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EXISTENCE OF VARIATIONAL SOLUTIONS IN NONCYLINDRICAL DOMAINS*

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Abstract. We study gradient flows of integral functionals in noncylindrical bounded domains $E \subset \mathbb{R}^n \times [0,T)$. The systems of differential equations take the form $\partial_t u - div D_\xi f(x,u,Du) = -D_u f(x,u,Du)$ on E, for an integrand $f(x,u,Du)$ that is convex and coercive with respect to the $W^{1,p}$-norm for $p > 1$. We prove the existence of variational solutions on noncylindrical domains under the only assumption that $\mathcal{L}^{n+1}(\partial E) = 0$, even for functionals that do not admit a growth condition from above. For nondecreasing domains, the solutions are unique and admit a time-derivative in $L^2(E)$. For domains that decrease the most with bounded speed and integrands that satisfy a p-growth condition, we prove that the constructed solutions are continuous in time with respect to the L^2-norm and solve the above system of differential equations in the weak sense. Under the additional assumption that the domain also increases the most at finite speed, we establish the uniqueness of solutions.

Key words. parabolic systems, variational solutions, noncylindrical domains, existence, continuity

AMS subject classifications. 35K40, 35A15, 35R05, 35A05

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1. Introduction.

The analysis of parabolic equations in noncylindrical domains is a classical problem that has attracted a lot of attention. Such a problem arises in applications such as fluid dynamics [27, 38], modeling of glacier formations [40], and mathematical biology, in particular in reaction-diffusion systems modeling pattern formation [13, 35]. For more details, we refer to the survey article [26]. The problem is also interesting from the mathematical point of view, since new phenomena may appear. In particular, for linear equations there is extensive literature on the existence theory of such problems; cf. [32, 4, 1, 12, 9, 33] and the references therein. Also, regularity issues for evolutionary problems in noncylindrical domains have attracted a lot of attention; see, for instance, [16, 21, 10, 6, 3].

De Giorgi became interested in evolutional problems in noncylindrical domains when he conjectured [14] that a modification of the method of *Minimizing Movements* is a proper tool to show the existence of weak solutions to linear parabolic equations in noncylindrical domains. The basic idea of this approach can be described as follows. First, one performs a discretization in time where the time interval $(0,T)$ is divided

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into small ones of length $0 < h \ll 1$. On the time slices, related elliptic minimization problems are solved. In order to obtain the correct boundary values in the noncylindrical domain, De Giorgi suggested adding a penalizing term in the elliptic functional. The minimizers are glued together to a function u_h that is piece-wise constant in time. De Giorgi conjectured that the approximating sequence u_h converges in some sense to a solution u of the parabolic equation on the prescribed noncylindrical domain.

This strategy was implemented by Gianazza and Savaré in [19] to show the existence of weak solutions to linear parabolic equations in an abstract setting. As an application, their result covers the case of noncylindrical domains that are nondecreasing in time, and thereby verifies De Giorgi's conjecture. It is plausible that the case of shrinking domains is significantly more difficult because the boundary value problem might become overdetermined if the domain shrinks too fast. In a subsequent paper [37], Savaré succeeded in treating uniformly $C^{1,1}$-domains that possibly decrease at a bounded speed in the sense that a certain distance function satisfies a Lipschitz condition; see section 2.4 for more details. In [8], Bonaccorsi and Guatteri weakened the assumption on the decreasing of the domains to a Hölder condition. The previously mentioned results are limited to the case of linear equations. Nonlinear operators were first considered by Paronetto [36]. He introduced a different kind of condition on the time-dependent domain by assuming that the time slices are Lipschitz domains that are regular deformations of each other. In a subsequent work, Calvo, Novaga, and Orlandi [11] proved the existence of weak solutions to parabolic p-Laplace type equations on domains that are Lipschitz regular in space and time. For the proof of uniqueness, they additionally rely on the assumptions in [36].

In this paper, we deal with boundary value problems to parabolic systems of the following type. For a noncylindrical domain $E = \bigcup_{t\in[0,T)} E^t \times \{t\} \subset \mathbb{R}^n \times [0,T)$ with $n \geq 2$ and $T > 0$, an initial datum $u_o \in L^2(E^0, \mathbb{R}^N)$ with $N \geq 1$, and an integrand $f\colon E \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to [0,\infty]$ satisfying (2.4) below, we consider the gradient flow for the associated integral functional, which can formally be written as

$$\begin{cases} \partial_t u - \operatorname{div} D_\xi f(x,u,Du) = -D_u f(x,u,Du) & \text{in } E, \\ u = 0 & \text{on } \displaystyle\bigcup_{t\in[0,T)} \partial E^t \times \{t\}, \\ u(\cdot,0) = u_o & \text{in } E^0. \end{cases} \tag{1.1}$$

We restrict ourselves to the model case of vanishing lateral boundary values in order not to overburden this article with additional technicalities. However, we expect that our methods can also be applied to nonzero lateral boundary values, even time-dependent lateral boundary values. More precisely, we expect to obtain existence for boundary values $g \in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ with $\partial_t g \in L^2(\Omega_T)$. This will be the objective of forthcoming research.

The goal of the present paper is twofold. First, we establish the existence of variational solutions of gradient flows under minimal assumptions on the domain and the underlying functional. More precisely, we only assume $\mathcal{L}^{n+1}(\partial E) = 0$ and convexity and coercivity of the integrand f. These assumptions are by far more general than the ones in any of the previously mentioned results. Moreover, our methods are flexible enough to include the case of systems. The key to obtaining existence in this generality is the notion of variational solutions, which goes back to Lichnewsky and Temam [30] and allows us to use techniques from the Calculus of Variations.

Second, we address questions like uniqueness and differentiability and continuity in time. Surprisingly, these properties turn out to be true in the case of nondecreasing domains without any further assumptions on the domain and the functional. These problems are much more delicate in the case of a domain possibly shrinking in time, in which the boundary value problem might be overdetermined. For this reason, it is natural that further assumptions are required that prevent the domain to shrink too fast. Since the desired regularity results cannot be expected for general functionals, we impose standard p-growth conditions on the integrand f. For technical reasons, at this point we have to assume $p > \frac{2(n+1)}{n+2}$, but we do not know whether this lower bound is optimal. Contrary to all previous works, we do not need to restrict ourselves to uniformly $C^{1,1}$ or Lipschitz domains, but we are able to deal with irregular domains whose time slices satisfy a uniform measure density condition.

In the following section we state the precise assumptions and results. The methods of proof and novelties of the used techniques will be explained in section 2.4.

2. Setting and statement of the main results.

2.1. Notation and setting. Before we state the main results of this paper, we shall introduce some notation. We let $n \geq 2$, $N \geq 1$, and $T > 0$ and consider a bounded relatively open *noncylindrical domain* $E \subset \mathbb{R}^n \times [0, T)$ satisfying

$$\mathcal{L}^{n+1}(\partial E) = 0 \quad \text{and} \quad E \subset \Omega_T := \Omega \times [0, T) \tag{2.1}$$

for some bounded and open set $\Omega \subset \mathbb{R}^n$. For fixed $t \in [0, T)$ we denote the section of E by

$$E^t := \{x \in \mathbb{R}^n : (x, t) \in E\} \subset \mathbb{R}^n,$$

so that

$$E \equiv \bigcup_{t \in [0,T)} E^t \times \{t\}.$$

In the following, we denote by V_t the space of those maps $v \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ that vanish outside of E^t, i.e.,

$$V_t := \left\{ v \in W_0^{1,p}(\Omega, \mathbb{R}^N) : v = 0 \text{ a.e. on } \Omega \setminus E^t \right\}.$$

Note that $V_t \cong W_0^{1,p}(E^t, \mathbb{R}^N)$ if E^t satisfies a measure density condition; see Remark 3.2 below for more details. Throughout the rest of the paper we will use the abbreviations

$$V^p(E) := \left\{ u \in L^p\big(0, T; W_0^{1,p}(\Omega, \mathbb{R}^N)\big) : u(t) \in V_t \text{ for a.e. } t \in [0, T) \right\} \tag{2.2}$$

and

$$V_2^p(E) := V^p(E) \cap L^2\big(\Omega_T, \mathbb{R}^N\big).$$

By $u(t)$ we denote the function $u(\cdot, t)$. Obviously, the distinction between the two spaces is relevant only in the subquadratic case $p < 2$. However, for the sake of a unified treatment of the cases $p \geq 2$ and $p < 2$, we will use the notation $V_2^p(E)$ also in the case $p \geq 2$. The above spaces are equipped with the norms

$$\|u\|_{V^p(E)} := \|Du\|_{L^p(\Omega_T)} \quad \text{and} \quad \|u\|_{V_2^p(E)} := \|Du\|_{L^p(\Omega_T)} + \|u\|_{L^2(\Omega_T)}. \tag{2.3}$$

The norm on the dual space $(V^p(E))'$ is defined as

$$\|L\|_{(V^p(E))'} := \sup_{0\neq u\in V^p(E)} \frac{|\langle L,u\rangle|}{\|u\|_{V^p(E)}} \qquad \text{for each } L\in (V^p(E))',$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(V^p(E))'$ and $V^p(E)$. The norm of $(V_2^p(E))'$ is defined analogously.

At this point it must be explicitly emphasized that the Dirichlet boundary condition $(1.1)_2$ for the solution u will be encoded in terms of the space $V_2^p(E)$, by simply requiring that $u\in V_2^p(E)$. Note that this ensures only that the Dirichlet boundary values are taken in horizontal direction. We consider as integrand a Carathéodory function $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to[0,\infty]$ fulfilling the following convexity, coercivity, and integrability assumptions:

$$\begin{cases} (\mathbb{R}^N,\mathbb{R}^{Nn})\ni(u,\xi)\mapsto f(x,u,\xi) \text{ is convex for a.e. } x\in\Omega,\\ \nu|\xi|^p\le f(x,u,\xi) \text{ for a.e. } x\in\Omega \text{ and all } (u,\xi)\in\mathbb{R}^N\times\mathbb{R}^{Nn}, \end{cases} \tag{2.4}$$

for some universal structural constant $\nu>0$ and some integrability exponent $1<p<\infty$. Furthermore, we assume that there exists a measurable function $g\colon\Omega\times[0,\infty)\to[0,\infty)$ such that for any fixed $M\ge 0$ the partial map $g(\cdot,M)$ is integrable, i.e., $g(\cdot,M)\in L^1(\Omega)$, and the pointwise bound

$$0\le f(x,u,\xi)\le g(x,M) \quad \text{for a.e. } x\in\Omega, \text{ provided } |u|\le M,\ |\xi|\le M, \tag{2.5}$$

holds true. Note that this assumption on the integrand f covers any reasonable growth condition of the form

$$0\le f(x,u,\xi)\le \alpha(x)\phi(|\xi|)+\beta(x)\psi(|u|)+\gamma(x),$$

with nonnegative continuous functions $\phi,\psi\colon[0,\infty)\to[0,\infty)$ and integrable functions $\alpha,\beta,\gamma\in L^1(\Omega)$. We emphasize that we do not require a p-growth condition from above for the mapping $\xi\mapsto f(x,u,\xi)$, i.e., we assume only p-coercivity as in $(2.4)_2$ and condition (2.5) ensures the integrability of $f(x,u,Du)$ for smooth functions u. For the initial datum u_o, we assume that

$$u_o\in L^2\big(E^0,\mathbb{R}^N\big).$$

Within this context we define variational solutions to the initial boundary value problem (1.1) in the noncylindrical domain E as follows.

Definition 2.1. *Suppose that $f\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to[0,\infty]$ is a Carathéodory function, that $E\subset\Omega\times[0,T)$ is a relatively open noncylindrical domain, and that $u_o\in L^2\big(E^0,\mathbb{R}^N\big)$. A measurable map $u\colon\Omega_T\to\mathbb{R}$ in the class*

$$u\in L^\infty\big(0,T;L^2(\Omega)\big)\cap V_2^p(E)$$

is called a variational solution (or pseudo solution) with initial datum u_o if and only if the variational inequality

$$\begin{aligned}\iint_{E\cap\Omega_\tau} f(x,u,Du)\,\mathrm{d}x\mathrm{d}t \le &\iint_{E\cap\Omega_\tau}\Big[\partial_t v\cdot(v-u)+f(x,v,Dv)\Big]\mathrm{d}x\mathrm{d}t\\ &-\tfrac12\big\|(v-u)(\tau)\big\|^2_{L^2(E^\tau,\mathbb{R}^N)}+\tfrac12\big\|v(0)-u_o\big\|^2_{L^2(E^0,\mathbb{R}^N)}\end{aligned} \tag{2.6}$$

holds true for a.e. $\tau\in[0,T)$ and any $v\in V_2^p(E)$ with $\partial_t v\in L^2\big(\Omega_T,\mathbb{R}^N\big)$.

Remark 2.2. The variational inequality (2.6) implies that the variational solution u attains the initial datum u_o in the L^2-sense, i.e., there holds

$$\lim_{h\downarrow 0} \tfrac{1}{h} \int_0^h \|u(t)-u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} \mathrm{d}t = 0.$$

For the proof we refer to Lemma 3.3 below. □

2.2. Existence of variational solutions in nondecreasing domains. Our first existence result is concerned with nondecreasing domains, in the sense that the sets E^t are not allowed to decrease in time. More specifically, we assume that

$$E^s \subseteq E^t \quad \text{for any } 0 \le s \le t \le T. \tag{2.7}$$

Within the context of nondecreasing domains, we are able to prove the existence of a unique variational solution for integrands as considered in (2.4). The precise statement is as follows.

THEOREM 2.3. *Suppose that $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to[0,\infty]$ is a variational integrand satisfying* (2.4) *and* (2.5) *and that the noncylindrical domain E satisfies* (2.1) *and* (2.7)*. Then, for any initial datum $u_o \in L^2(E^0,\mathbb{R}^N)$ there exists a unique variational solution u in the sense of Definition* 2.1*. Moreover, the variational solution satisfies*

$$u \in C^0\big([0,T); L^2(\Omega,\mathbb{R}^N)\big) \cap V_2^p(E).$$

Remark 2.4. Note that the variational solution constructed in Theorem 2.3 satisfies $u \in C^0\big([0,T); L^2(\Omega,\mathbb{R}^N)\big)$. This allows us to conclude that the variational inequality (2.6) holds true for any $\tau\in(0,T]$. Moreover, the initial datum u_o is attained in the stronger C^0–L^2-sense, i.e., we have that

$$\lim_{t\downarrow 0} \|u(t)-u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} = 0$$

holds true. Furthermore, for nondecreasing domains hypothesis (2.1) is, for instance, implied by the condition that

$$E^t \text{ satisfies } \mathcal{L}^n(\partial E^t)=0 \text{ for any } t\in[0,T); \tag{2.8}$$

see section 3.4. □

Remark 2.5. If u_o satisfies

$$u_o \in V_0 \quad \text{and} \quad \int_\Omega f(x,u_o,Du_o)\mathrm{d}x < \infty, \tag{2.9}$$

then we have $\partial_t u \in L^2(\Omega_T,\mathbb{R}^N)$; see Lemma 4.2.

2.3. Variational solutions in general variable domains. Next, we state our second main result which is concerned with the existence of variational solutions in general time-dependent domains, i.e., domains which are also allowed to decrease in time. Then, we still obtain an existence result for variational solutions. But, in general, uniqueness and the regularity in $C^0([0,T); L^2(\Omega,\mathbb{R}^N))$ can no longer be guaranteed. More precisely, we are able to prove the following existence result.

THEOREM 2.6. *Suppose that* $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to[0,\infty]$ *is a variational integrand satisfying* (2.4) *and* (2.5) *and that the noncylindrical domain* E *satisfies* (2.1). *Then, for any initial datum* $u_o \in L^2(E^0,\mathbb{R}^N)$, *there exists a variational solution* u *in the sense of Definition* 2.1.

If the domain decreases too fast with respect to time, it is not likely that variational solutions belong to the space $C^0([0,T);L^2(\Omega,\mathbb{R}^N))$. We would like to mention that a related phenomenon has also been observed in [31, 22, 6, 5] by using Petrovskiĭ-type constructions. In particular, these constructions yield examples of domains that shrink so fast that solutions of the parabolic p-Laplace equation can become discontinuous in one point of the boundary. This result suggests that also the weaker property $u\in C^0([0,T];L^2(\Omega,\mathbb{R}^N))$ cannot be obtained for arbitrary domains. More precisely, we need to impose the following additional assumptions on the domain E: We assume that the complement $\Omega\setminus E^t$ of E^t satisfies the *measure density condition*, i.e., there exists a constant $\delta>0$ such that for any $t\in[0,T)$ we have

$$\mathcal{L}^n\big((\mathbb{R}^n\setminus E^t)\cap B_r(x_o)\big)\ge\delta\mathcal{L}^n(B_r(x_o))\quad\text{for any } x_o\in\partial E^t \text{ and } r>0, \tag{2.10}$$

and a *one-sided growth condition* for the *complementary excess*

$$\mathbf{e}^c(E^s,E^t):=\sup_{x\in\Omega\setminus E^t}\operatorname{dist}(x,\Omega\setminus E^s), \tag{2.11}$$

of the form that there exists a constant $M>0$ such that there holds

$$\mathbf{e}^c(E^s,E^t)\le M(t-s)\quad\text{provided } 0\le s\le t<T. \tag{2.12}$$

Note that the measure density condition (2.10) implies that $V_t\cong W_0^{1,p}(E^t,\mathbb{R}^N)$ for any $t\in[0,T)$ (see Lemma 3.1), while the one-sided growth condition for the complementary excess (2.12) ensures that the domain E does not decrease too fast with respect to time. In particular, under the monotonicity assumption (2.7) we have $\mathbf{e}^c(E^s,E^t)=0$ for $0\le s\le t<T$. Additionally, we assume that the integrand f satisfies a standard coercivity and growth condition of the form

$$\nu|\xi|^p\le f(x,u,\xi)\le L\big(|\xi|^p+|u|^p+G(x)\big) \tag{2.13}$$

for a.e. $x\in\Omega$ and all $(u,\xi)\in\mathbb{R}^N\times\mathbb{R}^{Nn}$, with some universal structural constants $0<\nu\le L$ and some $G\in L^{p'}(\Omega_T)$. Note that assumption (2.13) implies (2.5).

Under these extra assumptions we shall prove that the variational solution lies in the space $C^0\big([0,T);L^2(\Omega,\mathbb{R}^N)\big)$. As a preliminary result, we deduce that the variational solution admits a time derivative in the dual space $\big(V^p(E)\big)'$. The precise statement is as follows.

