cdot\phi^\delta(a) &= a\cdot\partial\psi^\delta(a) \\ &\geqslant \psi^\delta(a) \quad \forall a\in\mathbb{R}^d \end{aligned}$$

and (cf. [32, Appendix A])

$$|\psi^\delta(a)-\psi(a)| \lesssim \delta(1+\psi(a)) \quad \forall a\in\mathbb{R}^d.$$

This implies

$$a\cdot\phi^\delta(a) \geqslant c\psi(a) - C\delta,$$

and thus, for $\delta>0$ small enough,

$$\begin{aligned} &\int_{\mathcal{O}} (X_r^{\varepsilon,\delta,n})^{[m-1]} \operatorname{div}\phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right) d\zeta \\ &= -(m-1)\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{m-2}\nabla X_r^{\varepsilon,\delta,n}\cdot\phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right) d\zeta \\ &\leqslant -c(m-1)\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{m-2}\psi(\nabla X_r^{\varepsilon,\delta,n})d\zeta + C(m-1)\delta\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{m-2}d\zeta. \end{aligned}$$

For $u\in W^{1,\infty}_{\mathrm{av}}$ we observe that

$$\begin{aligned} \int_{\mathcal{O}} |u|^{m-2}\psi(\nabla u)d\zeta &= \frac{1}{p}\int_{\mathcal{O}} |u|^{m-2}|\nabla u|^p d\zeta \\ &= \frac{1}{p}\int_{\mathcal{O}} ||u|^{\frac{m-2}{p}}\nabla u|^p d\zeta \\ &= c\int_{\mathcal{O}} |\nabla u^{[\frac{m-2+p}{p}]}|^p d\zeta \end{aligned}$$

for some generic constant $c=c(m,p)>0$. By Poincaré's inequality we obtain that

$$\int_{\mathcal{O}} |u|^{m-2}\psi(\nabla u)d\zeta \geqslant c\int_{\mathcal{O}} |u|^{p+m-2}d\zeta.$$

By smooth approximation, this inequality remains true for all $u\in H^1_{\mathrm{av}}$ with $u\in\bigcap_{m\geqslant 1} L^m$. Hence, using Step 1 and (3.4), we conclude that

$$\mathbb{E}\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{m-2}\psi(\nabla X_r^{\varepsilon,\delta,n})d\zeta \geqslant c\mathbb{E}\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{p+m-2}d\zeta.$$

Using the above yields that

$$\begin{aligned} &\frac{1}{m}\mathbb{E}\|X_t^{\varepsilon,\delta,n}\|_m^m + c\mathbb{E}\int_0^t \|X_r^{\varepsilon,\delta,n}\|_{p+m-2}^{p+m-2}dr \\ &\leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m + C\mathbb{E}\int_0^t \|X_r^{\varepsilon,\delta,n}\|_m^m dr + Ct + C(m-1)\delta\mathbb{E}\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{m-2}d\zeta. \end{aligned}$$

By Step 1

$$\delta\mathbb{E}\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{m-2}d\zeta \to 0$$

for $\delta \to 0$. In conclusion, by Fatou's lemma and (3.8),

$$\frac{1}{m}\mathbb{E}\|X_t^{\varepsilon,n}\|_m^m + c\int_0^t \mathbb{E}\|X_r^{\varepsilon,n}\|_{p+m-2}^{p+m-2}dr \leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m + C\mathbb{E}\int_0^t \|X_r^{\varepsilon,n}\|_m^m dr + Ct.$$

An application of Gronwall's lemma and then letting $\varepsilon \to 0$, $n \to \infty$ concludes the proof. ∎

The proof of Theorem 8 may now be concluded as in section 2. For the reader's convenience we give some details. Let u be the unique solution to

$$\begin{aligned} du_t &\in \operatorname{div}\left(|\nabla u_t|^{p-2}\nabla u_t\right)\, dt, \\ |\nabla u_t|^{p-2}\nabla u_t\cdot\nu &\ni 0 \quad \text{on } \partial\mathcal{O},\ t>0, \\ u_0 &= x_0 \in H. \end{aligned} \tag{3.12}$$

LEMMA 10. *Let $x_0 \in H$ and let u be the corresponding solution to* (3.12)*. Then there is a $C > 0$ such that*

$$\|u_t\|_2^2 \leqslant \frac{C}{t}\|x_0\|_{m_0}^{m_0},$$

where $m_0 = 4 - p \in (2, 3]$.

Proof. Using Lemma 9, the proof is analogous to that of Lemma 3. ∎

LEMMA 11. *Let $\varepsilon > 0$ and $x \in L^{m_1}(\Omega, \mathcal{F}_0; L^{m_1}(\mathcal{O}))$ with $m_1 = m_0+2-p \in (2, 4]$, $m_0 = 4 - p \in (2, 3]$. Then there is an $R = R(\varepsilon) > 0$ such that*

$$Q_T(x, B_R^{m_0}(0)) \geqslant 1 - \varepsilon$$

for all $T \geqslant 1$.

Proof. Using Lemma 9, the proof is analogous to that of Lemma 4. ∎

LEMMA 12. *For each $T \geqslant 0$, $\delta > 0$ we have*

$$\inf_{x\in B} \mathbb{P}\left(\sup_{t\in[0,T]} \|X_t^x - u_t^x\|_H^2 \leqslant \delta\right) > 0$$

for all bounded sets $B \subseteq H$.

Proof. The proof follows from [32, Lemma 6.6]. ∎

LEMMA 13. *Let $\varepsilon > 0$, $x \in L^{m_1}(\mathcal{O})$ with m_1 as before and $\delta > 0$. Then*

$$\liminf_{T\to\infty} Q_T(x, B_\delta(0)) > 0.$$

Proof. The proof is the same as that of Lemma 6. ∎

Proof of Theorem 8. The proof is the same as that of Theorem 1. ∎

4. Convergence of solutions: Nonlocal to local. In this section we investigate the convergence of the solutions to the stochastic nonlocal p-Laplace equation to solutions of the stochastic (local) p-Laplace equation, under appropriate rescaling of the kernel J. The convergence of the associated unique invariant measures will be considered in section 5 below.

In the following let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded, smooth domain with convex boundary $\partial\mathcal{O}$ and let $J : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative continuous radial function with compact support, $J(0) > 0$, $\int_{\mathbb{R}^d} J(z)\,dz = 1$ and $J(x) \geqslant J(y)$ for all $|x| \leqslant |y|$. Further, let W be a cylindrical Wiener process on H and $B \in L_2(H)$ symmetric with $B \in L_2(H, H^3_{\mathrm{av}})$. As above, let $H := L^2_{\mathrm{av}}(\mathcal{O})$ and $S = H^1_{\mathrm{av}}(\mathcal{O})$.

For $p \in (1,2)$, $\varepsilon > 0$, we consider the rescaled stochastic nonlocal p-Laplace equations of type

$$
\begin{aligned}
dX_t^\varepsilon &= \left(\int_{\mathcal{O}} J^\varepsilon(\cdot - \xi)|X_t^\varepsilon(\xi) - X_t^\varepsilon(\cdot)|^{p-2}(X_t^\varepsilon(\xi) - X_t^\varepsilon(\cdot))d\xi \right) dt + BdW_t,\\
X_0^\varepsilon &= x_0 \in L^2(\Omega, \mathcal{F}_0; H)
\end{aligned} \tag{4.1}
$$

with

$$
J^\varepsilon(\xi) := \frac{C_{J,p}}{\varepsilon^{p+d}} J\left(\frac{\xi}{\varepsilon}\right), \quad \xi \in \mathbb{R}^d,
$$

and corresponding energy

$$
\varphi^\varepsilon(u) := \frac{1}{2p} \int_{\mathcal{O}} \int_{\mathcal{O}} J^\varepsilon(\xi - \zeta)\,|u(\zeta) - u(\xi)|^p \, d\zeta d\xi
$$

for $u \in L^p(\mathcal{O})$, where

$$
C_{J,p}^{-1} := \frac{1}{2} \int_{\mathbb{R}^d} J(z)|z_d|^p\,dz.
$$

Furthermore, we set

$$
\varphi(u) := \begin{cases} \frac{1}{p}\int_{\mathcal{O}} |\nabla u|^p\,d\xi & \text{if } u \in W^{1,p}(\mathcal{O}),\\ +\infty & \text{if } u \in L^p(\mathcal{O}) \setminus W^{1,p}(\mathcal{O}). \end{cases}
$$

By [32, Theorem 4.1], for each $\varepsilon > 0$, there is a unique SVI solution X^ε to the stochastic nonlocal p-Laplace equation

$$
\begin{aligned}
dX_t^\varepsilon &= -\partial_{L^2}\varphi^\varepsilon(X_t^\varepsilon)\,dt + BdW_t,\\
X_0^\varepsilon &= x_0,
\end{aligned} \tag{4.2}
$$

and, by [32, Theorem 3.1], there is a unique SVI solution to the stochastic (local) p-Laplace equation

$$
\begin{aligned}
dX_t &= -\partial_{L^2}\varphi(X_t)\,dt + BdW_t,\\
X_0 &= x_0,
\end{aligned} \tag{4.3}
$$

where $\partial_{L^2}\varphi$ denotes the L^2 subgradient of φ restricted to L^2. In [32, section 5] weak convergence of $X^\varepsilon \rightharpoonup X$ in $L^2([0,T] \times \Omega; H)$ has been shown. The aim of this section is to strengthen this to pointwise-in-time weak convergence, that is,

$$
X_t^\varepsilon \rightharpoonup X_t \quad \text{weakly in } H \quad \forall t \geqslant 0, \mathbb{P}\text{-a.s.} \tag{4.4}
$$

This will be crucial in order to obtain the convergence of the associated semigroups $P_t^\varepsilon F(x) = \mathbb{E}F(X_t^{\varepsilon,x})$ to $P_t F(x) = \mathbb{E}F(X_t^x)$ for cylindrical functions $F \in \mathcal{F}C_b^1(H)$.

The strategy to prove the pointwise weak convergence (4.4) is based on considering the transformed equations (cf. Remark 18 below)

$$
\frac{d}{dt}Y_t = -\partial_{L^2}\varphi(Y_t + W_t^B(\omega)) = A(Y_t + W_t^B(\omega)) \tag{4.5}
$$

and

$$\frac{d}{dt}Y_t^\varepsilon = -\partial_{L^2}\varphi^\varepsilon(Y_t^\varepsilon + W_t^B(\omega)) = A^\varepsilon(Y_t^\varepsilon + W_t^B(\omega)) \tag{4.6}$$

and to prove the weak convergence $Y_t^\varepsilon \rightharpoonup Y_t$ in H for all $t \geqslant 0$ and a.a. $\omega \in \Omega$. The advantage of considering the transformed, random PDEs (4.5) and (4.6) is that Y^ε, Y enjoy better time regularity properties than X^ε, X, which may be used to deduce stronger convergence results.

THEOREM 14. *Let $x_0 \in L^2(\Omega, \mathcal{F}_0; H)$ and let X^ε, X be the unique solutions to* (4.2), (4.3), *respectively. Then, for each sequence $\varepsilon_n \to 0$,*

$$X_t^{\varepsilon_n} \rightharpoonup X_t \quad \text{for } n \to \infty$$

weakly in H for all $t \in [0,T], \mathbb{P}$-a.s. In particular,

$$P_t^\varepsilon F(x) \to P_t F(x) \quad \text{for } \varepsilon \to 0$$

for all $F \in \mathcal{F}C_b^1(H)$, $t \in [0,T]$, $x \in H$.

Motivated by [32], the general strategy of the proof of Theorem 14 is based on the SVI framework. Hence, we first briefly sketch well-posedness of SVI solutions to (4.5) and then proceed with the proof of Theorem 14.

Remark 15. In this section, we restrict ourselves to the case $p \in (1,2)$ for simplicity only. The interested reader will notice that the same arguments can be applied in the case $p = 1$ with only minor changes. The only difference is the treatment of the nonlocal, transformed random PDE (4.6). In the case $p = 1$ well-posedness of SVI solutions to (4.6) has to be shown as a first step. This can be done following the same arguments as in [32, section 3].

4.1. SVI approach to the transformed equation (4.5). Let $H := L^2_{\text{av}}(\mathcal{O})$, $S = H^1_{\text{av}}(\mathcal{O})$. Without loss of generality we may assume $W^B(\omega) \in C([0,T]; H^3_{\text{av}})$ for all $\omega \in \Omega$. In analogy to [32, Definition 2.1] we define the following.

DEFINITION 16. *Let $x_0 \in H$, $T > 0$. A map $Y \in L^2([0,T]; H)$ is said to be an* SVI solution *to* (4.5) *if*

(i) [regularity]

$$\operatorname{ess\,sup}_{t\in[0,T]} \|Y_t\|_H^2 + \int_0^T \varphi(Y_r + W_r^B)dr \leqslant \|x_0\|_H^2 + C \tag{4.7}$$

for some constants $C > 0$;

(ii) [variational inequality] *for each map $Z \in W^{1,2}([0,T]; S)$ we have*

$$\begin{aligned} &\|Y_t - Z_t\|_H^2 + 2\int_0^t \varphi(Y_r + W_r^B)\,dr \\ &\leqslant \mathbb{E}\|x_0 - Z_0\|_H^2 + 2\int_0^t \varphi(Z_r + W_r^B)\,dr - 2\int_0^t (G_r, Y_r - Z_r)_H\,dr \end{aligned} \tag{4.8}$$

for a.e. $t \in [0,T]$, where $G := \frac{d}{dt}Z$.

If, in addition, $Y \in C([0,T]; H)$, then Y is said to be a (time-)continuous SVI solution to (4.5).