THEOREM 2.7. *Suppose that* $f\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to[0,\infty]$ *is a variational integrand satisfying* $(2.4)_1$ *and* (2.13) *and that the noncylindrical domain* E *satisfies* (2.1), (2.10), (2.12), *and that* $u_o\in L^2\big(E^0,\mathbb{R}^N\big)$. *Then, the variational solution associated to the initial datum* u_o *obtained in Theorem* 2.6 *admits a distributional time derivative*

$$\partial_t u\in\big(V^p(E)\big)',$$

and the quantitative estimate

$$\|\partial_t u\|_{(V^p(E))'}\le c\left[T\int_\Omega\big(|G|+|G|^{p'}\big)\,\mathrm{d}x+\|u_o\|^2_{L^2(E^0,\mathbb{R}^N)}\right]^{\frac{1}{p'}}$$

holds true with a constant $c=c(p,\nu,L,\operatorname{diam}(\Omega))$.

Under certain weak assumptions on the domain E, the information $\partial_t u \in \big(V^p(E)\big)'$ is enough to prove continuity in time, as stated in the next theorem.

THEOREM 2.8. *Suppose that* $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn} \to [0,\infty]$ *is a variational integrand satisfying* $(2.4)_1$ *and* (2.13) *for some exponent* $p > \frac{2(n+1)}{n+2}$, *and that the noncylindrical domain* E *satisfies* (2.10) *and* (2.12). *Let* u *be a variational solution of* (2.6), *in the sense of Definition* 2.1. *If the solution satisfies* $\partial_t u \in \big(V^p(E)\big)'$, *then we have*

$$u \in C^0\big([0,T); L^2(\Omega,\mathbb{R}^N)\big),$$

and u *attains the prescribed boundary values, i.e.,* $u(0) = u_{\mathrm{o}}$.

Finally, under the assumptions of the preceding Theorem, we are able to show that, in the case of a differentiable integrand, every variational solution is a weak solution of the gradient flow to f; cf. Corollary 5.15.

For a uniqueness result on possibly decreasing domains, we strengthen the one-sided growth condition for the complementary excess (2.12) to a two-sided condition, which guarantees that the domains neither decrease nor increase too fast in time. More precisely, we assume

$$\mathbf{d}^c_H(E^s,E^t) \le M(t-s) \quad \text{for all } 0 \le s \le t < T, \tag{2.14}$$

with the *complementary Hausdorff distance*

$$\mathbf{d}^c_H(E^s,E^t) := \max\big\{\mathbf{e}^c(E^s,E^t), \mathbf{e}^c(E^t,E^s)\big\}.$$

Then we have the following.

THEOREM 2.9. *Consider a variational integrand* $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn} \to [0,\infty]$ *with the properties* $(2.4)_1$ *and* (2.13), *for some integrability exponent* $p > \frac{2(n+1)}{n+2}$. *For the noncylindrical domain* E *we assume* (2.10) *and* (2.14). *Then, there exists at most one variational solution in the sense of Definition* 2.1 *that satisfies* $\partial_t u \in \big(V^p(E)\big)'$.

2.4. Technical aspects and novelties. In the case of nondecreasing domains, we obtain the existence of variational solutions with a time derivative in $L^2(\Omega_T)$ with minimal assumptions on the domain and the integrand. The construction uses a modification of the method of minimizing movements. Contrary to the strategy proposed by De Giorgi, we omit the penalization term in the elliptic functional. Instead, we cover the domain by a union of cylinders of height $0 < h \ll 1$ and vary the class of admissible maps from step to step, which enables us to construct minimizers that admit the prescribed boundary values on the boundary of the approximating domain. In the limit $h \downarrow 0$, the approximating solutions converge to a solution that admits the desired boundary values on the given noncylindrical domain. Due to the notion of variational solutions we are able to pass to the limit without strong convergence of the solution or the spatial gradient. In order to establish the right boundary values of the limit function, the weak regularity assumption (2.1) on the boundary suffices. Since the domain is nondecreasing, the usual time mollification (see (4.1)) preserves the boundary values since this mollification is defined as a weighted mean of values in the past. Therefore, we are able to show continuity and weak differentiability in time by methods similar to the cylindrical case.

In the case of possibly decreasing domains, it turns out that a direct application of the minimizing movements method does not yield an existence result in the generality of Theorem 2.6. Instead, we employ a different strategy. Again, we approximate the noncylindrical domain from the outside by a union of cylinders of height $0 < h \ll 1$. On each of the cylinders, we do not solve an elliptic minimization problem as before, but somewhat similar to [11], we use the variational solution of a parabolic system as an approximation. In each step, we use the final state from the preceding step as new initial values. At this stage it is crucial that we are able to allow initial values in L^2, as done in Theorem 2.3. Again, the notion of variational solutions enables us to pass to the limit in the variational inequality only using weak convergence, and we obtain the prescribed boundary values on general domains with (2.1).

The next step is to establish that the solutions are contained in the space $C^0([0,T); L^2(\Omega,\mathbb{R}^N))$, which we regard as a regularity issue in the case of possibly decreasing domains. Here, of course we need additional assumptions for the domain that prevent the domain from decreasing too fast. In order to quantify the shrinking of the time slices E^t, Savaré [37] employed the one-sided excess

$$\mathbf{e}(E^s,E^t) := \sup_{x\in E^s} \operatorname{dist}(x,E^t) \qquad \text{for } 0 \le s \le t \le T,$$

and relied on the assumption that the domains E^t are uniformly $C^{1,1}$. The present work introduces a new way to measure the decreasing of the domains in time. In fact, our results show that it is more natural to bound the *complementary excess*

$$\mathbf{e}^c(E^s,E^t) := \sup_{x\in \Omega\setminus E^t} \operatorname{dist}(x,\Omega\setminus E^s) \qquad \text{for } 0 \le s \le t \le T,$$

instead of the excess $\mathbf{e}(E^s,E^t)$. In particular, the Lipschitz condition (2.12) on this excess rules out the possibility of a hole developing in the interior of the domain. This phenomenon is excluded in [37] by assuming that the time slices E^t are uniformly $C^{1,1}$. By using the complementary excess, we are able to deal with much more general domains that merely satisfy a measure density condition; see (2.10).

The first step in the proof of regularity is to show that the distributional time derivatives of the constructed solutions are contained in the dual space $(V^p(E))'$. For this we have to assume a standard p-growth condition for the functional. Nevertheless, we do not get $\partial_t u \in (V^p(E))'$ directly from the variational inequality (2.6) because we are not allowed to test with u or with time mollifications of u. Hence, we derive this property first for the approximating solutions and then show that it is preserved in the limit. Contrary to the cylindrical case, the property $\partial_t u \in (V^p(E))'$ does not immediately yield continuity in time by an embedding theorem. Since we do not know whether such an embedding exists for irregular domains as in Theorem 2.8, we are forced to employ a much more involved strategy. The key to the proof of continuity in time is an integration by parts formula of the form

$$\langle \partial_t u, \zeta u\rangle \le -\tfrac12 \iint_E |u|^2 \partial_t \zeta \,\mathrm{d}x\mathrm{d}t \tag{2.15}$$

for a nonnegative cutoff function in time $\zeta \in C_0^{0,1}(0,T)$. The fact that we do not obtain an identity in the preceding formula is caused by the one-sided excess assumption (2.12). In fact, under the two-sided assumption (2.14) we get an identity. The proof of the integration by parts inequality requires an intricate cutoff and mollification procedure and the application of Hardy's inequality. One of the terms that has to be controlled during the proof is of the type

$$\frac{1}{\sigma}\int_0^T \int_{E^t\setminus E^{t,\sigma}} |u|^2 \,\mathrm{d}x\mathrm{d}t$$

for $0<\sigma\ll 1$, where $E^{t,\sigma}:=\{x\in E^t\ :\mathrm{dist}(x,\partial E^t)>\sigma\}$ denotes the inner parallel set of E^t. This is the point where we need to require the lower-bound $p>\frac{2(n+1)}{n+2}$ on the growth exponent, since only in this case does a sufficiently strong Hardy–Sobolev inequality hold true that ensures that the above integrals vanish in the limit $\sigma\downarrow 0$.

The remaining part of the argument is divided into the proof of forward continuity and backward continuity in time. For the first part, we observe that in the interior of the domain we have continuity in time since we can apply the standard embedding on small cylinders compactly contained in E. It remains to show that there is no loss of L^2-norm at the boundary in the limit $\tau\uparrow\tau_{\rm o}$, which could happen if the domain shrank too fast. However, this loss of L^2-norm can be excluded by the inequality

$$\langle\partial_t u,\chi_{\Omega\times(\tau,\tau_{\rm o})}u\rangle\le\tfrac12\|u(\tau_{\rm o})\|^2_{L^2(\Omega)}-\tfrac12\|u(\tau)\|^2_{L^2(\Omega)}\tag{2.16}$$

for a.e. $\tau,\tau_{\rm o}\in(0,T)$ with $\tau<\tau_{\rm o}$, which is a consequence of (2.15).

The proof of the backward continuity exploits the fact that initial values are admitted in the L^2-sense. Hence, the crucial step in the proof is to show that the variational inequality can be localized to subdomains $E\cap\Omega\times(\tau_{\rm o},\tau)$ for every $\tau_{\rm o},\tau\in(0,T)$ with $\tau_{\rm o}<\tau$. For the proof of this localization property, we again rely on the integration by parts formula (2.15). Moreover, the backward continuity in time is needed at this stage since, otherwise, the localization would only be possible for a.e. $\tau_0<\tau$.

For the proof of uniqueness, we again rely on the integration by parts formula (2.15). At this stage, we use it to recast the variational inequality (2.6) into the form

$$\iint_{E\cap\Omega_\tau}f(x,u,Du)\mathrm{d}x\mathrm{d}t\le\langle\partial_t u,\chi_{\Omega_\tau}(v-u)\rangle+\iint_{E\cap\Omega_\tau}f(x,v,Dv)\mathrm{d}x\mathrm{d}t\tag{2.17}$$

for every $v\in V_2^p(E)$ and every $\tau\in(0,T)$. Since in this form the comparison map is not required to possess a weak time derivative in $L^2(\Omega_T,\mathbb{R}^N)$, it is possible to use another solution as a comparison map in this inequality. Hence, given two solutions u_i of the Cauchy–Dirichlet problem with $\partial_t u_i\in(V^p(E))'$ for $i=1,2$, a standard comparison procedure yields the estimate

$$\langle\partial_t(u_1-u_2),\chi_{\Omega_\tau}(u_1-u_2)\rangle\le 0$$

for every $\tau\in(0,T)$. Since u_1-u_2 vanishes at the initial time $\tau=0$, this yields the desired uniqueness if we have equality in the integration by parts formula (2.16) for u_1-u_2 in place of u. However, this identity holds under the two-sided condition (2.14) on the noncylindrical domain. This is the only point in the proof in which we also need to exclude domains that grow too fast in time.

Finally, we note that, in the case of differentiable integrands, the formulation (2.17) of the variational inequality implies that the constructed solutions are weak solutions, since we may use comparison maps of the form $v=u+s\varphi$ with $\varphi\in C_0^\infty(E,\mathbb{R}^N)$ and $s>0$ in (2.17). Passing to the limit $s\downarrow 0$ then yields the weak formulation of the gradient flow associated to the integrand f.

3. Preliminaries.

3.1. Notation. Throughout the paper we write $u(t)$ for the map $u(\cdot,t)$, i.e., for the restriction of $u\in L^1(0,T;L^1(\Omega,\mathbb{R}^N))$ to the time slice $\Omega\times\{t\}$ with $t\in[0,T)$. Moreover, for $t\in[0,T)$ and $\sigma>0$ we denote by

$$E^{t,\sigma}:=\big\{x\in E^t\ :\mathrm{dist}(x,\partial E^t)>\sigma\big\}\tag{3.1}$$

the inner parallel set of E^t.

3.2. Characterization of V_t. The question arises whether V_t can be identified with the Sobolev space $W_0^{1,p}(E^t,\mathbb{R}^N)$. In general, this will not be possible. However, if the complement $\mathbb{R}^n \setminus E^t$ of E^t satisfies the *measure density condition* (2.10), then this identification holds true. In fact, this statement is a special case of the much more refined result [18, Thm. 2.5*]. Nevertheless, for the convenience of the reader we include an easy proof that covers the present situation.

LEMMA 3.1. *Assume that* (2.1) *and the measure density condition* (2.10) *are in force. Then for any $t \in [0,T)$ the identification*

$$V_t \cong W_0^{1,p}(E^t,\mathbb{R}^N)$$

holds true.

Proof. Let $v \in V_t$ and consider a Lebesgue point x_o of v in ∂E^t, i.e., we have

$$v(x_o) = \lim_{r\downarrow 0} \fint_{B_r(x_o)} v\,\mathrm{d}x \quad \text{and} \quad \lim_{r\downarrow 0} \fint_{B_r(x_o)} |v(x) - v(x_o)|\,\mathrm{d}x = 0.$$

On the other hand, due to the measure density condition (2.10) and the fact that $v \equiv 0$ outside of E^t, we know for $0 < r < \operatorname{dist}(E^t, \partial\Omega)$ that

$$\begin{aligned}
\fint_{B_r(x_o)} |v(x) - v(x_o)|\,\mathrm{d}x &\geq \frac{1}{\mathcal{L}^n(B_r(x_o))} \int_{(\mathbb{R}^n\setminus E^t)\cap B_r(x_o)} |v(x) - v(x_o)|\,\mathrm{d}x \\
&= \frac{1}{\mathcal{L}^n(B_r(x_o))} \int_{(\mathbb{R}^n\setminus E^t)\cap B_r(x_o)} |v(x_o)|\,\mathrm{d}x \\
&= \frac{\mathcal{L}^n\big((\mathbb{R}^n \setminus E^t) \cap B_r(x_o)\big)}{\mathcal{L}^n(B_r(x_o))} |v(x_o)| \\
&\geq \delta |v(x_o)|
\end{aligned}$$

for the constant $\delta > 0$ from (2.10). This implies that $v(x_o) = 0$. Since p-quasievery point in $x_o \in \Omega$ is a Lebesgue point of v by [17, Chapter 9], we thus have shown that

$$0 = v(x_o) = \lim_{r\downarrow 0} \fint_{B_r(x_o)} v(x)\,\mathrm{d}x$$

for p-quasievery point $x_o \in \partial E^t$. By [2, Thm. 9.1.3] (see also [23, Thm. 1.1]) this ensures that $v \in W_0^{1,p}(E^t,\mathbb{R}^N)$. ∎

3.3. Hardy's inequality. The following inequalities will be crucial for controlling integrals over neighborhoods of the boundary that result from cutoff procedures.

LEMMA 3.2. *Assume that the measure density condition* (2.10) *is in force with a constant $\delta > 0$, and let $u \in V_2^p(E)$. Then there is a constant $c = c(n,p,\delta)$ such that Hardy's inequality*

$$\int_{E^t} \left(\frac{|u(x,t)|}{\operatorname{dist}(x,\partial E^t)}\right)^p \mathrm{d}x \leq c \int_{E^t} |Du(x,t)|^p\,\mathrm{d}x \tag{3.2}$$

holds true for a.e. $t \in [0,T)$. Moreover, in the case $p \in [\frac{2n}{n+2}, 2)$, the following Hardy–Sobolev inequality holds true:

$$\int_{E^t} \left(\frac{|u(x,t)|}{[\operatorname{dist}(x,\partial E^t)]^{1-\frac{n}{p}+\frac{n}{2}}}\right)^2 \mathrm{d}x \leq c \left(\int_{E^t} |Du(x,t)|^p\,\mathrm{d}x\right)^{\frac{2}{p}} \tag{3.3}$$

for a.e. $t \in [0,T)$, where, again, $c = c(n,p,\delta)$.

Proof. The Hardy inequality (3.2) follows from [29, Thm. 2]; cf. also [25, Cor. 3.11]. In fact, these results are applicable in the present situation since the assumption (2.10) implies that the complements $\mathbb{R}^n \setminus E^t$ are uniformly p-fat with a constant only depending on n, p, and δ. This is a straightforward consequence of an elementary relation between capacity and Lebesgue measure; see [15, Thm. 2 (vi), p. 151]. Moreover, we know from Lemma 3.1 that (2.10) and $u \in V_2^p(E)$ ensure that $u(t) \in W_0^{1,p}(E^t, \mathbb{R}^N)$ holds for a.e. $t \in [0, T)$.

In the case $p \in [\frac{2n}{n+2}, 2)$ we have $p < 2 \le p^* := \frac{np}{n-p}$. Therefore, the interpolation argument from [28, Thm. 2.1] with $q = 2$ and $\beta = 0$ implies that on any domain on which (3.2) holds true, we also have a Hardy–Sobolev inequality of the form (3.3). $\square$

3.4. A remark on nondecreasing domains. Here, we show that in the case of a nondecreasing domain E assumption (2.1) is implied by condition (2.8) for the slices. For $\ell \in \mathbb{N}$ and $i \in \{0, \dots, \ell\}$ we let $h_\ell := \frac{T}{\ell}$, $t_{\ell,i} := ih_\ell$ and

$$E_{(\ell)} := \Omega_T \cap \bigcup_{i=1}^{\ell} E^{t_{\ell,i-1}} \times I_{\ell,i} \quad \text{and} \quad E^{(\ell)} := \Omega_T \cap \bigcup_{i=1}^{\ell} E^{t_{\ell,i}} \times I_{\ell,i},$$

where $I_{\ell,i} = [t_{\ell,i-1}, t_{\ell,i})$. By construction we have $E_{(\ell)} \subset E \subset E^{(\ell)}$ for any $\ell \in \mathbb{N}$. Therefore, $\partial E \subset \overline{E^{(\ell)}} \setminus \operatorname{int}(E_{(\ell)})$ for any $\ell \in \mathbb{N}$. Since $\mathcal{L}^n(\partial E^t) = 0$ for any $t \in [0, T)$ by assumption (2.8), we have for any $\ell \in \mathbb{N}$ that

$$\begin{aligned}
\mathcal{L}^{n+1}(\partial E) \le \mathcal{L}^{n+1}\Big(\overline{E^{(\ell)}} \setminus \operatorname{int}(E_{(\ell)})\Big) &= \sum_{i=1}^{\ell} \mathcal{L}^n\Big(\overline{E^{t_{\ell,i}}} \setminus E^{t_{\ell,i-1}}\Big)\big(t_{\ell,i} - t_{\ell,i-1}\big) \\
&= \tfrac{T}{\ell} \sum_{i=1}^{\ell} \mathcal{L}^n\Big(\overline{E^{t_{\ell,i}}} \setminus E^{t_{\ell,i-1}}\Big) \\
&\le \tfrac{T}{\ell} \sum_{i=1}^{\ell} \mathcal{L}^n\Big(\partial E^{t_{\ell,i}}\Big) + \tfrac{T}{\ell} \sum_{i=1}^{\ell} \mathcal{L}^n\Big(E^{t_{\ell,i}} \setminus E^{t_{\ell,i-1}}\Big) \\
&= \tfrac{T}{\ell} \mathcal{L}^n\big(E^T \setminus E^0\big).
\end{aligned}$$

Letting $\ell \to \infty$ ensures the claim that $\mathcal{L}^{n+1}(\partial E) = 0$. $\square$

3.5. Properties of variational solutions. In this section we state some properties of variational solutions in the sense of Definition 2.1. Here, both cases, i.e., the case of nondecreasing domains as well as the case of possibly decreasing domains, are included.

3.5.1. Initial values. Here we shall establish that variational solutions in the sense of Definition 2.1 fulfill the initial condition $u(0) = u_o$ on Ω in the L^2-sense. We emphasize that the statement holds for general, possibly decreasing domains. The precise statement is as follows.

LEMMA 3.3. *Suppose that $f \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to [0, \infty]$ is a variational integrand satisfying* (2.4) *and* (2.5)*, that E is a bounded relatively open noncylindrical subdomain of $\mathbb{R}^n \times [0, T)$ and that $u_o \in L^2\big(E^0, \mathbb{R}^N\big)$. Then, any variational solution $u \in V_2^p(E)$ in the sense of Definition* 2.1 *satisfies the initial condition $u(0) = u_o$ in the L^2-sense, i.e.,*

$$\lim_{h \downarrow 0} \tfrac{1}{h} \int_0^h \|u(t) - u_o\|^2_{L^2(\Omega, \mathbb{R}^N)} \mathrm{d}t = 0.$$

Proof. For $\varepsilon > 0$ we consider the inner parallel set $E^{0,\varepsilon}$ of E^0 as defined in (3.1). Since E is relatively open and $\overline{E^{0,\varepsilon}}$ is compact, we find $t_\varepsilon > 0$ such that

$$E^{0,\varepsilon} \times [0, t_\varepsilon) \subset E. \tag{3.4}$$

Furthermore, we let $\phi \in C_0^\infty(B_1, \mathbb{R}_{\geq 0})$ be a standard mollifier and set $\phi_\varepsilon(x) := \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$, so that $\phi_\varepsilon \in C_0^\infty(B_\varepsilon, \mathbb{R}_{\geq 0})$. With these preparations we define the following mollification of the initial values:

$$u_o^{(\varepsilon)} := \big(u_o \chi_{E^{0,2\varepsilon}}\big) * \phi_\varepsilon.$$

Then, $u_o^{(\varepsilon)} \in C_0^\infty(E^{0,\varepsilon}, \mathbb{R}^N)$ and $u_o^{(\varepsilon)} \to u_o \in L^2(\Omega, \mathbb{R}^N)$ as $\varepsilon \downarrow 0$. Moreover, due to (3.4), we have that $\operatorname{spt} u_o^{(\varepsilon)} \subset E^t$ for any $t \in [0, t_\varepsilon)$. Furthermore, since $u_o^{(\varepsilon)}$ and $Du_o^{(\varepsilon)}$ are bounded, hypothesis (2.5) ensures that

$$0 \leq \int_\Omega f\big(x, u_o^{(\varepsilon)}, Du_o^{(\varepsilon)}\big)\,\mathrm{d}x < \infty, \tag{3.5}$$

for any $\varepsilon > 0$. Choosing the time-independent extension of $u_o^{(\varepsilon)}$ to $E \cap \Omega_\tau$ as comparison function in the variational inequality (2.6) on $E \cap \Omega_\tau$, we obtain

$$\tfrac{1}{2}\big\|u_o^{(\varepsilon)} - u(\tau)\big\|^2_{L^2(E^\tau,\mathbb{R}^N)} \leq \iint_{E\cap\Omega_\tau} f\big(x, u_o^{(\varepsilon)}, Du_o^{(\varepsilon)}\big)\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{2}\big\|u_o^{(\varepsilon)} - u_o\big\|^2_{L^2(E^0,\mathbb{R}^N)}$$

for a.e. $\tau \in (0, t_\varepsilon)$. In turn, this implies

$$\|u(\tau) - u_o\|^2_{L^2(E^\tau,\mathbb{R}^N)} \leq 4\tau \int_\Omega f\big(x, u_o^{(\varepsilon)}, Du_o^{(\varepsilon)}\big)\,\mathrm{d}x + 4\big\|u_o^{(\varepsilon)} - u_o\big\|^2_{L^2(E^0,\mathbb{R}^N)}.$$

Now, for $h \in (0, t_\varepsilon)$, we integrate the preceding inequality with respect to τ over the interval $[0, h]$ and divide both sides of the resulting inequality by h. In this way, we obtain

$$\tfrac{1}{h}\int_0^h \big\|u(\tau) - u_o\big\|^2_{L^2(E^\tau,\mathbb{R}^N)}\mathrm{d}\tau \leq 2h \int_\Omega f\big(x, u_o^{(\varepsilon)}, Du_o^{(\varepsilon)}\big)\,\mathrm{d}x + 4\big\|u_o^{(\varepsilon)} - u_o\big\|^2_{L^2(E^0,\mathbb{R}^N)}.$$

Passing to the limit $h \downarrow 0$ on both sides yields

$$\limsup_{h\downarrow 0} \tfrac{1}{h}\int_0^h \big\|u(\tau) - u_o\big\|^2_{L^2(E^\tau,\mathbb{R}^N)}\mathrm{d}\tau \leq 4\big\|u_o^{(\varepsilon)} - u_o\big\|^2_{L^2(E^0,\mathbb{R}^N)}$$

for any $0 < \varepsilon \ll 1$. Since the left-hand side does not depend on ε, we finally pass to the limit $\varepsilon \downarrow 0$ and obtain that

$$\lim_{h\downarrow 0} \tfrac{1}{h}\int_0^h \big\|u(\tau) - u_o\big\|^2_{L^2(E^\tau,\mathbb{R}^N)}\mathrm{d}\tau = 0$$

holds true. This proves the claim that $u(0) = u_o$ in the usual L^2-sense. ∎

3.5.2. Energy estimate. In this section we derive an energy estimate for variational solutions by testing the variational inequality with the admissible testing function $v \equiv 0$.

LEMMA 3.4. *Suppose that* $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to[0,\infty]$ *is a variational integrand satisfying* (2.4) *and* (2.5) *and that* E *is a bounded relatively open noncylindrical subdomain of* $\mathbb{R}^n\times[0,T)$ *and that* $u_o\in L^2\big(E^0,\mathbb{R}^N\big)$. *Then, any variational solution* $u\in V_2^p(E)$ *in the sense of Definition* 2.1 *satisfies the energy estimates*

$$\sup_{t\in(0,T)}\|u(t)\|^2_{L^2(E^t,\mathbb{R}^N)}\le 2T\int_\Omega g(x,0)\mathrm{d}x+\|u_o\|^2_{L^2(E^0,\mathbb{R}^N)}$$

and

$$\nu\iint_E|Du|^p\mathrm{d}x\mathrm{d}t\le\iint_E f(x,u,Du)\mathrm{d}x\mathrm{d}t\le T\int_\Omega g(x,0)\mathrm{d}x+\tfrac12\|u_o\|^2_{L^2(E^0,\mathbb{R}^N)}.$$

Proof. Choosing $v\equiv 0$ in the variational inequality (2.6) on $E\cap\Omega_\tau$ and using assumption (2.5), we obtain for a.e. $\tau\in[0,T)$ that

$$\begin{aligned}\tfrac12\|u(\tau)\|^2_{L^2(E^\tau,\mathbb{R}^N)}&+\iint_{E\cap\Omega_\tau}f(x,u,Du)\mathrm{d}x\mathrm{d}t\\&\le\iint_{E\cap\Omega_\tau}f(x,0,0)\mathrm{d}x\mathrm{d}t+\tfrac12\|u_o\|^2_{L^2(E^0,\mathbb{R}^N)}\\&\le\tau\int_\Omega g(x,0)\mathrm{d}x+\tfrac12\|u_o\|^2_{L^2(E^0,\mathbb{R}^N)}.\end{aligned}$$

Taking the supremum over $\tau\in[0,T)$ and discarding the nonnegative second integral on the left-hand side show that

$$\sup_{\tau\in(0,T)}\|u(\tau)\|^2_{L^2(E^\tau,\mathbb{R}^N)}\le 2T\int_\Omega g(x,0)\mathrm{d}x+\|u_o\|^2_{L^2(E^0,\mathbb{R}^N)},$$

while letting $\tau\uparrow T$ yields

$$\nu\iint_E|Du|^p\mathrm{d}x\mathrm{d}t\le\iint_E f(x,u,Du)\mathrm{d}x\mathrm{d}t\le T\int_\Omega g(x,0)\mathrm{d}x+\tfrac12\big\|u_o\big\|^2_{L^2(E^0,\mathbb{R}^N)}.$$

In the last line we used the coercivity assumption $(2.4)_2$ for the first inequality. This proves the claim. ∎

4. Variational solutions in nondecreasing domains. In nondecreasing domains we can use a standard time mollification that takes into account the values at previous times. To be more precise, for $T>0$, $h\in(0,T]$, $v_o\in L^1(\Omega,\mathbb{R}^N)$, and $v\in L^1(\Omega_T,\mathbb{R}^N)=L^1(0,T;L^1(\Omega,\mathbb{R}^N))$, we define

$$[\![v]\!]_h(t):=\mathrm{e}^{-\frac{t}{h}}v_o+\tfrac1h\int_0^t\mathrm{e}^{\frac{s-t}{h}}v(s)\mathrm{d}s\tag{4.1}$$

for any $t\in[0,T)$. The mollification $[\![v]\!]_h$ is constructed in such a way that it satisfies

$$\partial_t[\![v]\!]_h=-\tfrac1h\big([\![v]\!]_h-v\big).\tag{4.2}$$

For more information on the mollification we refer to [24, Lem. 2.2] and [7, Lems. 2.2 and 2.3]. Observe that in the case of a nondecreasing domain $E\subset\Omega_T$, i.e., when E satisfies (2.7), we have that $[\![v]\!]_h\in V_2^p(E)$ for any $v\in V_2^p(E)$.

4.1. Existence of variational solutions (Proof of Theorem 2.3). For the proof of Theorem 2.3 we shall proceed in several steps. The main tool will be the method of minimizing movements.

4.1.1. Construction of the time discretization. For $h > 0$, $v \in L^2(\Omega, \mathbb{R}^N)$, and $w \in L^2(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)$, we define the variational functional $\mathcal{F}_h$ by

$$\mathcal{F}_h\,[w, v] := \int_\Omega f(x, w, Dw)\,\mathrm{d}x + \tfrac{1}{2h} \int_\Omega |w - v|^2\,\mathrm{d}x.$$

Given $t > 0$ and $v \in L^2(\Omega, \mathbb{R}^N)$, there exists a unique minimizer $w \in V_t \cap L^2\big(\Omega, \mathbb{R}^N\big)$ of $\mathcal{F}_h$ satisfying

$$\int_\Omega f(x, w, Dw)\,\mathrm{d}x < \infty.$$

This can be deduced by the direct method of the calculus of variations. In fact, by [20, Thm. 4.5] (note that the required assumptions are in force by (2.4)) the functional

$$w \mapsto \int_\Omega f(x, w, Dw)\,\mathrm{d}x$$

is lower semicontinuous with respect to weak convergence in $W^{1,1}\big(\Omega, \mathbb{R}^N\big)$ and, furthermore, $w \mapsto \|w - v\|^2_{L^2(\Omega,\mathbb{R}^N)}$ is lower semicontinuous with respect to weak convergence in $L^2\big(\Omega, \mathbb{R}^N\big)$. Finally, by (2.5) we have

$$0 \le \mathcal{F}_h[0, v] = \int_\Omega f(x, 0, 0)\,\mathrm{d}x + \tfrac{1}{2h} \int_\Omega |v|^2\,\mathrm{d}x \le \int_\Omega g(x, 0)\,\mathrm{d}x + \tfrac{1}{2h} \int_\Omega |v|^2\,\mathrm{d}x < \infty.$$

Therefore, by the direct method $\mathcal{F}_h[\,\cdot\,, v]$ attains its (unique) minimum in the class of mappings $V_t \cap L^2\big(\Omega, \mathbb{R}^N\big)$.

Now, we fix $\ell \in \mathbb{N}$ and abbreviate $h_\ell := \frac{T}{\ell}$. The above reasoning allows us to construct inductively mappings $u_{\ell,0}, \ldots, u_{\ell,\ell}$, satisfying

$$\begin{cases} u_{\ell,0} := u_o \in L^2\big(\Omega, \mathbb{R}^N\big), \\ u_{\ell,i} \in V_{ih_\ell} \cap L^2\big(\Omega, \mathbb{R}^N\big), \\ \mathcal{F}_{h_\ell}\big[u_{\ell,i+1}, u_{\ell,i}\big] = \min\limits_{w \in V_{(i+1)h_\ell} \cap L^2(\Omega,\mathbb{R}^N)} \mathcal{F}_{h_\ell}\big[w, u_{\ell,i}\big] \end{cases} \tag{4.3}$$

for any $i \in \{0, \ldots, \ell\}$.

4.1.2. Reformulating the minimizing property. For $i \in \{0, \ldots, \ell - 1\}$, $s \in (0, 1]$, and $w \in V_{(i+1)h_\ell} \cap L^2(\Omega, \mathbb{R}^N)$ the convex combination of $u_{\ell,i+1}$ and w is an admissible comparison map in $(4.3)_3$, and therefore the minimality of $u_{\ell,i+1}$ implies

$$\mathcal{F}_{h_\ell}\big[u_{\ell,i+1}, u_{\ell,i}\big] \le \mathcal{F}_{h_\ell}\big[(1 - s)u_{\ell,i+1} + sw, u_{\ell,i}\big].$$

Using the convexity of f, we obtain

$$\begin{aligned} &\int_\Omega f\big(x, u_{\ell,i+1}, Du_{\ell,i+1}\big)\,\mathrm{d}x \\ &\qquad \le \int_\Omega \Big[(1 - s) f\big(x, u_{\ell,i+1}, Du_{\ell,i+1}\big) + s f(x, w, Dw)\Big]\,\mathrm{d}x \\ &\qquad\quad + \tfrac{1}{2h_\ell} \int_\Omega \Big[\big|(1 - s)u_{\ell,i+1} + sw - u_{\ell,i}\big|^2 - \big|u_{\ell,i+1} - u_{\ell,i}\big|^2\Big]\,\mathrm{d}x. \end{aligned}$$

Since $\int_\Omega f(x,u_{\ell,i+1},Du_{\ell,i+1})\,\mathrm{d}x<\infty$ we can reabsorb the first term of the right-hand side into the left. If we proceed in this way, using the identity

$$\begin{aligned}\big|(1-s)u_{\ell,i+1}+sw-u_{\ell,i}\big|^2-\big|u_{\ell,i+1}-u_{\ell,i}\big|^2\\ =s^2\big|u_{\ell,i+1}-w\big|^2+2s\big(u_{\ell,i+1}-u_{\ell,i}\big)\cdot\big(w-u_{\ell,i+1}\big),\end{aligned}$$

the following inequality is obtained by division by $s>0$:

$$\begin{aligned}&\int_\Omega f\big(x,u_{\ell,i+1},Du_{\ell,i+1}\big)\,\mathrm{d}x\\ &\quad\le\int_\Omega f(x,w,Dw)\,\mathrm{d}x+\tfrac{1}{h_\ell}\int_\Omega\Big[\tfrac{s}{2}\big|u_{\ell,i+1}-w\big|^2+\big(u_{\ell,i+1}-u_{\ell,i}\big)\cdot\big(w-u_{\ell,i+1}\big)\Big]\mathrm{d}x.\end{aligned}$$

Here we pass to the limit $s\downarrow 0$ and arrive at

$$\begin{aligned}&\int_\Omega f\big(x,u_{\ell,i+1},Du_{\ell,i+1}\big)\,\mathrm{d}x\\ &\quad\le\int_\Omega f(x,w,Dw)\,\mathrm{d}x+\tfrac{1}{h_\ell}\int_\Omega\big(u_{\ell,i+1}-u_{\ell,i}\big)\cdot\big(w-u_{\ell,i+1}\big)\,\mathrm{d}x\end{aligned}\tag{4.4}$$

for any $w\in V_{(i+1)h_\ell}\cap L^2(\Omega,\mathbb{R}^N)$.