PROPOSITION 17. *For each $x_0 \in H$, $\omega \in \Omega$ there is a unique time-continuous SVI solution $Y = Y(\omega)$ to (4.5). The process $(t,\omega) \mapsto Y_t(\omega)$ is $\mathcal{F}_t$-progressively measurable, and the constant $C = C(\omega)$ in (4.7) satisfies $C \in L^2(\Omega)$.*

Proof. The proof follows along the same line of arguments as the proof of [32, Theorem 3.1]. Hence, in the following we shall restrict ourselves to giving some details on required modifications of the proof. For notational convenience we set

$$\begin{aligned}\psi(z) :=&\frac{1}{p}|z|^p,\\ \phi(z) :=&\partial\psi(z)\end{aligned}$$

and let ψ^δ be the Moreau–Yosida approximation of ψ, $\phi^\delta := \partial\psi^\delta$. In analogy to (3.6) we consider the three-step approximation

$$\begin{aligned}\frac{d}{dt}Y_t^{\varepsilon,\delta,n} &= \varepsilon\Delta Y_t^{\varepsilon,\delta,n} + \operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))\right),\\ Y_0^{\varepsilon,\delta,n} &= x_n \in H^1_{\mathrm{av}}.\end{aligned} \tag{4.9}$$

Existence and uniqueness of a variational solution $Y^{\varepsilon,\delta,n}$ to (4.9) follows easily from [56, Theorem 4.2.4]. Progressive measurability of $(t,\omega) \mapsto Y_t^{\varepsilon,\delta,n}(\omega)$ follows as in [28, proof of Theorem 1.1].

Step 1. The first step consists in proving the existence of a strong solution to (4.9) and corresponding (uniform) energy bounds as in (3.7). We restrict ourselves to an informal derivation of these estimates; the rigorous justification proceeds as in [32, Theorem 3.1]. We set $\|v\|_{\dot H^1}^2 := \|\nabla v\|_2^2$ for $v \in H^1$. Informally, we compute

$$\begin{aligned}&\frac{d}{dt}\|Y_t^{\varepsilon,\delta,n}\|_{\dot H^1}^2\\ &= -2(\varepsilon\Delta Y_t^{\varepsilon,\delta,n} + \operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))\right), \Delta Y_t^{\varepsilon,\delta,n})_H\\ &= -2\varepsilon\|\Delta Y_t^{\varepsilon,\delta,n}\|_H^2 - 2\left(\operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))\right), \Delta(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))\right)_H\\ &+ 2(\operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))\right), \Delta W_t^B(\omega))_H.\end{aligned}$$

For $v \in H^2$ with $\nabla v \cdot \nu = 0$ on $\partial\mathcal{O}$, arguing as in [31, Example 7.11], we obtain that

$$\begin{aligned}-(v, \operatorname{div}\phi^\delta(\nabla v))_{\dot H^1} &= -(-\Delta v, \operatorname{div}\phi^\delta(\nabla v))_2\\ &= -\lim_{n\to\infty}(T_n v, \operatorname{div}\phi^\delta(\nabla v))_2\\ &= -\lim_{n\to\infty}(nu - nJ_n u, \operatorname{div}\phi^\delta(\nabla v))_2\\ &\leqslant \lim_{n\to\infty} n\left(\int_{\mathcal{O}}\psi^\delta(\nabla J_n u)d\xi - \int_{\mathcal{O}}\psi^\delta(\nabla u)d\xi\right)\\ &\leqslant 0,\end{aligned} \tag{4.10}$$

where T_n is the Yosida approximation and J_n the resolvent of the Neumann Laplacian $-\Delta$ on L^2. Here, the convexity of $\mathcal{O}$ is needed; see [31] for details. We next observe that

$$
\begin{aligned}
&2\left(\operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n}+W_t^B(\omega))\right),\Delta W_t^B(\omega)\right)_H\\
&=2\int_{\mathcal{O}}\operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n}+W_t^B(\omega))\right)\Delta W_t^B(\omega)d\xi\\
&\leqslant C\int_{\mathcal{O}}(1+|\nabla(Y_t^{\varepsilon,\delta,n}+W_t^B(\omega))|)|\nabla\Delta W_t^B(\omega)|d\xi\\
&\leqslant C\left(\|\nabla Y_t^{\varepsilon,\delta,n}\|_H^2+1+\|W_t^B(\omega)\|_{H^3}^2\right).
\end{aligned}
$$

Hence,

$$
\frac{d}{dt}\|Y_t^{\varepsilon,\delta,n}\|_{\dot H^1}^2\leqslant-2\varepsilon\|\Delta Y_t^{\varepsilon,\delta,n}\|_H^2+C(\|Y_t^{\varepsilon,\delta,n}\|_{\dot H^1}^2+1+\|W_t^B(\omega)\|_{H^3}).
$$

By Gronwall's lemma this implies

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|Y_t^{\varepsilon,\delta,n}\|_{\dot H^1}^2+2\varepsilon\int_0^T\|\Delta Y_r^{\varepsilon,\delta,n}\|_H^2dr\\
&\leqslant C\left(\|x_0\|_{\dot H^1}^2+\int_0^T\|W_r^B(\omega)\|_{H^3}dr+1\right)\\
&\leqslant C(\|x_0\|_{\dot H^1}^2+1),
\end{aligned}
\tag{4.11}
$$

which finishes the proof of the required energy bound.

Step 2. We next derive the variational inequality (4.8) and regularity estimate (4.7) for $Y^{\varepsilon,\delta,n}$. By the chain rule, and since $\phi^\delta=\partial\psi^\delta$, we have that

$$
\begin{aligned}
&\left\|Y_t^{\varepsilon,\delta,n}-Z_t\right\|_H^2\\
&=\|x_0^n-Z_0\|_H^2+2\int_0^t(\operatorname{div}\phi^\delta\left(\nabla(Y_r^{\varepsilon,\delta,n}+W_r^B(\omega))\right)+\varepsilon\Delta Y_r^{\varepsilon,\delta,n}-G_r,Y_r^{\varepsilon,\delta,n}-Z_r)_Hdr\\
&=\|x_0^n-Z_0\|_H^2+2\int_0^t\int_{\mathcal{O}}\phi^\delta\left(\nabla(Y_r^{\varepsilon,\delta,n}+W_r^B(\omega))\right)\cdot(\nabla Z_r-\nabla Y_r^{\varepsilon,\delta,n})d\xi dr\\
&\quad-2\varepsilon\int_0^t(\Delta Y_r^{\varepsilon,\delta,n},Z_r-Y_r^{\varepsilon,\delta,n})_Hdr-2\int_0^t(G_r,Y_r^{\varepsilon,\delta,n}-Z_r)_Hdr\\
&\leqslant\|x_0^n-Z_0\|_H^2-2\int_0^t\left(\int_{\mathcal{O}}\psi^\delta\left(\nabla(Y_r^{\varepsilon,\delta,n}+W_r^B(\omega))\right)d\xi+\frac{\varepsilon}{2}\left\|Y_r^{\varepsilon,\delta,n}\right\|_{\dot H^1}^2\right)dr\\
&\quad+\varepsilon\int_0^t\|Z_r\|_{\dot H^1}^2\,dr+2\int_0^t\int_{\mathcal{O}}\psi^\delta\left(\nabla(Z_r+W_r^B(\omega))\right)d\xi dr-2\int_0^t(G_r,Y_r^{\varepsilon,\delta,n}-Z_r)_Hdr
\end{aligned}
$$

for all $Z\in W^{1,2}([0,T];H)$ and $G:=\frac{d}{dt}Z$. In particular, choosing $Z\equiv0$ we obtain that

$$
\begin{aligned}
&\left\|Y_t^{\varepsilon,\delta,n}\right\|_H^2+2\int_0^t\int_{\mathcal{O}}\psi^\delta\left(\nabla(Y_r^{\varepsilon,\delta,n}+W_r^B(\omega))\right)d\xi\,dr\\
&\leqslant\|x_0\|_H^2+2\int_0^t\int_{\mathcal{O}}\psi^\delta\left(\nabla W_r^B(\omega)\right)\,dr\\
&\leqslant\|x_0\|_H^2+C(\omega),
\end{aligned}
$$

with $C\in L^2(\Omega)$.

Step 3. The next step is to take the limit $\delta \to 0$; i.e., we estimate

$$
\begin{aligned}
&\frac{d}{dt}\|Y_t^{\varepsilon,\delta_1,n} - Y_t^{\varepsilon,\delta_2,n}\|_H^2 \\
&= 2\varepsilon(\Delta Y_t^{\varepsilon,\delta_1,n} - \Delta Y_t^{\varepsilon,\delta_2,n}, Y_t^{\varepsilon,\delta_1,n} - Y_t^{\varepsilon,\delta_2,n})_H \\
&+ 2\left(\operatorname{div}\phi^{\delta_1}\left(\nabla(Y_t^{\varepsilon,\delta_1,n}+W_t^B(\omega))\right) - \operatorname{div}\phi^{\delta_2}\left(\nabla(Y_t^{\varepsilon,\delta_2,n}+W_t^B(\omega))\right), Y_t^{\varepsilon,\delta_1,n} - Y_t^{\varepsilon,\delta_2,n}\right)_H \\
&\leqslant 2\left(\operatorname{div}\phi^{\delta_1}\left(\nabla(Y_t^{\varepsilon,\delta_1,n}+W_t^B(\omega))\right) - \operatorname{div}\phi^{\delta_2}\left(\nabla(Y_t^{\varepsilon,\delta_2,n}+W_t^B(\omega))\right), Y_t^{\varepsilon,\delta_1,n} - Y_t^{\varepsilon,\delta_2,n}\right)_H .
\end{aligned}
$$

Using [32, Appendix A, (A.6)] we conclude that

$$
\begin{aligned}
&\frac{d}{dt}\|Y_t^{\varepsilon,\delta_1,n} - Y_t^{\varepsilon,\delta_2,n}\|_H^2 \\
&\leqslant C(\delta_1+\delta_2)(1 + \|Y_t^{\varepsilon,\delta_1,n}\|_{\dot H^1}^2 + \|Y_t^{\varepsilon,\delta_2,n}\|_{\dot H^1}^2 + \|W_t^B(\omega)\|_{\dot H^1}^2).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|Y_t^{\varepsilon,\delta_1,n} - Y_t^{\varepsilon,\delta_2,n}\|_H^2 \\
&\leqslant C(\delta_1+\delta_2)(1 + \int_0^t \|Y_r^{\varepsilon,\delta_1,n}\|_{\dot H^1}^2 dr + \int_0^t \|Y_r^{\varepsilon,\delta_2,n}\|_{\dot H^1}^2 dr + \int_0^t \|W_r^B(\omega)\|_{\dot H^1}^2 dr),
\end{aligned}
$$

which, using (4.11), implies convergence of $Y^{\varepsilon,\delta,n}$ in $C([0,T];H)$ for $\delta \to 0$.

Step 4. The limits $\varepsilon \to 0$, $n \to \infty$ can be justified precisely as in the proof of [32, Theorem 3.1], and the proof can be concluded as in [32, Theorem 3.1]. Progressive measurability of $(t,\omega) \mapsto Y_t(\omega)$ follows from the respective property of $Y^{\varepsilon,\delta,n}$. ∎

Remark 18.

(i) Let X be the unique SVI solution to (3.1) and let Y be the unique SVI solution to (4.5). Then $X = Y + W^B$, $\mathbb{P}$-a.s.

(ii) Let X^ε be the unique SVI solution to (4.1) and let Y^ε be the unique variational solution to (4.5) given by [56, Theorem 4.2.4]. Then $X^\varepsilon = Y^\varepsilon + W^B$, $\mathbb{P}$-a.s.

Proof. (i) If Y is the unique SVI solution to (4.5), then $\bar X := Y + W^B$ is progressively measurable, and taking the expectation in (4.7) yields $\bar X \in L^2([0,T]\times\Omega;H)$. It is then easy to see that $\bar X$ is an SVI solution to (3.1). Thus, uniqueness of SVI solutions implies $X = \bar X$, $\mathbb{P}$-a.s. (ii) Follows by the same proof as (i). In order to prove that $\bar X^\varepsilon$ is an SVI solution to (4.1), see the proof of Theorem 14 below, in particular (4.14), (4.15). ∎

4.2. Convergence of solutions.

Proof of Theorem 14. Let $\varepsilon_n \to 0$ and set $Y^n := Y^{\varepsilon_n}$, $J^n = J^{\varepsilon_n}$.

Step 1. *Weak convergence in* $L^2([0,T];H)$. By [56, Theorem 4.2.4] there is a unique variational solution $Y^n \in C([0,T];H)$ to (4.6) with respect to the trivial Gelfand triple $V = H = L^2$. Indeed, it is easy to see that A^n satisfies the required hemicontinuity and monotonicity on L^2. Concerning coercivity, using [5, Lemma 6.5], we note that

$$
\begin{aligned}
&{}_{V^*}\langle A^n(u), u\rangle_V \\
&= -\frac{1}{2}\int_{\mathcal{O}}\int_{\mathcal{O}} J^n(\zeta-\xi)\phi(u(\xi)-u(\zeta))(u(\xi)-u(\zeta))\,d\xi\,d\zeta \\
&= -\frac{1}{2}\int_{\mathcal{O}}\int_{\mathcal{O}} J^n(\zeta-\xi)|u(\xi)-u(\zeta)|^p\,d\xi\,d\zeta \\
&= -\frac{1}{2}\|u\|_{V_n}^p \\
&\leqslant 0
\end{aligned}
\tag{4.12}
$$

for all $u \in L^2$, where we have set $V_n := L^p_{J^n} \cap L^p_{\mathrm{av}}$. Moreover, using [5, Lemma 6.5] and Hölder's inequality, we note that

$$
\begin{aligned}
(A^n(u), v)_H &= \int_{\mathcal{O}}\int_{\mathcal{O}} J^n(\zeta-\xi)|u(\xi)-u(\zeta)|^{p-2}(u(\xi)-u(\zeta))(v(\xi)-v(\zeta))\,d\xi\,d\zeta \\
&\leqslant \left(\int_{\mathcal{O}}\int_{\mathcal{O}} J^n(\zeta-\xi)|u(\xi)-u(\zeta)|^p\,d\xi\,d\zeta\right)^{\frac{p-1}{p}} \\
&\quad\times \left(\int_{\mathcal{O}}\int_{\mathcal{O}} J^n(\zeta-\xi)|v(\xi)-v(\zeta)|^p\,d\xi\,d\zeta\right)^{\frac{1}{p}} \\
&\leqslant \|u\|_{V_n}^{p-1}\|v\|_{V_n} \quad \forall u, v \in L^2.
\end{aligned}
\tag{4.13}
$$