4.1.3. Energy estimates. We set $t_{\ell,i}:=ih_\ell$ for $i\in\{-1,\dots,\ell\}$ and define a function $u_\ell\colon\Omega\times(-h_\ell,T]\to\mathbb{R}^N$ as a piecewise constant function with respect to the time variable by

$$u_\ell(t):=u_{\ell,i}\quad\text{ for } t\in J_{\ell,i}:=(t_{\ell,i-1},t_{\ell,i}]\text{ with } i\in\{0,\dots,\ell\}.$$

In the following, we shall derive certain energy bounds for the functions u_ℓ. To achieve this, we choose in (4.4) the comparison map $w\equiv 0$, multiply the result by h_ℓ, and finally use the estimate

$$u_{\ell,i+1}\cdot u_{\ell,i}\le\tfrac12|u_{\ell,i+1}|^2+\tfrac12|u_{\ell,i}|^2$$

to obtain

$$\begin{aligned}&\iint_{\Omega\times J_{\ell,i+1}}f\big(x,u_{\ell,i+1},Du_{\ell,i+1}\big)\,\mathrm{d}x\mathrm{d}t+\tfrac12\int_\Omega|u_{\ell,i+1}|^2\mathrm{d}x\\ &\quad\le h_\ell\int_\Omega f(x,0,0)\,\mathrm{d}x+\tfrac12\int_\Omega|u_{\ell,i}|^2\mathrm{d}x\le h_\ell\int_\Omega g(x,0)\,\mathrm{d}x+\tfrac12\int_\Omega|u_{\ell,i}|^2\mathrm{d}x,\end{aligned}$$

where we have also used (2.5). We sum up the last inequalities from $i=0,...,m-1$ with $m\le\ell$ to obtain

$$\begin{aligned}&\iint_{\Omega\times(0,mh_\ell]}f\big(x,u_\ell,Du_\ell\big)\,\mathrm{d}x\mathrm{d}t+\tfrac12\int_\Omega\big|u_\ell(mh_\ell)\big|^2\mathrm{d}x\\ &\quad\le mh_\ell\int_\Omega g(x,0)\,\mathrm{d}x+\tfrac12\int_\Omega|u_{\mathrm{o}}|^2\mathrm{d}x.\end{aligned}$$

On the one hand, this shows that

$$\sup_{t\in[0,T)}\int_\Omega|u_\ell(t)|^2\mathrm{d}x\le 2T\int_\Omega g(x,0)\,\mathrm{d}x+\int_\Omega|u_{\mathrm{o}}|^2\mathrm{d}x,\tag{4.5}$$

while on the other hand we obtain for the choice $m = \ell$ that

$$\iint_{\Omega_T} |Du_\ell|^p \mathrm{d}x\mathrm{d}t \le \tfrac{1}{\nu}\bigg[T\int_\Omega g(x,0)\,\mathrm{d}x + \tfrac12\int_\Omega |u_{\mathrm{o}}|^2\,\mathrm{d}x\bigg]. \tag{4.6}$$

In the last line we also used the coercivity assumption $(2.4)_2$. Together, the estimates (4.5) and (4.6) ensure that the sequence $(u_\ell)_{\ell\in\mathbb{N}}$ is uniformly bounded in the spaces $L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big)$ and $L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big)$. Therefore, there exists a limit map

$$u \in L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big) \cap L^p\big(0,T;W^{1,p}_0(\Omega,\mathbb{R}^N)\big)$$

and a subsequence $\mathfrak{K}\subset\mathbb{N}$ such that in the limit $\mathfrak{K}\ni\ell\to\infty$ the following convergences hold true:

$$\begin{cases} u_\ell \overset{*}{\rightharpoondown} u & \text{weakly}^* \text{ in } L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big),\\ u_\ell \rightharpoondown u & \text{weakly in } L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big). \end{cases} \tag{4.7}$$

4.1.4. Boundary values. The main subject of this section is to show that the limit map u from (4.7) possesses the correct boundary values in the sense that $u(t) = 0$ a.e. on $\Omega\setminus E^t$ for a.e. $t\in[0,T)$, and therefore $u \in V^p_2(E)$. To this end, for $\ell\in\mathbb{N}$ we define

$$E^{(\ell)} := \bigcup_{i=1}^{\ell} Q_{\ell,i}, \quad \text{where } Q_{\ell,i} := E^{t_{\ell,i}}\times I_{\ell,i} \text{ and } I_{\ell,i} := [t_{\ell,i-1}, t_{\ell,i}).$$

By construction, $u_\ell\equiv 0$ a.e. in $\Omega_T\setminus E^{(\ell)}$ and $E\subset E^{(\ell)}$. Now let $z_{\mathrm{o}} := (x_{\mathrm{o}}, t_{\mathrm{o}}) \in \Omega_T\setminus\overline{E}$ be an arbitrary but fixed point. We claim that the following holds true:

$$\exists\,\varepsilon>0, \ell_{\mathrm{o}}\in\mathbb{N} \text{ such that } \forall\ell\ge\ell_{\mathrm{o}} : Q_\varepsilon(z_{\mathrm{o}}) \subset \Omega_T\setminus\overline{E^{(\ell)}}. \tag{4.8}$$

Here, we used the shorthand notation

$$Q_\varrho(z_{\mathrm{o}}) := B_\varrho(x_{\mathrm{o}})\times\Lambda_\varrho(t_{\mathrm{o}}), \quad \text{where } \Lambda_\varrho(t_{\mathrm{o}}) := (t_{\mathrm{o}}-\varrho, t_{\mathrm{o}}+\varrho)\cap[0,T).$$

Indeed, since $z_{\mathrm{o}}\in\Omega_T\setminus\overline{E}$ there exists $\varepsilon>0$ such that $Q_{2\varepsilon}(z_{\mathrm{o}})\subset\Omega_T\setminus\overline{E}$. Due to assumption (2.7) this implies that $B_{2\varepsilon}(x_{\mathrm{o}})\subset\Omega\setminus\overline{E^t}$ for any $t\in(0,t_{\mathrm{o}}+2\varepsilon)$. Assuming that $h_\ell<\varepsilon$ (which is true for any $\ell>\frac{T}{\varepsilon}$), we find an index $i_{\mathrm{o}}\in\{1,\dots,\ell\}$ such that $i_o h_\ell\in[t_{\mathrm{o}}+\varepsilon, t_{\mathrm{o}}+2\varepsilon)$. Then, we have that $B_{2\varepsilon}(x_{\mathrm{o}})\subset\Omega\setminus\overline{E^{t_{\ell,i}}}$ for any $i\le i_{\mathrm{o}}$, so that by the construction of $E^{(\ell)}$ we conclude that

$$Q_\varepsilon(z_{\mathrm{o}}) \subset B_{2\varepsilon}(x_{\mathrm{o}})\times\Lambda_\varepsilon(t_{\mathrm{o}}) \subset \Omega_T\setminus\overline{E^{(\ell)}}.$$

Hence, the claim (4.8) follows for any choice of $\ell_{\mathrm{o}}\in\mathbb{N}$ with $\ell_{\mathrm{o}}>\frac{T}{\varepsilon}$.

Now $(4.7)_2$ ensures that $u_\ell\rightharpoondown u$ weakly in $L^p\big(Q_\varepsilon(z_{\mathrm{o}}),\mathbb{R}^N\big)$. By lower semicontinuity, we conclude that

$$\iint_{Q_\varepsilon(z_{\mathrm{o}})} |u|^p\,\mathrm{d}x\mathrm{d}t \le \liminf_{\ell\to\infty}\iint_{Q_\varepsilon(z_{\mathrm{o}})} |u_\ell|^p\,\mathrm{d}x\mathrm{d}t = 0,$$

i.e., $u\equiv 0$ a.e. in $Q_\varepsilon(z_{\mathrm{o}})$. Since $z_{\mathrm{o}}\in\Omega_T\setminus\overline{E}$ was arbitrary, this ensures that $u\equiv 0$ a.e. in $\Omega_T\setminus\overline{E}$. By a Fubini-type argument we conclude that $u(t)=0$ a.e. on $\Omega\times\{t\}\setminus\overline{E}$ for a.e. $t\in[0,T)$. Noting that

$$0 = \mathcal{L}^{n+1}(\partial E) \equiv \mathcal{L}^{n+1}\big(\overline{E}\setminus E\big) = \int_0^T \mathcal{L}^n\Big(\Omega\times\{t\}\cap\big(\overline{E}\setminus E\big)\Big)\mathrm{d}t,$$

by assumption (2.1), we conclude that $\mathcal{L}^n(\Omega \times \{t\} \cap (\overline{E} \setminus E)) = 0$ for a.e. $t \in [0, T)$. This, together with the preceding argument and the identity

$$\big(\Omega \setminus E^t\big) \times \{t\} = \big[\,\Omega \times \{t\} \setminus \overline{E}\,\big] \cup \big[\,\Omega \times \{t\} \cap \big(\overline{E} \setminus E\big)\big]$$

ensures that $u(t) = 0$ a.e. on $\Omega \setminus E^t$ for a.e. $t \in [0, T)$. In turn, this yields that $u(t) \in V_t$ for those t. This proves the claim that u fulfills the correct Dirichlet boundary values in the sense of (2.2), i.e., that $u \in V_2^p(E)$.

4.1.5. Variational inequality. In this section we shall establish that u satisfies the variational inequality (2.6). This, finally, proves that u is a variational solution in the sense of Definition 2.1. To this end, we consider a comparison map $v \in V_2^p(E)$ with $\partial_t v \in L^2\big(\Omega_T, \mathbb{R}^N\big)$ and let $\tau \in [0, T)$. Due to the definition of u_ℓ and inequality (4.4), we already know that

$$\begin{aligned}\iint_{\Omega_\tau} f\big(x, u_\ell, Du_\ell\big)\mathrm{d}x\mathrm{d}t \le& \iint_{\Omega_\tau} f(x, w, Dw)\mathrm{d}x\mathrm{d}t \\ &+ \tfrac{1}{h_\ell}\iint_{\Omega_\tau} \big(u_\ell(t) - u_\ell(t - h_\ell)\big)\cdot\big(w(t) - u_\ell(t)\big)\mathrm{d}x\mathrm{d}t\end{aligned}$$

holds true for any map $w \in L^\infty\big(0, T; L^2(\Omega, \mathbb{R}^N)\big) \cap L^p\big(0, T; W_0^{1,p}(\Omega, \mathbb{R}^N)\big)$ satisfying $w(t) \in V_{h_\ell\lceil \frac{t}{h_\ell}\rceil}$ for a.e. $t \in [0, T)$. Since $V_s \subset V_t$ holds for $0 \le s \le t$ by hypothesis (2.7), we have that $v(t) \in V_{h_\ell\lceil \frac{t}{h_\ell}\rceil}$ for a.e. $t \in [0, T)$. Therefore, we may choose $w \equiv v$ as comparison map in the preceding inequality, so that

$$\iint_{\Omega_\tau} f\big(x, u_\ell, Du_\ell\big)\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_\tau} f(x, v, Dv)\mathrm{d}x\mathrm{d}t + \iint_{\Omega_\tau} \Delta_t^{h_\ell} u_\ell \cdot \big(v - u_\ell\big)\mathrm{d}x\mathrm{d}t,$$

where we have abbreviated the finite difference quotient in time by

$$\Delta_t^{h_\ell} u_\ell(t) := \tfrac{1}{h_\ell}\big(u_\ell(t) - u_\ell(t - h_\ell)\big).$$

To pass from the preceding inequality to the variational inequality, we need to modify the second integral on the right-hand side. We extend v to $\Omega \times (-h_\ell, T]$ by letting $v(t) \equiv v(0)$ for $t \in (-h_\ell, 0)$ and compute

$$\begin{aligned}\mathrm{I}_\ell :=& \iint_{\Omega_\tau} \Delta_t^{h_\ell} u_\ell \cdot \big(v - u_\ell\big)\mathrm{d}x\mathrm{d}t \\ =& \iint_{\Omega_\tau} \Delta_t^{h_\ell} v \cdot \big(v - u_\ell\big)\mathrm{d}x\mathrm{d}t - \iint_{\Omega_\tau} \Delta_t^{h_\ell}\big(v - u_\ell\big)\cdot\big(v - u_\ell\big)\mathrm{d}x\mathrm{d}t \\ =& \iint_{\Omega_\tau} \Delta_t^{h_\ell} v \cdot \big(v - u_\ell\big)\mathrm{d}x\mathrm{d}t - \tfrac{1}{2}\iint_{\Omega_\tau} \Delta_t^{h_\ell}\big|v - u_\ell\big|^2\mathrm{d}x\mathrm{d}t \\ &- \tfrac{h_\ell}{2}\iint_{\Omega_\tau} \big|\Delta_t^{h_\ell}\big(v - u_\ell\big)\big|^2\mathrm{d}x\mathrm{d}t \\ \le& \iint_{\Omega_\tau} \Delta_t^{h_\ell} v \cdot \big(v - u_\ell\big)\mathrm{d}x\mathrm{d}t - \tfrac{1}{2}\iint_{\Omega_\tau} \Delta_t^{h_\ell}\big|v - u_\ell\big|^2\mathrm{d}x\mathrm{d}t =: \mathrm{II}_\ell - \mathrm{III}_\ell,\end{aligned}$$

with the obvious meaning of II_ℓ and III_ℓ. Since

$$
\begin{aligned}
\mathrm{III}_\ell &= \tfrac{1}{2h_\ell}\iint_{\Omega_\tau}\Big[\big|v(t)-u_\ell(t)\big|^2-\big|v(t-h_\ell)-u_\ell(t-h_\ell)\big|^2\Big]\mathrm{d}x\mathrm{d}t\\
&= \tfrac{1}{2h_\ell}\iint_{\Omega\times(\tau-h_\ell,\tau]}\big|v-u_\ell\big|^2\mathrm{d}x\mathrm{d}t-\tfrac{1}{2h_\ell}\iint_{\Omega\times(-h_\ell,0]}\big|v-u_\ell\big|^2\mathrm{d}x\mathrm{d}t\\
&= \tfrac{1}{2h_\ell}\iint_{\Omega\times(\tau-h_\ell,\tau]}\big|v-u_\ell\big|^2\mathrm{d}x\mathrm{d}t-\tfrac{1}{2}\int_{\Omega}\big|v(0)-u_{\mathrm{o}}\big|^2\mathrm{d}x,
\end{aligned}
$$

we have that

$$
\mathrm{I}_\ell \le \mathrm{II}_\ell - \tfrac{1}{2h_\ell}\iint_{\Omega\times(\tau-h_\ell,\tau]}\big|v-u_\ell\big|^2\mathrm{d}x\mathrm{d}t+\tfrac12\big\|v(0)-u_{\mathrm{o}}\big\|^2_{L^2(\Omega,\mathbb{R}^N)}
$$

holds true. We insert this above and obtain

$$
\begin{aligned}
&\iint_{\Omega_\tau} f\big(x,u_\ell,Du_\ell\big)\mathrm{d}x\mathrm{d}t+\tfrac{1}{2h_\ell}\iint_{\Omega\times(\tau-h_\ell,\tau]}\big|v-u_\ell\big|^2\mathrm{d}x\mathrm{d}t\\
&\qquad\le \iint_{\Omega_\tau}\Big[f(x,v,Dv)+\Delta_t^{h_\ell}v\cdot\big(v-u_\ell\big)\Big]\mathrm{d}x\mathrm{d}t+\tfrac12\big\|v(0)-u_{\mathrm{o}}\big\|^2_{L^2(\Omega,\mathbb{R}^N)}.
\end{aligned}
$$

Integrating this inequality with respect to τ over $(t_{\mathrm{o}},t_{\mathrm{o}}+\delta)\subset[0,T)$ and dividing by δ, we obtain by Fubini's theorem that

$$
\begin{aligned}
&\iint_{\Omega_{t_{\mathrm{o}}}} f\big(x,u_\ell,Du_\ell\big)\mathrm{d}x\mathrm{d}t+\tfrac{1}{2\delta}\iint_{\Omega\times(t_{\mathrm{o}}-h_\ell,t_{\mathrm{o}}+\delta)}\big|v-u_\ell\big|^2\mathrm{d}x\mathrm{d}t\\
&\qquad\le \fint_{t_{\mathrm{o}}}^{t_{\mathrm{o}}+\delta}\iint_{\Omega_\tau}\Big[f(x,v,Dv)+\Delta_t^{h_\ell}v\cdot\big(v-u_\ell\big)\Big]\mathrm{d}x\mathrm{d}t\mathrm{d}\tau+\tfrac12\big\|v(0)-u_{\mathrm{o}}\big\|^2_{L^2(\Omega,\mathbb{R}^N)}.
\end{aligned}
$$

For the first integral appearing on the left-hand side we use the weak lower-semicontinuity of the variational integral with respect to the weak convergence $u_\ell \rightharpoonup u$ in $L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big)$ and Fatou's lemma. For the second integral appearing on the left-hand side we use the lower semicontinuity with respect to the weak convergence of $u_\ell \rightharpoonup u$ in $L^2\big(\Omega_T,\mathbb{R}^N\big)$. Finally, for the first integral on the right-hand side we use that $\Delta_t^{h_\ell}v\to\partial_t v$ strongly in $L^2(\Omega_T,\mathbb{R}^N)$, since $\partial_t v\in L^2(\Omega_\tau,\mathbb{R}^N)$, together with the weak convergence $u_\ell\rightharpoonup u$ in $L^2\big(\Omega_\tau,\mathbb{R}^N\big)$. Therefore, in the limit $\mathfrak{K}\ni\ell\to\infty$ we obtain that

$$
\begin{aligned}
&\iint_{\Omega_{t_{\mathrm{o}}}} f(x,u,Du)\mathrm{d}x\mathrm{d}t+\tfrac{1}{2\delta}\iint_{\Omega\times(t_{\mathrm{o}},t_{\mathrm{o}}+\delta)}|v-u|^2\mathrm{d}x\mathrm{d}t\\
&\qquad\le \fint_{t_{\mathrm{o}}}^{t_{\mathrm{o}}+\delta}\iint_{\Omega_\tau}\Big[f(x,v,Dv)+\partial_t v\cdot(v-u)\Big]\mathrm{d}x\mathrm{d}t\mathrm{d}\tau+\tfrac12\big\|v(0)-u_{\mathrm{o}}\big\|^2_{L^2(\Omega,\mathbb{R}^N)}
\end{aligned}
$$

holds true. In the preceding inequality we finally pass to the limit $\delta\downarrow 0$. This yields the variational inequality

$$
\begin{aligned}
&\iint_{\Omega_{t_{\mathrm{o}}}} f(x,u,Du)\mathrm{d}x\mathrm{d}t+\tfrac12\big\|(v-u)(t_{\mathrm{o}})\big\|^2_{L^2(\Omega,\mathbb{R}^N)}\\
&\qquad\le \iint_{\Omega_{t_{\mathrm{o}}}}\Big[f(\cdot,v,Dv)+\partial_t v\cdot(v-u)\Big]\mathrm{d}x\mathrm{d}t+\tfrac12\big\|v(0)-u_{\mathrm{o}}\big\|^2_{L^2(\Omega,\mathbb{R}^N)}
\end{aligned}
$$

for a.e. $t_{\mathrm{o}}\in[0,T)$ and any comparison map $v\in V_2^p(E_T)$ with $\partial_t v\in L^2(\Omega_T,\mathbb{R}^N)$. This proves that u is a variational solution in the sense of Definition 2.1. The uniqueness follows from Lemma 4.1 below, and therefore it remains to establish that $u\in C^0\big([0,T);L^2(\Omega)\big)$.