It is easy to see that Y^n is an SVI solution to (4.6): Indeed, by the chain rule we have that

$$
\begin{aligned}
\|Y_t^n - Z_t\|_H^2 &\leqslant \|x_0^n - Z_0\|_H^2 + 2\int_0^t \varphi^n(Z_r + W_r^B)\,dr - 2\int_0^t \varphi^n(Y_r^n + W_r^B)\,dr \\
&\quad - \int_0^t (G_r, Y_r^n - Z_r)dr
\end{aligned}
\tag{4.14}
$$

for all $Z \in W^{1,2}([0,T];H)$ and $G := \frac{d}{dt}Z$. In particular, choosing $Z \equiv 0$ we obtain that

$$
\|Y_t^n\|_H^2 + 2\int_0^t \varphi^n(Y_r^n + W_r^B)\,dr \leqslant \|x_0\|_H^2 + 2\int_0^t \varphi^n(W_r^B)\,dr.
$$

By [32, p. 24, first equation] we have

$$
\varphi^n(v) \leqslant C\|v\|_{\dot{W}^{1,p}} \leqslant C(1 + \|v\|_{\dot{H}^1}^2) \quad \forall v \in H^1_{\mathrm{av}}.
$$

Hence,

$$
\|Y_t^n\|_H^2 + 2\int_0^t \varphi^n(Y_r^n + W_r^B)\,dr \lesssim \|x_0\|_H^2 + \int_0^t \|W_r^B\|_{\dot{H}^1}^2\,dr + 1,
\tag{4.15}
$$

and we may extract a subsequence (again denoted by Y^n) such that

$$
Y^n \rightharpoonup Y \quad \text{in } L^2([0,T];H).
$$

Using the Mosco convergence of $\varphi^n \to \varphi$ and $\limsup_{n\to\infty} \varphi^n(u) \leqslant \varphi(u)$ (cf. [32, Proposition 5.2]) and Mosco convergence of integral functionals (cf. [32, Appendix B]), it is easy to see that Y is an SVI solution to (4.5). Since SVI solutions to (4.5) are unique by Proposition 17 this implies weak convergence of the whole sequence Y^n to Y in $L^2([0,T];H)$.

Step 2. By the chain rule, (4.12), and (4.13) we have that

$$\begin{aligned}
\|Y_t^n\|_H^2 &= \|x_0\|_H^2 + 2\int_0^t (A^n(Y_r^n + W_r^B(\omega)), Y_r^n)_H \, dr \\
&= \|x_0\|_H^2 + 2\int_0^t (A^n(Y_r^n + W_r^B(\omega)), Y_r^n + W_r^B(\omega))_H \, dr \\
&\quad - 2\int_0^t (A^n(Y_r^n + W_r^B(\omega)), W_r^B(\omega))_H \, dr \\
&\leqslant \|x_0\|_H^2 - 2\int_0^t \|Y_r^n + W_r^B(\omega)\|_{V_n}^p \, dr \\
&\quad + 2\int_0^t \|Y_r^n + W_r^B(\omega)\|_{V_n}^{p-1} \|W_r^B(\omega)\|_{V_n} \, dr \\
&\leqslant \|x_0\|_H^2 - \int_0^t \|Y_r^n + W_r^B(\omega)\|_{V_n}^p \, dr + C\int_0^t \|W_r^B(\omega)\|_{V_n}^p \, dr.
\end{aligned}$$

Hence, using (A.1),

$$\sup_{t\in[0,T]} \|Y_t^n\|_H^2 + \int_0^T \|Y_r^n + W_r^B(\omega)\|_{V_n}^p \, dr \leqslant \|x_0\|_H^2 + C\int_0^T \|W_r^B(\omega)\|_{H^1}^p \, dr. \tag{4.16}$$

We continue with an argument from [27]: Consider the set

$$\mathcal{K} := \{(Y^n, v)_H \ : \ v \in W^{1,p} \cap H, \ \|v\|_H \vee \|v\|_{W^{1,p}} \leqslant 1, \ n \in \mathbb{N}\} \subset C([0,T]).$$

By (4.16), $\mathcal{K}$ is bounded in $C([0,T])$. Moreover, by (4.13) and (4.16),

$$\begin{aligned}
(Y_{t+s}^n - Y_t^n, v)_H &= \int_t^{t+s} ((Y^n)'_r, v)_H \, dr \\
&= \int_t^{t+s} (A^n(Y_r^n + W_r^B(\omega)), v)_H \, dr \\
&\leqslant \int_t^{t+s} \|Y_r^n + W_r^B(\omega)\|_{V_n}^{p-1} \|v\|_{V_n} \, dr \\
&\leqslant C s^{1/p} \int_0^T \|Y_r^n + W_r^B(\omega)\|_{V_n}^p \, dr \\
&\leqslant C s^{1/p},
\end{aligned}$$

where we used (A.1). Hence, $\mathcal{K}$ is a set of equibounded, equicontinuous functions and thus is relatively compact in $C([0,T])$ by the Arzelà–Ascoli theorem. Thus, for every $v \in W^{1,p} \cap H$, $\|v\|_H \vee \|v\|_{W^{1,p}} \leqslant 1$, there is a subsequence (again denoted by Y^n) such that $(Y^n, v)_H \to g$ in $C([0,T])$. Since also $(Y^n, v)_H \to (Y, v)_H$ in $L^2([0,T])$ by Step 1, we have $g = (Y, v)_H$. Hence, for each $v \in W^{1,p} \cap H$, the whole sequence $(Y^n, v)_H$ converges to $(Y, v)_H$ in $C([0,T])$.

For $h \in H$, $\varepsilon > 0$ we can choose $v^\varepsilon \in W^{1,p} \cap H$ such that $\|h - v^\varepsilon\| \leqslant \varepsilon$. Then

$$\begin{aligned}(Y_t^n - Y_t, h)_H &= (Y_t^n - Y_t, h - v^\varepsilon)_H + (Y_t^n - Y_t, v^\varepsilon)_H \\ &\leqslant \|Y_t^n - Y_t\|_H \|h - v^\varepsilon\|_H + (Y_t^n - Y_t, v^\varepsilon)_H \\ &\leqslant C\varepsilon + (Y_t^n - Y_t, v^\varepsilon)_H.\end{aligned}$$

Hence, choosing n large enough implies

$$(Y_t^n - Y_t, h)_H \leqslant C\varepsilon \quad \forall n \geqslant n_0(\varepsilon),$$

that is, $Y_t^n \rightharpoonup Y_t$ weakly in H for $n \to \infty$.

Since $Y^n = X^n - W^B$, $Y = X - W^B$, $\mathbb{P}$-a.s., this implies weak convergence of X_t^n to X_t, $\mathbb{P}$-a.s..

Step 3. We next prove the convergence of the associated semigroups P_t^ε, P_t. Let $F \in \mathcal{F}C_b^1(E)$ with $F = f(l_1, \ldots, l_k)$ and let $t \geqslant 0$, $x \in H$. Further let $\varepsilon_n \to 0$ and set $P_t^n := P_t^{\varepsilon_n}$. Then

$$\begin{aligned}P_t^n F(x) &= \mathbb{E}F(X_t^{n,x}) \\ &= \mathbb{E}f(l_1(X_t^{n,x}), \ldots, l_k(X_t^{x,n})) \\ &\to \mathbb{E}f(l_1(X_t^x), \ldots, l_k(X_t^x)) \\ &= \mathbb{E}F(X_t^x) \\ &= P_t F(x),\end{aligned}$$

as $n \to \infty$, by Lebesgue's dominated convergence theorem. ∎

5. Convergence of invariant measures: Nonlocal to local. By Theorem 1, for each $\varepsilon > 0$, there exists a unique invariant measure μ^ε to the stochastic nonlocal p-Laplace equation (4.2), and by Theorem 8 there is a unique invariant measure μ to the (local) stochastic p-Laplace equation (4.3). In this section we prove weak* convergence of μ^ε to μ in a suitable topology for $p \in (1, 2)$.

Several difficulties appear, due to the nonlocal and singular-degenerate nature of the SPDE (4.2). First, we expect tightness of μ^ε on $H = L^2_{\mathrm{av}}$ only under stringent dimensional restrictions. Indeed, for the (expected) limit μ we only know $\mu(W^{1,p}_{\mathrm{av}}) = 1$, which, roughly speaking, would lead to assuming that the embedding $W^{1,p}_{\mathrm{av}} \hookrightarrow L^2_{\mathrm{av}}$ is compact, and thus we restrict ourselves to one spatial dimension, i.e., $d = 1$, in general. Second, we only have weak convergence $X^\varepsilon \rightharpoonup X$ in H. Therefore, the convergence of the associated semigroups $P_t^n F$ for general $F \in \mathrm{Lip}_b(H)$—a crucial ingredient in previously available methods—is unclear.

The first problem is overcome in this section by considering weak convergence of μ^ε on $E = L^p_{\mathrm{av}}$ rather than on L^2_{av}. Again, for the limit μ we know $\mu(W^{1,p}_{\mathrm{av}}) = 1$. Hence, by compactness of the embedding $W^{1,p}_{\mathrm{av}} \hookrightarrow L^p_{\mathrm{av}}$, μ is concentrated on compact sets in L^p_{av}, which suggests that tightness of μ^ε on L^p_{av} should hold without restrictions on the dimension. Indeed, this is established in Proposition 24 below. The resulting difficulty of working with two topologies, weak* convergence of μ^ε on L^p_{av} versus weak convergence of X^ε on L^2_{av}, is solved in Lemma 21. The second problem is overcome by first considering cylindrical functions on H. For a cylindrical function F, weak convergence $X_t^n \rightharpoonup X_t$ in H is enough to deduce $P_t^n F \to P_t F$. It turns out that this is sufficient to deduce the weak* convergence $\mu^\varepsilon \rightharpoonup^* \mu$ by means of a monotone class argument (cf. proof of Theorem 19). The following theorem is the main result of this section.

THEOREM 19. *Let μ^ε be the unique invariant measure to* (4.2) *and let μ be the unique invariant measure to* (4.3). *Then $\mu^\varepsilon \rightharpoonup^* \mu$ for $\varepsilon \to 0$ weakly* in the set of probability measures on $L^p_{\mathrm{av}}(\mathcal{O})$; that is, for each bounded, Lipschitz continuous function F on $L^p_{\mathrm{av}}(\mathcal{O})$ we have $\mu^\varepsilon(F) \to \mu(F)$ for $\varepsilon \to 0$.*

5.1. Asymptotic invariance. In this section we provide a general result on the convergence of invariant measures for convergent semigroups. Compared to previous results [17] the main novelty here is to work with two distinct topologies corresponding to the convergence of the invariant measures on the one hand and to the convergence of the semigroups on the other hand.

DEFINITION 20. *Let E be a Banach space, let $\mathcal{G} \subseteq \mathcal{B}_b(E)$ be a set of bounded, measurable functions on E, and let P_t be a semigroup on E. Then a probability measure μ on E is said to be $\mathcal{G}$-invariant if*

$$\int_E P_t G \, d\mu = \int_E G \, d\mu \quad \forall G \in \mathcal{G}.$$

LEMMA 21. *Let E, H be Banach spaces with $H \hookrightarrow E$ dense. Further let $\mathcal{G} \subset \mathrm{Lip}_b(E)$, let P^n_t, P_t be Feller semigroups on H, and let μ^n be $\mathcal{G}$-invariant probability measures for P^n_t for all $n \in \mathbb{N}$. Suppose that $\mu_n \rightharpoonup^* \mu$ as probability measures on E, the semigroups P^n_t satisfy a uniform e-property, that is, there exists a $C > 0$ such that for all $F \in \mathrm{Lip}_b(E)$, $x, y \in E$,*

$$|P^n_t F(x) - P^n_t F(y)| \leqslant C \,\mathrm{Lip}(F) \,\|x - y\|_E \quad \forall n \in \mathbb{N}, t \geqslant 0,$$

and that for every $G \in \mathcal{G}$, $t \geqslant 0$, $x \in H$,

$$\lim_{n\to\infty} P^n_t G(x) = P_t G(x).$$

Then μ is $\mathcal{G}$-invariant, i.e.,

$$\int_E P_t G \, d\mu = \int_E G \, d\mu \quad \textit{for all } G \in \mathcal{G}, t \geqslant 0.$$

Proof. For two (Borel) probability measures ν_1, ν_2 on $(E, \mathcal{B}(E))$, denote by $\beta_E(\nu_1, \nu_2)$ the bounded Lipschitz distance between them, that is,

$$\beta_E(\nu_1,\nu_2) := \sup \left\{ \left| \int_E F \, d(\nu_1 - \nu_2) \right| \; : \; F \in \mathrm{Lip}_b(E), \; \|F\|_{E,\infty} + \mathrm{Lip}_E(F) \leqslant 1 \right\}.$$

We have $\mathrm{Lip}_b(E) \subseteq \mathrm{Lip}_b(H)$, and by continuous extension we can identify

$$\{F \in \mathrm{Lip}_b(H) \; : \; \exists C > 0 \text{ s.t. } \|F(x) - F(y)\|_E \leqslant C\|x - y\|_E \;\; \forall x, y \in E\} = \mathrm{Lip}_b(E).$$

Accordingly, due to the e-property, $P^n_t : \mathrm{Lip}_b(E) \to \mathrm{Lip}_b(E)$. Let $G \in \mathcal{G}$, $t \geqslant 0$. We have that

$$\begin{aligned}
&\left| \int_E G \, d\mu - \int_E P_t G \, d\mu \right| \\
&\leqslant \left| \int_E G \, d\mu - \int_E P^n_t G \, d\mu_n \right| + \left| \int_E P^n_t G \, d\mu_n - \int_E P^n_t G \, d\mu \right| \\
&+ \left| \int_E P^n_t G \, d\mu - \int_E P_t G \, d\mu \right|.
\end{aligned}$$