4.1.6. Continuity in time. Exactly as in the proof of Lemma 3.3 we construct the regularization of the initial values u_o by

$$u_\mathrm{o}^{(\varepsilon)} := \big(u_\mathrm{o}\chi_{E^{0,2\varepsilon}}\big) * \phi_\varepsilon.$$

Again, $u_\mathrm{o}^{(\varepsilon)} \in C_0^\infty\big(E^{0,\varepsilon}, \mathbb{R}^N\big)$, $u_\mathrm{o}^{(\varepsilon)} \to u_\mathrm{o} \in L^2(\Omega, \mathbb{R}^N)$, and $u_\mathrm{o}^{(\varepsilon)}$ has finite energy, i.e., (3.5) holds true. Moreover, from (2.7), i.e., the fact that the sets E^t are nondecreasing with respect to t, we know that $\operatorname{spt} u_\mathrm{o}^{(\varepsilon)} \subset E^t$ for any $t \in [0, T)$. We test the variational inequality (2.6) with $v = [\![u]\!]_{\lambda,\varepsilon}$, where $[\![u]\!]_{\lambda,\varepsilon}$ denotes the mollification with respect to time from (4.1) with the choice $v_\mathrm{o} \equiv u_\mathrm{o}^{(\varepsilon)}$ as initial value. This leads to

$$\begin{aligned}
&\tfrac12\big\|\big([\![u]\!]_{\lambda,\varepsilon} - u\big)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} + \iint_{E\cap\Omega_\tau} f(x,u,Du)\,\mathrm{d}x\mathrm{d}t \\
&\le \iint_{E\cap\Omega_\tau} \Big[\partial_t[\![u]\!]_{\lambda,\varepsilon}\cdot\big([\![u]\!]_{\lambda,\varepsilon} - u\big) + f\big(x, [\![u]\!]_{\lambda,\varepsilon}, D[\![u]\!]_{\lambda,\varepsilon}\big)\Big]\mathrm{d}x\mathrm{d}t \\
&\quad + \tfrac12\big\|u_\mathrm{o}^{(\varepsilon)} - u_\mathrm{o}\big\|^2_{L^2(\Omega,\mathbb{R}^N)} \\
&\le \iint_{E\cap\Omega_\tau} \big[\!\big[f(x,u,Du)\big]\!\big]_{\lambda,\varepsilon}\mathrm{d}x\mathrm{d}t + \tfrac12\big\|u_\mathrm{o}^{(\varepsilon)} - u_\mathrm{o}\big\|^2_{L^2(\Omega,\mathbb{R}^N)}
\end{aligned}$$

for a.e. $\tau \in [0, T)$. In the last line we used identity (4.2) for the time mollification, which allowed us to discard the negative first integral and [7, Lem. 2.3]. The latter is possible because on the one hand f is convex with respect to (u, ξ) and on the other hand (3.5) holds. Note that $[\![f(x,u,Du)]\!]_{\lambda,\varepsilon}$ is defined according to (4.1) with the choice $v_\mathrm{o} = f\big(x, u_\mathrm{o}^{(\varepsilon)}, Du_\mathrm{o}^{(\varepsilon)}\big)$. In the first term appearing on the left-hand side of the preceding inequality we now replace $[\![u]\!]_{\lambda,\varepsilon}$ by $[\![u]\!]_\lambda$, where, according to (4.1), $[\![u]\!]_\lambda$ is defined with initial values $v_\mathrm{o} \equiv u_\mathrm{o}$. This leads to

$$\begin{aligned}
&\big\|\big([\![u]\!]_\lambda - u\big)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} \\
&\quad\le 4\iint_{E\cap\Omega_\tau}\Big[\big[\!\big[f(x,u,Du)\big]\!\big]_{\lambda,\varepsilon} - f(x,u,Du)\Big]\mathrm{d}x\mathrm{d}t + 4\big\|u_\mathrm{o}^{(\varepsilon)} - u_\mathrm{o}\big\|^2_{L^2(\Omega,\mathbb{R}^N)} \\
&\quad= 4\iint_{\Omega_\tau}\Big[\big[\!\big[f(x,u,Du)\big]\!\big]_{\lambda,\varepsilon} - f(x,u,Du)\Big]\mathrm{d}x\mathrm{d}t + 4\big\|u_\mathrm{o}^{(\varepsilon)} - u_\mathrm{o}\big\|^2_{L^2(\Omega,\mathbb{R}^N)} \\
&\quad= -4\lambda\iint_{\Omega_\tau}\partial_t\big[\!\big[f(x,u,Du)\big]\!\big]_{\lambda,\varepsilon}\mathrm{d}x\mathrm{d}t + 4\big\|u_\mathrm{o}^{(\varepsilon)} - u_\mathrm{o}\big\|^2_{L^2(\Omega,\mathbb{R}^N)} \\
&\quad\le 4\lambda\int_\Omega f\big(x, u_\mathrm{o}^{(\varepsilon)}, Du_\mathrm{o}^{(\varepsilon)}\big)\,\mathrm{d}x + 4\big\|u_\mathrm{o}^{(\varepsilon)} - u_\mathrm{o}\big\|^2_{L^2(\Omega,\mathbb{R}^N)}
\end{aligned} \tag{4.9}$$

for a.e. $\tau \in [0, T)$. In the second to last line we again used identity (4.2); in fact, we only need to replace v by $f(x, u, Du)$. In the third to last line we used the fact that for $z = (x_\mathrm{o}, t_\mathrm{o}) \in \Omega_T \setminus E$ the integrand vanishes. Indeed, we have

$$\begin{aligned}
&\Big[\big[\!\big[f(x,u,Du)\big]\!\big]_{\lambda,\varepsilon} - f(x,u,Du)\Big]\Big|_{(x_\mathrm{o},t_\mathrm{o})} \\
&\quad= \mathrm{e}^{-\frac{t_\mathrm{o}}{\lambda}} f\big(x_\mathrm{o}, u_\mathrm{o}^{(\varepsilon)}(x_\mathrm{o}), Du_\mathrm{o}^{(\varepsilon)}(x_\mathrm{o})\big) + \tfrac1\lambda\int_0^{t_\mathrm{o}} \mathrm{e}^{\frac{s-t_\mathrm{o}}{\lambda}} f\big(x_\mathrm{o}, u(x_\mathrm{o}, s), Du(x_\mathrm{o}, s)\big)\,\mathrm{d}s \\
&\qquad - f\big(x_\mathrm{o}, u(x_\mathrm{o}, t_\mathrm{o}), Du(x_\mathrm{o}, t_\mathrm{o})\big) \\
&\quad= \mathrm{e}^{-\frac{t_\mathrm{o}}{\lambda}} f\big(x_\mathrm{o}, 0, 0\big) + \tfrac1\lambda\int_0^{t_\mathrm{o}} \mathrm{e}^{\frac{s-t_\mathrm{o}}{\lambda}} f\big(x_\mathrm{o}, 0, 0\big)\,\mathrm{d}s - f\big(x_\mathrm{o}, 0, 0\big) = 0.
\end{aligned}$$

Now, we consider a sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ with $\varepsilon_i \downarrow 0$ and choose

$$\lambda_i := \min\left\{\varepsilon_i, \left[\int_\Omega f\big(x, u_o^{(\varepsilon_i)}, Du_o^{(\varepsilon_i)}\big)\,\mathrm{d}x\right]^{-2}\right\},$$

so that also $\lambda_i \downarrow 0$. Using (4.9) with λ_i, we obtain for any $i \in \mathbb{N}$ that

$$\sup_{\tau\in[0,T)} \big\|\big([\![u]\!]_{\lambda_i} - u\big)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} \le 4\sqrt{\lambda_i} + 4\big\|u_o^{(\varepsilon_i)} - u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)}$$

holds true. Since the left-hand side converges to 0 as $i \to \infty$, we conclude that

$$\lim_{i\to\infty} \sup_{\tau\in[0,T)} \big\|\big([\![u]\!]_{\lambda_i} - u\big)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} = 0.$$

Keeping in mind that $[\![u]\!]_{\lambda_i} \in C^0\big([0,T); L^2(\Omega,\mathbb{R}^N)\big)$ is true for any $i \in \mathbb{N}$, we deduce from the above convergence that also $u \in C^0\big([0,T); L^2(\Omega,\mathbb{R}^N)\big)$ is true. This proves the desired continuity property with respect to time, and finishes the proof of Theorem 2.3.

4.2. Properties of variational solutions in nondecreasing domains. In the case of nondecreasing domains satisfying (2.7), we obtain some stronger properties of variational solutions than in the general case.

4.2.1. Uniqueness of variational solutions in nondecreasing domains. The precise statement is as follows.

LEMMA 4.1. *Suppose that* $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn} \to [0,\infty]$ *is a variational integrand satisfying* (2.4) *and* (2.5) *and that the noncylindrical domain* E *satisfies* (2.7)*. Then, for any initial datum* $u_o \in L^2\big(E^0,\mathbb{R}^N\big)$ *there exists at most one variational solution in the sense of Definition* 2.1*.*

Proof. Assume that u_1 and u_2 are two variational solutions with initial datum u_o in the sense of Definition 2.1, then for $i = 1, 2$ the variational inequality

$$\begin{aligned}\iint_{E\cap\Omega_\tau} f(x,u_i,Du_i)\,\mathrm{d}z \le \iint_{E\cap\Omega_\tau} \Big[\partial_t v\cdot(v-u_i) + f(x,v,Dv)\Big]\,\mathrm{d}z \\ - \tfrac12\big\|(v-u_i)(\tau)\big\|^2_{L^2(E^\tau,\mathbb{R}^N)} + \tfrac12\big\|v(0)-u_o\big\|^2_{L^2(E^0,\mathbb{R}^N)}\end{aligned}$$

holds true for a.e. $\tau \in [0,T)$ and any $v \in V_2^p(E)$ with $\partial_t v \in L^2(\Omega_T,\mathbb{R}^N)$. Again, we define $u_o^{(\varepsilon)}$ as in the proof of Lemma 3.3. By $[\![u_i]\!]_h$ for $i \in \{1,2\}$ we denote the time mollification of u_i as in (4.1) with initial value $v_o \equiv u_o^{(\varepsilon)}$. Note that due to assumption (2.7) we have that $[\![u_i]\!]_h \in V_2^p(E)$ and $\partial_t[\![u_i]\!]_h \in L^2\big(\Omega_T,\mathbb{R}^N\big)$. With these prerequisites we take $v = \frac12\big([\![u_1]\!]_h + [\![u_2]\!]_h\big)$ as a comparison map in the above variational inequalities. We recall the identity (4.2) and estimate

$$\partial_t v\cdot(v-u_1) + \partial_t v\cdot(v-u_2) = -\tfrac{1}{2h}\Big|[\![u_1]\!]_h - u_1 + [\![u_2]\!]_h - u_2\Big|^2 \le 0.$$

We add the variational inequalities for u_1 and u_2 and use in turn the last estimate, the convexity of f, and the choice $v_o = u_o^{(\varepsilon)}$ as initial value, so that $v(0) = u_o^{(\varepsilon)}$. This procedure leads to

$$\begin{aligned}
&\iint_{E\cap\Omega_\tau}\Big[f(x,u_1,Du_1)+f(x,u_2,Du_2)\Big]\,\mathrm{d}z\\
&\qquad+\tfrac12\big\|(v-u_1)(\tau)\big\|^2_{L^2(E^\tau,\mathbb{R}^N)}+\tfrac12\big\|(v-u_2)(\tau)\big\|^2_{L^2(E^\tau,\mathbb{R}^N)}\\
&\qquad\le\iint_{E\cap\Omega_\tau}2f(x,v,Dv)\,\mathrm{d}z+\big\|u_{\mathrm{o}}^{(\varepsilon)}-u_{\mathrm{o}}\big\|^2_{L^2(E^0,\mathbb{R}^N)}\\
&\qquad\le\iint_{E\cap\Omega_\tau}\Big[f\big(x,[\![u_1]\!]_h,D[\![u_1]\!]_h\big)+f\big(x,[\![u_2]\!]_h,D[\![u_2]\!]_h\big)\Big]\,\mathrm{d}z\\
&\qquad\quad+\big\|u_{\mathrm{o}}^{(\varepsilon)}-u_{\mathrm{o}}\big\|^2_{L^2(E^0,\mathbb{R}^N)}.
\end{aligned}\tag{4.10}$$

By the energy estimate from Lemma 3.4 we know for $i=1,2$ that u_i has finite energy, i.e., that $\iint_E f(x,u_i,Du_i)\,\mathrm{d}z<\infty$ holds true. Also, $u_{\mathrm{o}}^{(\varepsilon)}$ has finite energy due to assumption (2.5); see (3.5). Therefore, the hypotheses of [7, Lem. 2.3] are fulfilled and we conclude that

$$\lim_{h\downarrow 0}\iint_{E\cap\Omega_\tau}f\big(x,[\![u_i]\!]_h,D[\![u_i]\!]_h\big)\mathrm{d}x\mathrm{d}t=\iint_{E\cap\Omega_\tau}f(x,u_i,Du_i)\mathrm{d}x\mathrm{d}t.$$

Consequently, in the limit $h\downarrow 0$ inequality (4.10) simplifies to

$$\big\|(u_1-u_2)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)}\le 4\big\|u_{\mathrm{o}}^{(\varepsilon)}-u_{\mathrm{o}}\big\|^2_{L^2(E^0)}$$

for a.e. $\tau\in[0,T)$. In the preceding inequality we let $\varepsilon\downarrow 0$. This proves the desired claim, that $u_1=u_2$ a.e. on E. ∎

4.2.2. Time derivative. In this section we shall prove that every variational solution u in the sense of Definition 2.1 satisfies $\partial_t u\in L^2\big(\Omega_T,\mathbb{R}^N\big)$ if the domains E^t are nondecreasing in time in the spirit of (2.7), and, moreover, the initial datum u_{o} has finite energy in the sense of (2.9). We note that in the case of linear systems, the regularity property $\partial_t u\in L^2\big(\Omega_T,\mathbb{R}^N\big)$ is known from [19]. Here we are considering, much more generally, gradient flows for functionals that only satisfy (2.4) and (2.5). As a byproduct of the proof, we obtain that the integral functionals $\int_{\Omega\times\{t\}}f(x,u,Du)\mathrm{d}x$ depend monotonically decreasing on the time $t\in[0,T)$, as to be expected for a gradient flow. More precisely, we will show the following result.

LEMMA 4.2. *Suppose that* $f\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to[0,\infty]$ *is a variational integrand satisfying* (2.4) *and* (2.5) *and that the noncylindrical domain* E *satisfies* (2.7)*. If the initial datum* $u_{\mathrm{o}}\in L^2\big(E^0,\mathbb{R}^N\big)$ *satisfies* (2.9)*, then* $\partial_t u\in L^2\big(\Omega_T,\mathbb{R}^N\big)$ *and* $f(\cdot,u,Du)\in L^\infty(0,T;L^1(\Omega))$ *together with the estimates*

$$\iint_{\Omega_T}|\partial_t u|^2\mathrm{d}x\mathrm{d}t\le\int_\Omega f(x,u_{\mathrm{o}},Du_{\mathrm{o}})\mathrm{d}x$$

and

$$\int_{\Omega\times\{t\}}f(x,u,Du)\mathrm{d}x\le\int_\Omega f(x,u_{\mathrm{o}},Du_{\mathrm{o}})\mathrm{d}x\tag{4.11}$$

for a.e. $t\in[0,T)$.

Proof. We will use the mollification in time which was introduced in (4.1). Since the domains E^t are nondecreasing in time, we can test the variational inequality (2.6) with $v=[\![u]\!]_h$, where we take $v_{\mathrm{o}}=u_{\mathrm{o}}$ as initial values. Then, clearly $[\![u]\!]_h(0)=u_{\mathrm{o}}$ holds and we obtain

$$\iint_{\Omega_\tau} f(x,u,Du)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_\tau} \Big[\partial_t [\![u]\!]_h \cdot \big([\![u]\!]_h - u\big) + f\big(x, [\![u]\!]_h, D[\![u]\!]_h\big)\Big]\mathrm{d}x\mathrm{d}t$$

for a.e. $\tau \in [0,T)$. Here, we used that $[\![u]\!]_h \equiv 0$ on $\Omega_T \setminus E$ and therefore $f(x,[\![u]\!]_h, D[\![u]\!]_h)=f(x,0,0)=f(x,u,Du)$ on $\Omega_T\setminus E$. The preceding inequality, however, implies

$$\begin{aligned}
\iint_{\Omega_\tau} \big|\partial_t [\![u]\!]_h\big|^2 \mathrm{d}x\mathrm{d}t &= -\tfrac{1}{h}\iint_{\Omega_\tau} \partial_t [\![u]\!]_h \cdot \big([u]_h - u\big)\,\mathrm{d}x\mathrm{d}t \\
&\le \tfrac{1}{h}\iint_{\Omega_\tau} \Big[f\big(x,[\![u]\!]_h, D[\![u]\!]_h\big) - f(x,u,Du)\Big]\mathrm{d}x\mathrm{d}t \\
&\le \tfrac{1}{h}\iint_{\Omega_\tau} \Big[[\![f(x,u,Du)]\!]_h - f(x,u,Du)\Big]\mathrm{d}x\mathrm{d}t \\
&= -\iint_{\Omega_\tau} \partial_t [\![f(x,u,Du)]\!]_h\,\mathrm{d}x\mathrm{d}t \\
&= \int_\Omega f(x,u_o,Du_o)\,\mathrm{d}x - \int_{\Omega\times\{\tau\}} [\![f(x,u,Du)]\!]_h\,\mathrm{d}x \\
&\le \int_\Omega f(x,u_o,Du_o)\,\mathrm{d}x < \infty,
\end{aligned} \tag{4.12}$$

where $[\![f(x,u,Du)]\!]_h$ is defined according to (4.1) with v_o replaced by $f(x,u_o,Du_o)$. Note that we used in turn (4.2) and

$$f\big(x,[\![u]\!]_h, D[\![u]\!]_h\big) \le [\![f(x,u,Du)]\!]_h,$$

which holds due to the convexity of f; cf. [7, Lem. 2.3]. Taking the supremum over $\tau \in [0,T)$ in (4.12), we infer that the integrals $\iint_{\Omega_T} |\partial_t [\![u]\!]_h|^2\,\mathrm{d}x\mathrm{d}t$ are bounded independently from $h>0$. This ensures that the time derivative $\partial_t u$ exists with $\partial_t u \in L^2\big(\Omega_T,\mathbb{R}^N\big)$ together with the quantitative estimate

$$\iint_{\Omega_T} |\partial_t u|^2 \mathrm{d}x\mathrm{d}t \le \int_\Omega f(x,u_o,Du_o)\,\mathrm{d}x.$$

Moreover, since the left-hand side of (4.12) is nonnegative, we deduce

$$\int_{\Omega\times\{\tau\}} [\![f(x,u,Du)]\!]_h\,\mathrm{d}x \le \int_\Omega f(x,u_o,Du_o)\,\mathrm{d}x$$

for a.e. $\tau \in [0,T)$ and every $h>0$. Letting $h \downarrow 0$, we conclude the second asserted estimate (4.11). This finishes the proof of the lemma. ◻

4.2.3. Time derivative in the dual space. In this section we prove that variational solutions admit a time derivative $\partial_t u$ in the dual space $(V^p(E))'$, provided that the integrand f additionally satisfies the standard coercivity and p-growth conditions (2.13). Note that (2.13) already implies condition (2.5). Furthermore, [34, Lem. 2.1] implies the Lipschitz condition

$$\begin{aligned}
&\big|f(x,u_1,\xi_1) - f(x,u_2,\xi_2)\big| \\
&\quad\le c\Big[\big(|\xi_1|+|\xi_2|+|u_1|+|u_2|\big)^{p-1} + |G(x)|\Big]\big(|u_1-u_2| + |\xi_1-\xi_2|\big)
\end{aligned} \tag{4.13}$$

for all $x\in\Omega$ and $(u_1,\xi_1),(u_2,\xi_2) \in \mathbb{R}^N\times\mathbb{R}^{Nn}$ and a constant $c=c(p,L)$. For the proof of $\partial_t u \in (V^p(E))'$, we rely on the following notion of parabolic minimizers, which was introduced by Wieser [41].

DEFINITION 4.3 (parabolic minimizer). Suppose that $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to[0,\infty]$ is a Carathéodory function and that $E\subset\Omega\times[0,T)$ is a relatively open noncylindrical domain. A map $u\in V^p(E)$ is termed *parabolic minimizer* associated to the variational integrand f if and only if the minimality condition

$$\iint_E [u\cdot\partial_t\varphi + f(x,u,Du)]\,\mathrm{d}x\mathrm{d}t \le \iint_E f(x,u+\varphi,Du+D\varphi)\mathrm{d}x\mathrm{d}t \tag{4.14}$$

holds true for any $\varphi\in C_0^\infty\big(E,\mathbb{R}^N\big)$. □

LEMMA 4.4. *Suppose that $f\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to[0,\infty]$ is a variational integrand satisfying* (2.4)$_1$ *and* (2.13) *and that the noncylindrical domain E satisfies* (2.7) *and that $u_o\in L^2\big(E^0,\mathbb{R}^N\big)$. Then, any variational solution associated to the initial datum u_o in the sense of Definition* 2.1 *is also a parabolic minimizer in the sense of Definition* 4.3.