By the property of being $\mathcal{G}$-invariant measures, the first term equals $\int_E G\,d(\mu-\mu_n)$ and hence tends to zero as $n \to \infty$. By the e-property, the second term can be bounded (with $\|F\|_{E,\infty} := \sup_{x\in E}|F(x)|$) as

$$\beta_E(\mu_n,\mu)\left[\|P_t^n G\|_{E,\infty} + \mathrm{Lip}_E(P_t^n G)\right] \leqslant \beta_E(\mu_n,\mu_0)\left[\|G\|_{E,\infty} + C\,\mathrm{Lip}_E(G)\right]$$

and hence in turn tends to zero as $n \to \infty$ by weak convergence of μ_n to μ and Lebesgue's dominated convergence, since (in Polish spaces) the bounded Lipschitz metric generates the weak topology; see, e.g., [62, Chapter 1, section 12, pp. 73–74]. Since $\mu^n(H) = 1$ and $\mu_n \rightharpoonup^* \mu$, we have $\mu(H) = 1$. Thus, the third term converges to zero by convergence of semigroups and Lebesgue's dominated convergence theorem. ∎

5.2. Tightness. Below, taking complements of sets refers to the Polish space E; that is, we denote $A^c := E \setminus A$ for any set $A \subset E$.

Definition 22. *A sequence of probability measures μ_n on a Polish space E is called* asymptotically tight *if for each $\eta > 0$ there exists a compact set K_η such that for each $\delta > 0$ it holds that*

$$\limsup_{n\to\infty} \mu_n((K_\eta^\delta)^c) < \eta,$$

where $K_\eta^\delta \supset K_\eta$ is the open δ-enlargement of K_η.

The next result can be found in [62, Theorem 1.3.9].

Lemma 23. *If μ_n is asymptotically tight, then it is weakly relatively compact.*

Let μ^ε, $\varepsilon > 0$ be the unique invariant measure associated to (4.2).

Proposition 24. *Let $\varepsilon_n \searrow 0$ as $n \to \infty$ and set $\mu_n := \mu_{\varepsilon_n}$. Then μ_n is asymptotically tight on $E := L^p_{\mathrm{av}}(\mathcal{O})$.*

Proof. Let $\eta > 0$ and $C := \|B\|^2_{L_2(H)}$. Recall $\varphi_\varepsilon \leqslant K\varphi$ for all $\varepsilon \in (0,1]$ for some constant $K > 0$. Then, by Poincaré's inequality,

$$K_\eta := \left\{x \in L^p_{\mathrm{av}} : \varphi(x) \leqslant \frac{2C}{\eta K}\right\}$$

is a bounded set in $W^{1,p}_{\mathrm{av}}(\mathcal{O})$ and hence compact in $L^p_{\mathrm{av}}(\mathcal{O})$. For $\delta > 0$, let K_η^δ be the open δ-enlargement of K_η in E. Let

$$G_n := \left\{x \in L^p_{\mathrm{av}} : \varphi^{\varepsilon_n}(x) \leqslant \frac{2C}{\eta}\right\}$$

for some $\varepsilon_n \searrow 0$, $n \to \infty$. Since $\varphi_\varepsilon \leqslant K\varphi$ for all $\varepsilon \in (0,1]$, it holds that $G_n \supset K_\eta$ for $n \in \mathbb{N}$.

We claim that for each $\delta > 0$ there exists an $n_0 \in \mathbb{N}$ such that $G_n \subset K_\eta^\delta$ for all $n \geqslant n_0$. We argue by contradiction. If there exists $\delta_0 > 0$, such that for all $n \in \mathbb{N}$ it holds that $G_n \not\subset K_\eta^{\delta_0}$, then we can find a sequence $x_n \in G_n \setminus K_\eta^{\delta_0}$ such that $\mathrm{dist}_E(x_n, K_\eta) \geqslant \delta_0$ for every n. By the definition of G_n and [5, Theorem 6.11 (2)], $\{x_n\}$ is relatively compact in $L^p_{\mathrm{av}}(\mathcal{O})$. Hence, there exists a subsequence (denoted by $\{x_n\}$) such that $x_n \to \bar{x}$ in $L^p_{\mathrm{av}}(\mathcal{O})$ and $\bar{x} \in W^{1,p}_{\mathrm{av}}(\mathcal{O})$. By the Mosco convergence of $\varphi_\varepsilon \to \varphi$ on L^p (cf. [32, Proposition 5.2]) we obtain that

$$\varphi(\bar{x}) \leqslant \liminf_{n\to\infty} \varphi^{\varepsilon_n}(x_n) \leqslant \frac{2C}{\eta},$$

and thus $\bar{x} \in K_\eta$. Hence,

$$\delta_0 \leqslant \operatorname{dist}_E(x_n, K_\eta) \leqslant \|\bar{x} - x_n\|_{L^p_{\mathrm{av}}(\mathcal{O})} \underset{n\to\infty}{\longrightarrow} 0,$$

the desired contradiction.

Now, by Theorem 1, for each $n \geqslant n_0(\delta)$,

$$\begin{aligned}\mu_n((K_\eta^\delta)^c) &\leqslant \mu_n(G_n^c) \leqslant \frac{\eta}{2C}\int \varphi^{\varepsilon_n}(z)\,\mu_n(dz)\\ &\leqslant \frac{\eta}{2} < \eta.\end{aligned}$$

The proof is completed by taking the limsup as $n \to \infty$. ∎

5.3. Proof of Theorem 19. We aim to apply Lemma 21 with $E = L^p_{\mathrm{av}}(\mathcal{O})$, $H = L^2_{\mathrm{av}}(\mathcal{O})$. Since $p \in (1,2)$, we have that $H \subseteq E$. Let $\mathcal{G}$ be the space of cylindrical functions on E, that is,

$$\mathcal{G} = \mathcal{F}C_b^1(E).$$

Let $\varepsilon_n \to 0$, set $\mu_n := \mu_{\varepsilon_n}$, and let $t \geqslant 0$ be arbitrary, fixed. By Proposition 24 μ_n is asymptotically tight and thus has a weakly* convergent subsequence (again denoted by μ_n) such that $\mu_n \rightharpoonup \nu$. The uniform e-property for $P_t^n := P_t^{\varepsilon_n}$ on E has been verified in Lemma 7, and by Theorem 14 we have

$$P_t^n F(x) \to P_t F(x) \quad \text{for } n \to \infty$$

for all $F \in \mathcal{G}$, $t \geqslant 0$, $x \in H$. An application of Lemma 21 thus yields that ν is $\mathcal{G}$-invariant.