Proof. We consider a testing function $\varphi\in C_0^\infty\big(E,\mathbb{R}^N\big)$. Our aim in the following is to prove that inequality (4.14) holds true. Since u is not regular enough to serve as a comparison function, we have to use a certain mollification with respect to time. More precisely, we define as in the proof of Lemma 3.3 the mollification $[\![u]\!]_h$ according to (4.1) with the choice $u_o^{(\varepsilon)}$ for v_o, while $[\![\varphi]\!]_h$ is defined in the same way but with the choice $v_o=0$. Assumption (2.13) ensures that $u_o^{(\varepsilon)}$ has finite energy, i.e., that (3.5) holds true. In (2.6) we let $\tau\uparrow T$, and then choose the comparison map $v=v_h$, where $v_h:=[\![u]\!]_h+s[\![\varphi]\!]_h$ with $s>0$. Note that $\operatorname{spt} v_h\subset E$. Then, for the first term on the right-hand side of (2.6) we deduce that

$$\begin{aligned}
&\iint_E \partial_t v_h\cdot(v_h-u)\,\mathrm{d}x\mathrm{d}t\\
&\quad=\iint_{\Omega_T}\big[\partial_t[\![u]\!]_h\cdot([\![u]\!]_h-u)+s\partial_t[\![u]\!]_h\cdot[\![\varphi]\!]_h+s\partial_t[\![\varphi]\!]_h\cdot(v_h-u)\big]\,\mathrm{d}x\mathrm{d}t\\
&\quad=\iint_{\Omega_T}\Big[-\tfrac1h\big|[\![u]\!]_h-u\big|^2-s[\![u]\!]_h\cdot\partial_t[\![\varphi]\!]_h+s\,(v_h-u)\cdot\partial_t[\![\varphi]\!]_h\Big]\,\mathrm{d}x\mathrm{d}t\\
&\qquad+s\int_\Omega\left([\![u]\!]_h\cdot[\![\varphi]\!]_h\right)(\cdot,T)\mathrm{d}x\\
&\quad\le\iint_E s\,(s[\![\varphi]\!]_h-u)\cdot\partial_t[\![\varphi]\!]_h\mathrm{d}x\mathrm{d}t+s\int_{E^T}\left([\![u]\!]_h\cdot[\![\varphi]\!]_h\right)(\cdot,T)\mathrm{d}x\,.
\end{aligned}$$

In the second identity we used (4.2) and implemented an integration by parts with respect to time. Note that at $t=0$ no boundary term occurs, since $[\![\varphi]\!]_h(\cdot,0)=0$. Inserting this into the variational inequality (2.6), discarding the nonnegative term $\|(v_h-u)(\cdot,\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)}$, and letting $\tau\uparrow T$, we get

$$\begin{aligned}
\iint_E f(x,u,Du)\mathrm{d}x\mathrm{d}t\le{}&\iint_E\left[s\,(s[\![\varphi]\!]_h-u)\cdot\partial_t[\![\varphi]\!]_h+f\left(x,v_h,Dv_h\right)\right]\mathrm{d}x\mathrm{d}t\\
&+s\int_{E^T}\left([\![u]\!]_h\cdot[\![\varphi]\!]_h\right)(\cdot,T)\mathrm{d}x.
\end{aligned}$$

Here, we pass to the limit $h\downarrow0$. Note that the boundary term disappears in the limit, since $\varphi(\cdot,T)\equiv0$ on E^T. In this way, we obtain

$$\begin{aligned}
&\iint_E f(x,u,Du)\mathrm{d}x\mathrm{d}t\\
&\quad\le \iint_E \left[s(s\varphi-u)\cdot\partial_t\varphi + f\left(x,u+s\varphi,Du+sD\varphi\right)\right]\mathrm{d}x\mathrm{d}t\\
&\quad\le \iint_E \left[s(s\varphi-u)\cdot\partial_t\varphi + (1-s)f(x,u,Du)+sf(x,u+\varphi,Du+D\varphi)\right]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

In the last line we used the convexity of f. Reabsorbing the second integral of the right-hand side on the left and dividing by $s>0$ we get

$$\iint_E f(x,u,Du)\mathrm{d}x\mathrm{d}t \le \iint_E \left[(s\varphi-u)\cdot\partial_t\varphi + f(x,u+\varphi,Du+D\varphi)\right]\mathrm{d}x\mathrm{d}t.$$

Here, we pass to the limit $s\downarrow 0$ and finally come up with the inequality

$$\iint_E \left[u\cdot\partial_t\varphi + f(x,u,Du)\right]\mathrm{d}x\mathrm{d}t \le \iint_E f(x,u+\varphi,Du+D\varphi)\mathrm{d}x\mathrm{d}t.$$

Recall that the previous inequality holds true for any $\varphi\in C_0^\infty\big(E,\mathbb{R}^N\big)$, which means that u is a parabolic minimizer in the sense of Definition 4.3. This completes the proof. ∎

LEMMA 4.5. *Suppose that $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to[0,\infty]$ is a variational integrand satisfying* $(2.4)_1$ *and* (2.13) *and that the noncylindrical domain E satisfies* (2.7) *and* (2.10) *and that $u_{\mathrm{o}}\in L^2\big(E^0,\mathbb{R}^N\big)$. Then, any variational solution associated to the initial datum u_{o} in the sense of Definition* 2.1 *admits a distributional time derivative*

$$\partial_t u\in\big(V^p(E)\big)'$$

with

$$\|\partial_t u\|_{(V_2^p(E))'}\le c\left[\|u\|_{V_2^p(E)}^{p-1}+\|G\|_{L^{p'}(E)}\right]$$

with a constant $c=c(p,L)$. Moreover, for any $\varphi\in V^p(E)$ there holds

$$\left|\iint_E u\cdot\partial_t\varphi\mathrm{d}x\mathrm{d}t\right|\le c\iint_E\left[|Du|^{p-1}+|u|^{p-1}+|G|\right]\left[|D\varphi|+|\varphi|\right]\mathrm{d}x\mathrm{d}t$$

with a constant $c=c(p,L)$.

Proof. From Lemma 4.4 we already know that a variational solution u is also a parabolic minimizer in the sense of Definition 4.3. Therefore, taking $s\varphi$ instead of φ in (4.14), we get for any $\varphi\in C_0^\infty\big(E,\mathbb{R}^N\big)$ and $s\in(0,1)$ that

$$\begin{aligned}
&\left|\iint_E u\cdot\partial_t\varphi\mathrm{d}x\mathrm{d}t\right|\\
&\quad\le\left|\iint_E \tfrac{1}{s}\Big[f(x,u+s\varphi,Du+sD\varphi)-f(x,u,Du)\Big]\mathrm{d}x\mathrm{d}t\right|\\
&\quad\le c\iint_E\Big[\Big(|Du|+|Du+sD\varphi|+|u|+|u+s\varphi|\Big)^{p-1}+|G|\Big]\big[|D\varphi|+|\varphi|\big]\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where we used the Lipschitz property (4.13) of f for the last estimate. Passing to the limit $s\downarrow 0$, we deduce

$$\begin{aligned}\left|\iint_E u\cdot\partial_t\varphi \mathrm{d}x\mathrm{d}t\right| &\le c\iint_E\Big[|Du|^{p-1}+|u|^{p-1}+|G|\Big]\big[|D\varphi|+|\varphi|\big]\mathrm{d}x\mathrm{d}t\\ &\le c\bigg[\iint_E\Big[|Du|^p+|u|^p+|G|^{p'}\Big]\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{p'}}\|\varphi\|_{V^p(E)}\\ &\le c\Big[\|u\|^{p-1}_{V^p(E)}+\|G\|_{L^{p'}(E)}\Big]\|\varphi\|_{V^p(E)}.\end{aligned}$$

The previous estimate holds true for any testing function $\varphi\in C_0^\infty\big(E,\mathbb{R}^N\big)$. From Lemma 5.3 below, we know that $C_0^\infty\big(E,\mathbb{R}^N\big)$ is dense in $V^p(E)$. Note that (2.12) is automatically fulfilled, since we are dealing with nondecreasing sets E^t; recall that assumption (2.7) is in force. Therefore, $\partial_t u\in\big(V^p(E)\big)'$ is valid together with the alleged quantitative estimate for the $\big(V^p(E)\big)'$-norm. This proves the claim. ◻

5. Variational solutions in general domains.

5.1. Existence of variational solutions (Proof of Theorem 2.6).

In this section we prove Theorem 2.6. We will proceed in several steps.

5.1.1. Construction of the approximating sequence. For given $\ell\in\mathbb{N}$ we define $h_\ell:=\frac{T}{\ell}$ and $t_{\ell,i}:=ih_\ell$ for $i\in\{0,\dots,\ell\}$, as well as $I_{\ell,i}:=[t_{\ell,i-1},t_{\ell,i})$ for $i\in\{1,\dots,\ell\}$. We define the open sets $E_{\ell,i}$ and associated cylinders $Q_{\ell,i}$ by

$$E_{\ell,i}:=\bigcup_{t\in I_{\ell,i}}E^t\subset\mathbb{R}^n,\qquad Q_{\ell,i}:=E_{\ell,i}\times I_{\ell,i}.$$

By construction we have

$$E\subset\bigcup_{i=1}^{\ell}Q_{\ell,i}. \tag{5.1}$$

By an inductive procedure we shall define a sequence $(u_{\ell,i})_{i=1,\dots,\ell}$ of variational solutions $u_{\ell,i}\colon Q_{\ell,i}\to\mathbb{R}^N$ defined on the cylindrical domains $Q_{\ell,i}$. The precise construction is as follows: Let

$$u_{\ell,1}\in C^0\big(\overline{I_{\ell,1}};L^2(E_{\ell,1},\mathbb{R}^N)\big)\cap L^p\big(I_{\ell,1};W_0^{1,p}(E_{\ell,1},\mathbb{R}^N)\big)$$

be the unique variational solution in the sense of Definition 2.1 from Theorem 2.3 on the cylindrical domain $Q_{\ell,1}$ with initial datum u_{o}. Note that $E^0\subset E_{\ell,1}$ by construction and that u_{o} vanishes on $E_{\ell,1}\setminus E^0$. Now we assume that the mapping $u_{\ell,i-1}$ is already defined for some index $i\in\{2,\dots,\ell\}$. Then, we denote by

$$u_{\ell,i}\in C^0\big(\overline{I_{\ell,i}};L^2(E_{\ell,i},\mathbb{R}^N)\big)\cap L^p\big(I_{\ell,i};W_0^{1,p}(E_{\ell,i},\mathbb{R}^N)\big)$$

the unique variational solution from Theorem 2.3 on the cylindrical domain $Q_{\ell,i}$ with initial datum

$$w_{\ell,i}:=u_{\ell,i-1}\big(t_{\ell,i-1}\big)\chi_{E_{\ell,i}}\in L^2\big(E_{\ell,i},\mathbb{R}^N\big).$$

For $i\in\{1,\dots,\ell\}$ we extend $u_{\ell,i}$ to $\big(\Omega\setminus E_{\ell,i}\big)\times\overline{I_{\ell,i}}$ by zero. Finally, we glue the mappings $u_{\ell,i}$ together to form one mapping $u_\ell\colon\Omega_T\to\mathbb{R}^N$ defined on the whole cylinder Ω_T by letting

$$u_\ell(x,t):=u_{\ell,i}(x,t)\quad\text{for } (x,t)\in\Omega\times I_{\ell,i}\text{ with } i\in\{1,\dots\ell\}.$$

5.1.2. Energy estimates. For $\ell \in \mathbb{N}$ and $i \in \{1, \dots, \ell\}$ the variational inequality (2.6) for $u_{\ell,i}$ reads as follows:

$$
\begin{aligned}
&\iint_{\Omega\times[t_{\ell,i-1},\tau)} f\big(x, u_{\ell,i}, Du_{\ell,i}\big)\,\mathrm{d}x\mathrm{d}t \\
&\qquad \le \iint_{\Omega\times[t_{\ell,i-1},\tau)} \Big[\partial_t v\cdot\big(v-u_{\ell,i}\big) + f(x,v,Dv)\Big]\,\mathrm{d}x\mathrm{d}t \\
&\qquad\quad - \tfrac12\big\|v(\tau)-u_{\ell,i}(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} + \tfrac12\big\|v(t_{\ell,i-1})-w_{\ell,i}\big\|^2_{L^2(\Omega,\mathbb{R}^N)}
\end{aligned}
\tag{5.2}
$$

for a.e. $\tau \in I_{\ell,i}$ and any comparison map $v \in L^p\big(I_{\ell,i}; W_0^{1,p}(E_{\ell,i},\mathbb{R}^N)\big) \cap L^2(Q_{\ell,i},\mathbb{R}^N)$ with $\partial_t v \in L^2\big(Q_{\ell,i},\mathbb{R}^N\big)$ and $v \equiv 0$ in $\big(\Omega\setminus E_{\ell,i}\big)\times I_{\ell,i}$. We choose $v \equiv 0$ as comparison map in (5.2), which, of course, is a possible choice, and use hypothesis (2.5). This leads us to

$$
\begin{aligned}
\tfrac12\int_\Omega \big|u_{\ell,i}(t_{\ell,i})\big|^2\mathrm{d}x &+ \iint_{\Omega\times I_{\ell,i}} f\big(x,u_{\ell,i},Du_{\ell,i}\big)\,\mathrm{d}x\mathrm{d}t \\
&\le \tfrac12\int_\Omega \big|u_{\ell,i-1}(t_{\ell,i-1})\chi_{E_{\ell,i}}\big|^2\mathrm{d}x + \iint_{\Omega\times I_{\ell,i}} f(x,0,0)\,\mathrm{d}x\mathrm{d}t \\
&\le \tfrac12\int_\Omega \big|u_{\ell,i-1}(t_{\ell,i-1})\big|^2\mathrm{d}x + h_\ell\int_\Omega g(x,0)\,\mathrm{d}x.
\end{aligned}
$$

In this context we abbreviate $u_{\ell,0}(t_{\ell,0}) \equiv u_{\mathrm{o}}$. Note that $t_{\ell,0} = 0$. Summing up the preceding inequalities over $i = 1, \dots, m$ with $m \le \ell$ and recalling the definition of u_ℓ, we get

$$
\tfrac12\int_\Omega \big|u_\ell(t_{\ell,m})\big|^2\mathrm{d}x + \iint_{\Omega_{t_{\ell,m}}} f\big(x,u_\ell,Du_\ell\big)\,\mathrm{d}x\mathrm{d}t \le T\int_\Omega g(x,0)\,\mathrm{d}x + \tfrac12\int_\Omega |u_{\mathrm{o}}|^2\,\mathrm{d}x.
$$

Using the coercivity $(2.4)_2$, we obtain for the choice $m = \ell$ the energy estimate

$$
\iint_{\Omega_T} \big|Du_\ell\big|^p \le \tfrac1\nu\Big[T\int_\Omega g(x,0)\,\mathrm{d}x + \tfrac12\int_\Omega |u_{\mathrm{o}}|^2\,\mathrm{d}x\Big].
\tag{5.3}
$$

Moreover, for $\tau \in [0,T)$ we find $i \in \{1,\dots,\ell\}$ such that $\tau \in I_{\ell,i}$. Therefore, applying Lemma 3.4 to $u_{\ell,i}$ (more precisely, we use the first estimate from Lemma 3.4), we find that

$$
\begin{aligned}
\int_\Omega \big|u_\ell(\tau)\big|^2\mathrm{d}x &\le 2h_\ell\int_\Omega g(x,0)\,\mathrm{d}x + \int_\Omega \big|u_\ell(t_{\ell,i-1})\big|^2\mathrm{d}x \\
&\le 4T\int_\Omega g(x,0)\,\mathrm{d}x + \int_\Omega |u_{\mathrm{o}}|^2\,\mathrm{d}x,
\end{aligned}
$$

which implies the L^∞–L^2-estimate

$$
\sup_{t\in[0,T)} \|u_\ell(\cdot,t)\|^2_{L^2(\Omega,\mathbb{R}^N)} \le 4T\int_\Omega g(x,0)\,\mathrm{d}x + \|u_{\mathrm{o}}\|^2_{L^2(\Omega,\mathbb{R}^N)}.
\tag{5.4}
$$

Together, the inequalities (5.3) and (5.4) ensure that the sequence $(u_\ell)_{\ell\in\mathbb{N}}$ is uniformly bounded in $L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big) \cap L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big)$. Therefore, there exists a subsequence $\mathfrak{K} \subset \mathbb{N}$ and a map

$$
u \in L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big) \cap L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big)
\tag{5.5}
$$

such that in the limit $\mathfrak{K} \ni \ell \to \infty$ we have

$$\begin{cases} u_\ell \xrightharpoonup{*} u & \text{weakly}^* \text{ in } L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big), \\ u_\ell \rightharpoonup u & \text{weakly in } L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big). \end{cases}$$

5.1.3. Boundary values. The main subject of the section is to show that the limit map u from (5.5) possesses the correct boundary values, i.e., that $u \equiv 0$ a.e. in $\Omega_T \setminus \overline{E}$ which in turn implies $u \in V_2^p(E)$. To this end, for $\ell \in \mathbb{N}$ we define

$$E^{(\ell)} := \bigcup_{i=1}^{\ell} Q_{\ell,i}.$$

By construction, we know that $u_\ell \equiv 0$ a.e. in $\Omega_T \setminus E^{(\ell)}$ and that $E \subset E^{(\ell)}$. Now let $z_o := (x_o, t_o) \in \Omega_T \setminus \overline{E}$. We claim that the following holds true:

$$\exists\, \varepsilon > 0, \ell_o \in \mathbb{N} \text{ such that } \forall \ell \geq \ell_o : Q_\varepsilon(z_o) \subset \Omega_T \setminus \overline{E^{(\ell)}}. \tag{5.6}$$

Here, we used the shorthand notation $Q_\varrho(z_o) := B_\varrho(x_o) \times \Lambda_\rho(t_o)$, where $\Lambda_\rho(t_o) := (t_o - \rho, t_o + \rho) \cap [0,T)$. Indeed, since $z_o \in \Omega_T \setminus \overline{E}$ there exists $\varepsilon > 0$ such that $Q_{2\varepsilon}(z_o) \subset \Omega_T \setminus \overline{E}$. This, however, implies that $B_{2\varepsilon}(x_o) \subset \Omega \setminus \overline{E^t}$ for any $t \in \Lambda_{2\varepsilon}(t_o)$. But that means

$$B_{2\varepsilon}(x_o) \subset \bigcap_{t \in \Lambda_{2\varepsilon}(t_o)} \big(\Omega \setminus \overline{E^t}\big) = \Omega \Big\backslash \bigcup_{t \in \Lambda_{2\varepsilon}(t_o)} \overline{E^t}.$$

Of course, this implies that

$$B_\varepsilon(x_o) \subset \Omega \Big\backslash \overline{\bigcup_{t \in \Lambda_{2\varepsilon}(t_o)} \overline{E^t}} \subset \Omega \Big\backslash \overline{\bigcup_{t \in \Lambda_{2\varepsilon}(t_o)} E^t}.$$

Furthermore, if one assumes that $h_\ell < \varepsilon$ (which is true for any $\ell \geq \frac{T}{\varepsilon}$), then one finds indices $1 \leq i_o < i_1 \leq \ell$ such that

$$\Lambda_\varepsilon(t_o) \subset \bigcup_{i_o \leq i \leq i_1} I_{\ell,i} \subset \Lambda_{2\varepsilon}(t_o).$$

But this implies for any index $i \in \{i_o, \dots, i_1\}$ that

$$\begin{aligned} B_\varepsilon(x_o) \times I_{\ell,i} &\subset \Bigg[\Omega \Big\backslash \overline{\bigcup_{t \in \Lambda_{2\varepsilon}(t_o)} E^t}\Bigg] \times I_{\ell,i} \subset \Bigg[\Omega \Big\backslash \overline{\bigcup_{t \in I_{\ell,i}} E^t}\Bigg] \times I_{\ell,i} \\ &= \Omega \times I_{\ell,i} \Big\backslash \Bigg[\overline{\bigcup_{t \in I_{\ell,i}} E^t} \times I_{\ell,i}\Bigg] = \Omega \times I_{\ell,i} \setminus \overline{Q_{\ell,i}} \end{aligned}$$

holds true. This implies that

$$Q_\varepsilon(z_o) \subset B_\varepsilon(x_o) \times \bigcup_{i_o \leq i \leq i_1} I_{\ell,i} \subset \bigcup_{i_o \leq i \leq i_1} \Big[\Omega \times I_{\ell,i} \setminus \overline{Q_{\ell,i}}\Big] \subset \Omega_T \setminus \overline{E^{(\ell)}}.$$

Hence, claim (5.6) follows for any choice of $\ell_o \in \mathbb{N}$ with $\ell_o > \frac{T}{\varepsilon}$.

Now $(4.7)_2$ ensures that $u_\ell \rightharpoonup u$ weakly in $L^p\big(Q_\varepsilon(z_o), \mathbb{R}^N\big)$. By lower semicontinuity, we conclude that

$$\iint_{Q_\varepsilon(z_o)} |u|^p \,\mathrm{d}x\mathrm{d}t \le \liminf_{\ell\to\infty} \iint_{Q_\varepsilon(z_o)} |u_\ell|^p \,\mathrm{d}x\mathrm{d}t = 0,$$

i.e., $u \equiv 0$ a.e. in $Q_\varepsilon(z_o)$. Since $z_o \in \Omega_T \setminus \overline{E}$ was arbitrary, this ensures that $u \equiv 0$ a.e. in $\Omega_T \setminus \overline{E}$. Now, we argue exactly as in section 4.1.4 to conclude that $u \in V_2^p(E)$, i.e., u fulfills the Dirichlet boundary condition in the sense of (2.2).

5.1.4. Variational inequality for the limit map. In this section we shall establish that the limit map u from (5.5) solves the variational inequality in the sense of Definition 2.1. We consider $v \in V_2^p(E)$ with $\partial_t v \in L^2(\Omega_T, \mathbb{R}^N)$. Note that $v \in C^0\big([0,T); L^2(\Omega,\mathbb{R}^N)\big)$. With $v_{\ell,i}\colon \Omega \times I_{\ell,i} \to \mathbb{R}^N$ we denote for $\ell \in \mathbb{N}$ and $i \in \{1,\dots,\ell\}$ the restriction of v to the subcylinder $\Omega \times I_{\ell,i}$. Then, by (5.1) $v_{\ell,i}$ is an admissible comparison map in (5.2) and by the very definition of $u_{\ell,i}$ and the inequality

$$\begin{aligned}\tfrac12\big\|v(t_{\ell,i-1}) - w_{\ell,i}\big\|^2_{L^2(\Omega,\mathbb{R}^N)} &= \tfrac12\big\|\big[v(t_{\ell,i-1}) - u_{\ell,i-1}(t_{\ell,i-1})\big]\chi_{E_{\ell,i}}\big\|^2_{L^2(\Omega,\mathbb{R}^N)}\\ &\le \tfrac12\big\|v(t_{\ell,i-1}) - u_{\ell,i-1}(t_{\ell,i-1})\big\|^2_{L^2(\Omega,\mathbb{R}^N)}\end{aligned}$$

note that by construction $E^{t_{\ell,i-1}} \subset E_{\ell,i}$, we get for any $i \in \{1,\dots,\ell\}$ that

$$\begin{aligned}\tfrac12\big\|v_{\ell,i}(\tau) - u_{\ell,i}(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} &+ \iint_{\Omega\times[t_{\ell,i-1},\tau)} f\big(x,u_{\ell,i},Du_{\ell,i}\big)\,\mathrm{d}x\mathrm{d}t\\ &\le \iint_{\Omega\times[t_{\ell,i-1},\tau)} \Big[\partial_t v_{\ell,i}\cdot\big(v_{\ell,i}-u_{\ell,i}\big) + f\big(x,v_{\ell,i},Dv_{\ell,i}\big)\Big]\mathrm{d}x\mathrm{d}t\\ &\quad + \tfrac12\big\|v_{\ell,i}(t_{\ell,i-1}) - u_{\ell,i-1}(t_{\ell,i-1})\big\|^2_{L^2(\Omega,\mathbb{R}^N)}\end{aligned} \tag{5.7}$$

for a.e. $\tau \in I_{\ell,i}$. We recall that $u_{\ell,0}(t_{\ell,0}) \equiv u_o$. Now, for a fixed $\sigma \in [0,T)$ we find $m \in \{0,\dots,\ell-1\}$ such that $\sigma \in [t_{\ell,m}, t_{\ell,m+1}) = I_{\ell,m+1}$. We sum up the inequalities $(5.7)_i$ with $\tau = t_{\ell,i}$ from $i = 1,\dots,m$ (which means that $\Omega\times[t_{\ell,i-1},\tau) \equiv \Omega \times I_{\ell,i}$) and keep in mind that $v_{\ell,i}(t_{\ell,i-1}) = v_{\ell,i-1}(t_{\ell,i-1})$. This leads us to

$$\begin{aligned}\tfrac12\big\|v(t_{\ell,m}) - u_\ell(t_{\ell,m})\big\|^2_{L^2(\Omega,\mathbb{R}^N)} &+ \iint_{\Omega_{t_{\ell,m}}} f\big(x,u_\ell,Du_\ell\big)\,\mathrm{d}x\mathrm{d}t\\ &\le \iint_{\Omega_{t_{\ell,m}}} \Big[\partial_t v\cdot\big(v-u_\ell\big) + f(x,v,Dv)\Big]\mathrm{d}x\mathrm{d}t + \tfrac12\|v(0)-u_o\|^2_{L^2(\Omega,\mathbb{R}^N)}.\end{aligned}$$

To the preceding inequality we add the inequality $(5.7)_{m+1}$ with $\tau = \sigma$ and obtain

$$\begin{aligned}\tfrac12\big\|v(\sigma) - u_\ell(\sigma)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} &+ \iint_{\Omega_\sigma} f\big(x,u_\ell,Du_\ell\big)\,\mathrm{d}x\mathrm{d}t\\ &\le \iint_{\Omega_\sigma} \Big[\partial_t v\cdot\big(v-u_\ell\big) + f(x,v,Dv)\Big]\mathrm{d}x\mathrm{d}t + \tfrac12\|v(0)-u_o\|^2_{L^2(\Omega,\mathbb{R}^N)}\end{aligned}$$

for a.e. $\sigma \in [0,T)$ and any comparison function v as defined above. Integrating this inequality with respect to σ over $[t_o, t_o+\delta) \subset [0,T)$ and dividing by δ, we obtain that

$$\begin{aligned}\iint_{\Omega_{t_o}} f\big(x,u_\ell,Du_\ell\big)\,\mathrm{d}x\mathrm{d}t &+ \tfrac{1}{2\delta}\iint_{\Omega\times(t_o-h_\ell,t_o+\delta)} \big|v-u_\ell\big|^2\mathrm{d}x\mathrm{d}t\\ &\le \fint_{t_o}^{t_o+\delta}\iint_{\Omega_\sigma} \Big[\partial_t v\cdot\big(v-u_\ell\big) + f(x,v,Dv)\Big]\mathrm{d}x\mathrm{d}t\mathrm{d}\sigma + \tfrac12\big\|v(0)-u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)}.\end{aligned}$$

On the left-hand side we use the lower semicontinuity of the first integral with respect to the weak convergence of $u_\ell \rightharpoonup u$ in $L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$, respectively, the lower semicontinuity of the second integral with respect to the weak convergence $u_\ell \rightharpoonup u$ in $L^2(\Omega_T,\mathbb{R}^N)$. For the first integral on the right-hand side, we use the weak convergence $u_\ell \rightharpoonup u$ in $L^2\big(\Omega_\tau,\mathbb{R}^N\big)$. Therefore, in the limit $\mathfrak{K} \ni \ell \to \infty$ we obtain that

$$\begin{aligned}
&\iint_{\Omega_{t_o}} f(x,u,Du)\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{2\delta}\iint_{\Omega\times(t_o,t_o+\delta)} |v-u|^2\,\mathrm{d}x\mathrm{d}t\\
&\qquad \le \fint_{t_o}^{t_o+\delta}\iint_{\Omega_\sigma}\Big[\partial_t v\cdot(v-u)+f(x,v,Dv)\Big]\mathrm{d}x\mathrm{d}t\mathrm{d}\sigma + \tfrac12\big\|v(0)-u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)}.
\end{aligned}$$

Here, we pass to the limit $\delta\downarrow 0$. This implies that for a.e. $t_o\in[0,T)$ there holds

$$\begin{aligned}
&\iint_{\Omega_{t_o}} f(x,u,Du)\,\mathrm{d}x\mathrm{d}t + \tfrac12\big\|(v-u)(t_o)\big\|^2_{L^2(\Omega,\mathbb{R}^N)}\\
&\qquad \le \iint_{\Omega_{t_o}}\Big[\partial_t v\cdot(v-u)+f(x,v,Dv)\Big]\mathrm{d}x\mathrm{d}t + \tfrac12\big\|v(0)-u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)}
\end{aligned}$$

for any $v\in V_2^p(E)$ with $\partial_t v\in L^2\big(\Omega_T,\mathbb{R}^N\big)$. This proves the claim that u solves the variational inequality (2.6) and finishes the proof of Theorem 2.6.

5.2. Time derivative in the dual space (Proof of Theorem 2.7). Under the stronger assumptions of the Theorem, we are able to improve the energy estimates obtained in section 5.1.2. To this aim, we will use the notation of the proof of Theorem 2.6 and comment only on the additional information that can be proved for the approximating functions u_ℓ and the limit function u. From Lemma 4.5 we know that $\partial_t u_{\ell,i}\in\big(V^p(Q_{\ell,i})\big)'$ for any $\ell\in\mathbb{N}$ and $i\in\{1,\dots,\ell\}$, and that for any $\varphi\in V^p(Q_{\ell,i})$ the estimate

$$\left|\iint_{Q_{\ell,i}} u_{\ell,i}\cdot\partial_t\varphi\,\mathrm{d}x\mathrm{d}t\right| \le c\iint_{Q_{\ell,i}}\Big[\big|Du_{\ell,i}\big|^{p-1}+\big|u_{\ell,i}\big|^{p-1}+|G|\Big]\big[|D\varphi|+|\varphi|\big]\mathrm{d}x\mathrm{d}t$$

holds true with a constant $c=c(p,L)$. Now, we consider $\varphi\in C_0^\infty(E)$. For $\ell\in\mathbb{N}$ and $\varepsilon\in(0,\frac{h_\ell}{4}]$ we define cutoff functions $\zeta_{\ell,i}^{(\varepsilon)}\in C^{0,1}(0,T)$, $i\in\{1,\dots,\ell\}$ by

$$\zeta_{\ell,i}^{(\varepsilon)}(t) := \begin{cases} 0 & \text{for } t\in[0,t_{\ell,i-1}],\\ \frac{t-t_{\ell,i-1}}{\varepsilon} & \text{for } t\in(t_{\ell,i-1},t_{\ell,i-1}+\varepsilon),\\ 1 & \text{for } t\in[t_{\ell,i-1}+\varepsilon,t_{\ell,i}-\varepsilon],\\ \frac{t_{\ell,i}-t}{\varepsilon} & \text{for } t\in(t_{\ell,i}-\varepsilon,t_{\ell,i}),\\ 0 & \text{for } t\in[t_{\ell,i},T].\end{cases}$$

Since $E\cap(\Omega\times I_{\ell,i})\subset Q_{\ell,i}$, we have that $\varphi_{\ell,i}^{(\varepsilon)} := \zeta_{\ell,i}^{(\varepsilon)}\varphi\in C_0^{0,1}(Q_{\ell,i})$ for any $i\in\{1,\dots,\ell\}$. By an approximation argument we see that $\varphi_{\ell,i}^{(\varepsilon)}$ is admissible above, so that

$$\begin{aligned}
&\left|\iint_{Q_{\ell,i}} u_{\ell,i}\cdot\partial_t\varphi_{\ell,i}^{(\varepsilon)}\,\mathrm{d}x\mathrm{d}t\right|\\
&\qquad\le c\iint_{Q_{\ell,i}}\Big[\big|Du_{\ell,i}\big|^{p-1}+\big|u_{\ell,i}\big|^{p-1}+|G|\Big]\Big[\big|D\varphi_{\ell,i}^{(\varepsilon)}\big|+\big|\varphi_{\ell,i}^{(\varepsilon)}\big|\Big]\mathrm{d}x\mathrm{d}t\\
&\qquad\le c\iint_{Q_{\ell,i}}\Big[\big|Du_{\ell,i}\big|^{p-1}+\big|u_{\ell,i}\big|^{p-1}+|G|\Big]\big[|D\varphi|+|\varphi|\big]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Summing up from $i=1,\dots,\ell$ yields

$$
\begin{aligned}
\bigg|\iint_{\Omega_T} & u_\ell\cdot\partial_t\varphi\,\mathrm{d}x\mathrm{d}t\bigg| \\
&=\bigg|\sum_{i=1}^{\ell}\iint_{\Omega\times I_{\ell,i}} u_\ell\cdot\partial_t\varphi\,\mathrm{d}x\mathrm{d}t\bigg| \\
&\le\sum_{i=1}^{\ell}\bigg|\iint_{Q_{\ell,i}} u_{\ell,i}\cdot\partial_t\varphi^{(\varepsilon)}_{\ell,i}\,\mathrm{d}x\mathrm{d}t\bigg|+\mathrm{R}^{(\varepsilon)}_\ell \\
&\le c\sum_{i=1}^{\ell}\iint_{Q_{\ell,i}}\Big[\big|Du_{\ell,i}\big|^{p-1}+\big|u_{\ell,i}\big|^{p-1}+|G|\Big]\big[|D\varphi|+|\varphi|\big]\,\mathrm{d}x\mathrm{d}t+\mathrm{R}^{(\varepsilon)}_\ell \\
&=c\iint_{\Omega_T}\Big[\big|Du_\ell\big|^{p-1}+\big|u_\ell\big|^{p-1}+|G|\Big]\big[|D\varphi|+|\varphi|\big]\,\mathrm{d}x\mathrm{d}t+\mathrm{R}^{(\varepsilon)}_\ell,
\end{aligned}
$$

where the remainder term is defined by

$$
\mathrm{R}^{(\varepsilon)}_\ell:=\bigg|\sum_{i=1}^{\ell}\iint_{Q_{\ell,i}} u_{\ell,i}\cdot\Big(\partial_t\varphi-\partial_t\varphi^{(\varepsilon)}_{\ell,i}\Big)\,\mathrm{d}x\mathrm{d}t\bigg|.
$$

For the remainder $\mathrm{R}^{(\varepsilon)}_\ell$, we have that

$$
\begin{aligned}
\mathrm{R}^{(\varepsilon)}_\ell&=\bigg|\sum_{i=1}^{\ell}\iint_{Q_{\ell,i}}\Big[\big(1-\zeta^{(\varepsilon)}_{\ell,i}\big)u_{\ell,i}\cdot\partial_t\varphi-\partial_t\zeta^{(\varepsilon)}_{\ell,i}u_{\ell,i}\cdot\varphi\Big]\,\mathrm{d}x\mathrm{d}t\bigg| \\
&\to\bigg|\sum_{i=1}^{\ell}\bigg[\int_\Omega u_{\ell,i}(t_{\ell,i})\cdot\varphi(t_{\ell,i})\,\mathrm{d}x-\int_\Omega u_{\ell,i}(t_{\ell,i-1})\cdot\varphi(t_{\ell,i-1})\,\mathrm{d}x\bigg]\bigg|,
\end{aligned}
$$

in the limit $\varepsilon\downarrow 0$, where we used the regularity $u_{\ell,i}\in C^0\big(\overline{I_{\ell,i}};L^2(E_{\ell,i},\mathbb{R}^N)\big)$. Now, since $\operatorname{spt}\varphi(t_{\ell,i-1})\subset E_{\ell,i}$ we conclude that

$$
u_{\ell,i}(t_{\ell,i-1})\cdot\varphi(t_{\ell,i-1})=u_{\ell,i-1}(t_{\ell,i-1})\chi_{E_{\ell,i}}\cdot\varphi(t_{\ell,i-1})=u_{\ell,i-1}(t_{\ell,i-1})\cdot\varphi(t_{\ell,i-1}).
$$

Furthermore, since $\varphi\in C^\infty_0(E)$ we have $u_{\ell,1}(0)\cdot\varphi(0)=0=u_{\ell,\ell}(T)\cdot\varphi(T)$. We use these informations in the formula for the remainder from above, to conclude that the right-hand side vanishes, i.e., we have that $\lim_{\varepsilon\downarrow 0}\mathrm{R}^{(\varepsilon)}_\ell=0$. Inserting this above and using Hölder's inequality and Poincaré's inequality slice-wise, we find that

$$
\begin{aligned}
\bigg|\iint_{\Omega_T} u_\ell\cdot\partial_t\varphi\,\mathrm{d}x\mathrm{d}t\bigg|&\le c\iint_{\Omega_T}\Big[\big|Du_\ell\big|^{p-1}+\big|u_\ell\big|^{p-1}+|G|\Big]\big[|D\varphi|+|\varphi|\big]\,\mathrm{d}x\mathrm{d}t \\
&\le c\bigg[\iint_{\Omega_T}\Big[\big|Du_\ell\big|^{p}+\big|u_\ell\big|^{p}+|G|^{p'}\Big]\,\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{p'}}\|\varphi\|_{V^p(E)} \\
&\le c\bigg[\iint_{\Omega_T}\Big[\big|Du_\ell\big|^{p}+|G|^{p'}\Big]\,\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{p'}}\|\varphi\|_{V^p(E)},
\end{aligned}
$$

where $c=c(p,L,\operatorname{diam}(\Omega))$. At this point we use inequality (5.3) with the choice $g(x,M):=L\big(2M^p+|G(x)|\big)$, which is possible under assumption (2.13). In this way, we obtain

$$\left|\iint_{\Omega_T} u_\ell \cdot \partial_t\varphi \mathrm{d}x\mathrm{d}t\right| \le c\left[T\int_\Omega |G|\mathrm{d}x + \iint_{\Omega_T}|G|^{p'}\mathrm{d}x\mathrm{d}t + \int_\Omega |u_{\mathrm{o}}|^2\mathrm{d}x\right]^{\frac{1}{p'}}\|\varphi\|_{V^p(E)}$$

for any $\varphi \in C_0^\infty(E,\mathbb{R}^N)$, where $c = c(p,\nu,L,\mathrm{diam}(\Omega))$. Since $u_\ell \rightharpoonup u$ weakly in $L^p(\Omega_T,\mathbb{R}^N)$, we may pass to the limit $\ell\to\infty$ on the left-hand side of the preceding inequality and infer that

$$\left|\iint_{\Omega_T} u \cdot \partial_t\varphi \mathrm{d}x\mathrm{d}t\right| \le c\left[T\int_\Omega \big(|G| + |G|^{p'}\big)\mathrm{d}x + \int_\Omega |u_{\mathrm{o}}|^2\mathrm{d}x\right]^{\frac{1}{p'}}\|\varphi\|_{V^p(E)}$$

holds true for any $\varphi \in C_0^\infty(E,\mathbb{R}^N)$. From Lemma 5.3, more precisely from Remark 5.4 below, we know that $C_0^\infty(E,\mathbb{R}^N)$ is dense in $V^p(E)$. We have thus proved that $\partial_t u \in \big(V^p(E)\big)'$ together with the estimate

$$\|\partial_t u\|_{(V^p(E))'} \le c\left[T\int_\Omega \big(|G| + |G|^{p'}\big)\mathrm{d}x + \|u_{\mathrm{o}}\|^2_{L^2(E^0,\mathbb{R}^N)}\right]^{\frac{1}{p'}},$$

where $c = c(p,\nu,L,\mathrm{diam}(\Omega))$.