We show next that this implies that ν is an invariant measure for P_t. First note that $\mathcal{G}$ is an algebra (with respect to pointwise multiplication) of bounded real-valued functions on E that contains the constant functions. By [51, Chapter II, section 3 (a), p. 54], $\mathcal{G}$ separates points of E, which by [13, Theorem 6.8.9] implies that $\mathcal{G}$ generates the Borel σ-algebra $\mathcal{B}(E)$. Set

$$\mathcal{H} := \mathcal{H}(\nu, t) := \left\{F \in \mathcal{B}_b(E) \ : \ \int_E P_t F\,d\nu = \int_E F\,d\nu\right\}.$$

Clearly, $1 \in \mathcal{H}$ and $\mathcal{H}$ is closed under monotone convergence by the Markov property and Beppo Levi's monotone convergence lemma. Further, $\mathcal{H}$ is closed under uniform convergence by Lebesgue's dominated convergence theorem and the Markov property. We have already shown $\mathcal{G} \subset \mathcal{H}$. Hence, by the monotone class theorem [13, Theorem 2.12.9 (ii)], $\mathcal{B}_b(E) \subset \mathcal{H}$ and therefore $\mathcal{B}_b(E) = \mathcal{H}$. Since $\nu(H) = 1$ this implies that ν is an invariant measure for P_t. By Theorem 8 there is a unique invariant measure μ for P_t. Thus $\mu = \nu$ and, by uniqueness, the whole sequence μ_n converges weakly* to μ.

Appendix A. Notation. We work with generic constants $C \geqslant 0$, $c > 0$ that are allowed to change value from line to line, and we write

$$A \lesssim B$$

if there is a constant $C \geqslant 0$ such that $A \leqslant CB$. For a metric space (E, d), $R > 0$, $x \in E$ we let $B_R(x)$ denote the open ball of radius R in E centered at x. Moreover, we let $\mathcal{B}(E)$ denote the Borel sigma algebra and $\mathcal{B}_b(E)$ the space of bounded, Borel-measurable functions on E. The $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ is denoted by S^{d-1}. For notational convenience we set

$$a^{[m]} := |a|^{m-1}a \quad \text{for } a \in \mathbb{R}, m \geqslant 0$$

and

$$|\xi|^{-1}\xi := \begin{cases} |\xi|^{-1}\xi & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ \bar{B}_1(0) & \text{if } \xi = 0. \end{cases}$$

For $m \geqslant 1$ we set $L^m(\mathcal{O})$ to be the usual Lebesgue spaces with norm $\|\cdot\|_{L^m}$, and we shall often use the shorthand notation $L^m := L^m(\mathcal{O}), \|\cdot\|_m := \|\cdot\|_{L^m(\mathcal{O})}$. We let $B^m_R(x)$ be the open ball in L^m of radius $R > 0$ centered at $x \in L^m$. We further define $L^m_{\mathrm{av}} := L^m_{\mathrm{av}}(\mathcal{O})$ to be the space of all functions in L^m with zero average, that is,

$$L^m_{\mathrm{av}}(\mathcal{O}) := \left\{ v \in L^m(\mathcal{O}) : \int_{\mathcal{O}} v d\xi = 0 \right\}$$

and $H^k_{\mathrm{av}} := H^k \cap L^2_{\mathrm{av}}$, where H^k are the usual Sobolev spaces. For a function $v \in L^m(\mathcal{O})$ we define its extension to all of $\mathbb{R}^d$ by

$$\bar{v}(\xi) = \begin{cases} v(\xi) & \text{if } \xi \in \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $J : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative, continuous, radial function with compact support, $J(0) > 0$, $\int_{\mathbb{R}^d} J(z)\, dz = 1$. We then consider the following nonlocal averaged Sobolev-type spaces: For $\varepsilon > 0$, $m \geqslant 1$, let $V_\varepsilon := L^m_{J^\varepsilon}(\mathcal{O})$ be equal to $L^m_{\mathrm{av}}(\mathcal{O})$ with the topology coming from the norm

$$\begin{aligned} \|v\|^m_{J^\varepsilon} &:= \frac{C_{J,m}}{2m\varepsilon^d} \int_{\mathcal{O}} \int_{\mathcal{O}} J\left(\frac{\xi - \zeta}{\varepsilon}\right) \left| \frac{v(\zeta) - v(\xi)}{\varepsilon} \right|^m d\zeta\, d\xi \\ &= \frac{C_{J,m}}{2m} \int_{\mathcal{O}} \int_{\mathbb{R}^d} J(z) 1_{\mathcal{O}}(\xi + \varepsilon z) \left| \frac{\bar{v}(\xi + \varepsilon z) - v(\xi)}{\varepsilon} \right|^m dz\, d\xi, \end{aligned}$$

where $C_{J,m}$ is a normalization constant given by

$$C^{-1}_{J,m} := \frac{1}{2} \int_{\mathbb{R}^d} J(z)|z_d|^m\, dz.$$

For notational convenience we set

$$J^\varepsilon(\xi) := \frac{C_{J,m}}{\varepsilon^{d+m}} J\left(\frac{\xi}{\varepsilon}\right) \quad \forall \xi \in \mathbb{R}^d.$$

By [5, Proposition 6.25] the norm $\|v\|_{J^\varepsilon}$ is equivalent to $\|v\|_m$. In particular, $L^m_{J^\varepsilon}(\mathcal{O})$ is a reflexive Banach space for all $m \in (1, \infty)$. Moreover, by [15],

$$\|\cdot\|_{V_\varepsilon} = \|\cdot\|_{J^\varepsilon} \leqslant C \|\cdot\|_{W^{1,m}} \tag{A.1}$$

for some constant $C > 0$ independent of $\varepsilon > 0$.

We say that a function $X \in L^1([0,T] \times \Omega; H)$ is $\mathcal{F}_t$-progressively measurable if $X1_{[0,t]} \in L^1(\mathcal{B}([0,t]) \otimes \mathcal{F}_t; H)$ for all $t \geqslant 0$.

Let E be a Banach space. For a Feller semigroup P_t on $\mathcal{B}_b(E)$ we define the dual semigroup on the space of probability measures $\mathcal{M}_1(E)$ on E by

$$P_t^*\mu(B) := \int_E P_t 1_B(x) d\mu(x)$$

and the time averages

$$Q_T(x, B) := \frac{1}{T} \int_0^T P_t 1_B(x) dt \quad \forall B \in \mathcal{B}(E).$$

We further set

$$Q_T\mu(B) := \int_E Q_T(x, B) d\mu(x).$$

We say that a probability measure μ on E is invariant for P_t if $P_t^*\mu = \mu$ for all $t \geqslant 0$. For an invariant probability measure μ we define its basin of attraction by

$$\mathcal{T}(\mu) := \left\{ x \in E : Q_T(x, \cdot) = \frac{1}{T} \int_0^T P_t(x, \cdot) dt \rightharpoonup^* \mu \text{ for } T \to \infty \right\} \subset E.$$

We say that P_t satisfies the e-property if, for some constant $C > 0$,

$$\|P_t F(x) - P_t F(y)\|_E \leqslant C \operatorname{Lip}(F) \|x - y\|_E \quad \forall x, y \in \mathbb{E},\ F \in \operatorname{Lip}(E).$$

For a Banach space E we define the space of cylindrical functions on E by

$$\mathcal{F}C_b^1(E) := \{ f(l_1, \ldots, l_k) : k \in \mathbb{N},\ l_1, \ldots, l_k \in E^*,\ f \in C_b^1(\mathbb{R}^k) \}.$$

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