5.3. Continuity in time. The aim of this section is to prove continuity in time direction for variational solutions. Since we are in the setting of general, possibly decreasing domains it is not likely that these properties hold under the weak assumptions of Theorem 2.6. Therefore, throughout this section we assume that the integrand satisfies the p-growth condition (2.13) and that the domain E additionally fulfills assumptions (2.10) and (2.12).

5.3.1. Density of $C_0^\infty(E,\mathbb{R}^N)$ in $V_2^p(E)$. Our goal is to prove density of $C_0^\infty(E,\mathbb{R}^N)$ in $V_2^p(E)$. We begin with a result for the intermediate space

$$V^{p,2}_{cpt}(E) := \big\{u \in V_2^p(E) : \exists\, \sigma > 0 \text{ with } \mathrm{spt}(u(t)) \subset \overline{E^{t,\sigma}} \text{ for a.e. } t\in[0,T)\big\},$$

where $E^{t,\sigma}$ denotes the inner parallel set of E^t; see (3.1).

LEMMA 5.1. *Assume that* (2.12) *is in force. Then the space* $C_0^\infty(E,\mathbb{R}^N)$ *is dense in* $V^{p,2}_{cpt}(E)$ *with respect to the norm* $\|\cdot\|_{V_2^p}$ *defined in* (2.3). *Moreover, if* $u \in V^{p,2}_{cpt}(E)$ *is given, then we can choose an approximating sequence* $\varphi_k \in C_0^\infty(E,\mathbb{R}^N)$ *with* $\varphi_k \to u$ *in* $V_2^p(E)$ *in such a way that there exists* $\sigma > 0$ *with* $\mathrm{spt}(\varphi_k(t)) \subset \overline{E^{t,\sigma}}$ *for every* $k\in\mathbb{N}$ *and* $t\in[0,T)$.

Proof. By definition, for a given $u \in V^{p,2}_{cpt}(E)$ there exists a $\sigma > 0$ with $\mathrm{spt}(u(t)) \subset \overline{E^{t,2\sigma}}$ for a.e. $t\in[0,T)$. Moreover, we may assume that $u\equiv 0$ on $\Omega\times[0,2\delta]$ and on $\Omega\times[T-2\delta,T]$ for some $\delta>0$, because the space of such maps is dense in $V^{p,2}_{cpt}(E)$ with respect to the norm in $V_2^p(E)$. We choose a backward-in-time mollifying kernel $\zeta\in C_0^\infty\big(B_1\times(0,1),[0,\infty)\big)$ with $\iint\zeta\,\mathrm{d}x\mathrm{d}t = 1$ and let $\zeta_\varepsilon(x,t) := \varepsilon^{-n-1}\zeta\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\big)$ for $0<\varepsilon<\min\{\delta,\frac{\sigma}{M+1}\}$. The maps $u_\varepsilon := u*\zeta_\varepsilon$ satisfy $u_\varepsilon\in C^\infty(\Omega_T,\mathbb{R}^N)$ and converge to u strongly in $L^p\big(0,T;W_0^{1,p}(\Omega,\mathbb{R}^N)\big)$ and in $L^2(\Omega_T)$. We claim that $\mathrm{spt}(u_\varepsilon(t))\subset\overline{E^{t,\sigma}}$ for every $t\in[\delta,T-\delta]$. For the proof of this claim, we fix a time $t\in[\delta,T-\delta]$ and $x\in\Omega\setminus\overline{E^{t,\sigma}}$ and note that

$$u_\varepsilon(x,t) = \iint_{B_\varepsilon(x)\times(t-\varepsilon,t)} u(y,s)\zeta_\varepsilon(x-y,t-s)\mathrm{d}y\mathrm{d}s. \tag{5.8}$$

Next, we consider a time $s \in (t-\varepsilon, t)$ and a point $y \in B_\varepsilon(x)$. Since $u = 0$ almost everywhere on $(B_\varepsilon(x) \times (t-\varepsilon, t)) \setminus E$, it is enough to consider the case $y \in B_\varepsilon(x) \cap E^s$. We choose a point $\tilde{x} \in \Omega \setminus E^t$ with $\operatorname{dist}(x, \Omega \setminus E^t) = |x - \tilde{x}|$. Since $s < t$, we can exploit condition (2.12) to estimate

$$\begin{aligned}
\operatorname{dist}(y, \partial E^s) &= \operatorname{dist}(y, \Omega \setminus E^s) \\
&\leq \operatorname{dist}(\tilde{x}, \Omega \setminus E^s) + |y - \tilde{x}| \\
&\leq \mathbf{e}^c(E^s, E^t) + |y - x| + |x - \tilde{x}| \\
&\leq M(t-s) + |y - x| + \operatorname{dist}(x, \Omega \setminus E^t) \\
&\leq (M+1)\varepsilon + \sigma.
\end{aligned}$$

Since $\varepsilon < \frac{\sigma}{M+1}$, this implies $y \in E^s \setminus \overline{E^{s,2\sigma}}$, on which set the function $u(s)$ vanishes almost everywhere for a.e. $s \in (t-\varepsilon, t)$. We conclude that the integrand in (5.8) vanishes almost everywhere, so that $u_\varepsilon(x,t) = 0$ if $t \in [\delta, T-\delta]$ and $x \in \Omega \setminus \overline{E^{t,\sigma}}$. Moreover, because of $\varepsilon < \delta$ it is clear from the construction that $u_\varepsilon \equiv 0$ on $\Omega \times [0, \delta]$ and on $\Omega \times [T-\delta, T)$. This proves that $\operatorname{spt}(u_\varepsilon(t)) \subset \overline{E^{t,\sigma}}$ holds for a.e. $t \in [0,T)$. In particular, we know $u_\varepsilon = 0$ on ∂E. Therefore, by a standard cutoff argument, the maps u_ε can be approximated by maps in $C_0^\infty(E, \mathbb{R}^N)$, i.e., by maps with compact support in E, in the $C^1(E, \mathbb{R}^N)$-norm. This implies that $C_0^\infty(E, \mathbb{R}^N)$ is dense in $V_{cpt}^{p,2}(E, \mathbb{R}^N)$ with respect to the norm in $V_2^p(E)$. The second claim is clear from the construction. ∎

The next step is to prove that $V_{cpt}^{p,2}(E)$ is dense in $V_2^p(E)$. For this purpose, we define a cutoff function $\tilde{\eta} \in C^{0,1}(\mathbb{R})$ by letting $\tilde{\eta} \equiv 0$ on $(-\infty, 1]$, $\tilde{\eta}(r) := r - 1$ for $r \in (1,2)$ and $\tilde{\eta} \equiv 1$ on $[2, \infty)$. For any $\sigma > 0$, we define

$$\eta_\sigma(x,t) := \tilde{\eta}\left(\frac{\operatorname{dist}(x, \Omega \setminus E^t)}{\sigma}\right) \qquad \text{for } (x,t) \in \Omega_T. \tag{5.9}$$

These cutoff functions are useful for our purposes because of the following result.

LEMMA 5.2. *Assume that E satisfies* (2.10)*, let $u \in V_2^p(E)$, and consider the cutoff functions η_σ introduced in* (5.9)*. Then, we have*

$$\eta_\sigma u \rightharpoonup u \text{ weakly in } V_2^p(E) \text{ as } \sigma \downarrow 0.$$

Proof. First, we observe that $\eta_\sigma u \to u$ strongly in $L^{\max\{p,2\}}(E, \mathbb{R}^N)$, in the limit $\sigma \downarrow 0$. Next, using the bound $|D\eta_\sigma| \leq \frac{1}{\sigma}$ and Hardy's inequality from Lemma 3.2, we estimate

$$\begin{aligned}
\iint_{\Omega_T} |D(\eta_\sigma u)|^p \,\mathrm{d}x\mathrm{d}t &\leq 2^{p-1} \iint_E |Du|^p \,\mathrm{d}x\mathrm{d}t + 2^{p-1} \int_0^T \int_{E^{t,\sigma} \setminus E^{t,2\sigma}} |D\eta_\sigma|^p |u|^p \,\mathrm{d}x\mathrm{d}t \\
&\leq 2^{p-1} \iint_E |Du|^p \,\mathrm{d}x\mathrm{d}t + 2^{2p-1} \int_0^T \int_{E^t} \left(\frac{|u|}{\operatorname{dist}(x, \partial E^t)}\right)^p \mathrm{d}x\mathrm{d}t \\
&\leq c(n,p,\delta) \iint_E |Du|^p \,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

We deduce that the family $(\eta_\sigma u)_{\sigma > 0}$ is bounded in $V_2^p(E)$. In view of the convergence $\eta_\sigma u \to u$ in $L^{\max\{p,2\}}(E, \mathbb{R}^N)$, this implies the asserted weak convergence. ∎

LEMMA 5.3. *Assume that assumptions* (2.10) *and* (2.12) *are in force. Then, the space $C_0^\infty(E, \mathbb{R}^N)$ is dense in $V_2^p(E)$ with respect to the norm topology in $V_2^p(E)$.*

Proof. We fix a map $u \in V_2^p(E)$ and let $u_\sigma := \eta_\sigma u$ for $\sigma > 0$, with the cutoff functions η_σ defined in (5.9). Then, we clearly have $u_\sigma \in V_{cpt}^{p,2}(E)$, and under assumption (2.10), Lemma 5.2 yields the convergence $u_\sigma \rightharpoonup u$ weakly in $V_2^p(E)$, as $\sigma \downarrow 0$. Therefore, Mazur's lemma provides us with convex combinations $v_k := \sum_{\ell=1}^{N_k} \lambda_\ell u_{\sigma_\ell}$ for suitable $N_k \in \mathbb{N}$, $\lambda_\ell \in [0,1]$, and $\sigma_\ell > 0$ for $\ell \in \{1, \dots, N_k\}$ such that $v_k \to u$ strongly in $V_2^p(E)$. Since $v_k \in V_{cpt}^{p,2}(E)$ for each $k \in \mathbb{N}$, this proves that $V_{cpt}^{p,2}(E)$ is dense in $V_2^p(E)$. Since we know from Lemma 5.1 that (2.12) is sufficient to ensure the density of $C_0^\infty(E, \mathbb{R}^N)$ in $V_{cpt}^{p,2}(E)$, the proof is complete. ∎

Remark 5.4. The same arguments also show that $C_0^\infty(E, \mathbb{R}^N)$ is dense in $V^p(E)$ with respect to the norm topology in $V^p(E)$. □

5.3.2. An integration by parts formula. Here and in the following $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(V_2^p(E))'$ and $V_2^p(E)$. The goal of this subsection is the proof of the following theorem.

THEOREM 5.5. *Suppose that E satisfies conditions* (2.10) *and* (2.12). *For $p > \frac{2(n+1)}{n+2}$ we consider a map $u \in V_2^p(E)$ with $\partial_t u \in \big(V_2^p(E)\big)'$. In the case $p < 2$, we assume, moreover, that $u \in L^\infty\big(0,T; L^2(\Omega, \mathbb{R}^N)\big)$. Then, we have*

$$\langle \partial_t u, \zeta u \rangle \le -\tfrac{1}{2} \iint_E \partial_t \zeta |u|^2 \,\mathrm{d}x\mathrm{d}t \tag{5.10}$$

for any cutoff function in time $\zeta \in C_0^{0,1}(0,T)$ with $\zeta \ge 0$ on $(0,T)$.

The reason that there is no equality in the partial integration formula is due to the fact that we only excluded a too fast shrinking of the domain in time, but the domain may increase arbitrarily. In particular, E may jump outward. If we, oppositely, impose a condition on the increasing of the domain, we obtain the opposite inequality.

COROLLARY 5.6. *Suppose that the assumptions of the preceding theorem are in force, but with* (2.12) *replaced by the condition*

$$\mathbf{e}^c(E^t, E^s) \le M(t-s) \quad \textit{provided } 0 \le s \le t < T. \tag{5.11}$$

Then we have

$$\langle \partial_t u, \zeta u \rangle \ge -\tfrac{1}{2} \iint_E |u|^2 \partial_t \zeta \,\mathrm{d}x\mathrm{d}t \tag{5.12}$$

for any cutoff function in time $\zeta \in C_0^{0,1}(0,T)$ with $\zeta \ge 0$ on $(0,T)$.

Proof. We apply Theorem 5.5 to the maps $\tilde{u}(x,t) := u(x, T-t)$, which satisfy $\tilde{u} \in V^p(\widetilde{E})$ and $\partial_t \tilde{u} \in (V^p(\widetilde{E}))'$, where

$$\widetilde{E} := \bigcup_{t \in (0,T)} E^{T-t} \times \{t\}.$$

Under assumption (5.11) on E, the set $\widetilde{E}$ satisfies (2.12). Therefore, Theorem 5.5, applied with $\tilde{u}$ and $\tilde{\zeta}(t) := \zeta(T-t)$, implies

$$-\langle \partial_t u, \zeta u \rangle = \langle \partial_t \tilde{u}, \tilde{\zeta} \tilde{u} \rangle \le -\tfrac{1}{2} \iint_{\widetilde{E}} |\tilde{u}|^2 \partial_t \tilde{\zeta} \,\mathrm{d}x\mathrm{d}t = \tfrac{1}{2} \iint_E |u|^2 \partial_t \zeta \,\mathrm{d}x\mathrm{d}t.$$

This yields the asserted inequality (5.12). ∎

In the following lemma we prove the assertion of Theorem 5.5 under the additional assumption that $u \in V^{p,2}_{cpt}(E)$.

LEMMA 5.7. *Suppose that E satisfies condition* (2.12). *Then, the assertion of Theorem* 5.5 *holds true if we additionally assume that $u \in V^{p,2}_{cpt}(E)$.*

Proof. We define forward-in-time Steklov averages

$$[u]_h(x,t) := \tfrac{1}{h}\int_t^{t+h} u(x,s)\,\mathrm{d}s \quad \forall (x,t) \in \Omega_T,$$

where we extend u by zero on $\Omega \times [T, T+h]$ for the definition. We claim that

$$\lim_{h\downarrow 0} \iint_E \zeta\partial_t[u]_h \cdot u\,\mathrm{d}x\mathrm{d}t = \langle \partial_t u, \zeta u\rangle. \tag{5.13}$$

Lemma 5.1 provides us with an approximating sequence $C_0^\infty(E,\mathbb{R}^N) \ni \varphi_\ell \to \zeta u$ in $V_2^p(E)$ as $\ell \to \infty$ that has the property $\mathrm{spt}(\varphi_\ell(t)) \subset \overline{E^{t,\sigma}}$ for every $t \in [0,T)$, for some fixed $\sigma > 0$ and every $\ell \in \mathbb{N}$. Since ζ has compact support in $(0,T)$, we can also achieve, after diminishing $\sigma > 0$ if necessary, that $\mathrm{spt}(\varphi_\ell) \subset \Omega \times (\sigma, T-\sigma)$ for every $\ell \in \mathbb{N}$. Now, we decompose the difference of the two terms appearing in (5.13) as follows:

$$\begin{aligned}\iint_E \zeta\partial_t[u]_h \cdot u\,\mathrm{d}x\mathrm{d}t - \langle \partial_t u, \zeta u\rangle &= \Big[\iint_E \partial_t[u]_h \cdot \varphi_\ell\,\mathrm{d}x\mathrm{d}t - \langle \partial_t u, \varphi_\ell\rangle\Big] \\ &\quad + \iint_E \partial_t[u]_h \cdot \big(\zeta u - \varphi_\ell\big)\,\mathrm{d}x\mathrm{d}t + \langle \partial_t u, \varphi_\ell - \zeta u\rangle \\ &=: \mathrm{I}_{h,\ell} + \mathrm{II}_{h,\ell} + \mathrm{III}_\ell.\end{aligned} \tag{5.14}$$

Since $\mathrm{spt}(\varphi_\ell) \subset E$, we have

$$\begin{aligned}\iint_E \partial_t[u]_h \cdot \varphi_\ell\,\mathrm{d}x\mathrm{d}t &= \iint_{\Omega_T} \partial_t[u]_h \cdot \varphi_\ell\,\mathrm{d}x\mathrm{d}t = -\iint_{\Omega_T} [u]_h \cdot \partial_t\varphi_\ell\,\mathrm{d}x\mathrm{d}t \\ &\to -\iint_{\Omega_T} u \cdot \partial_t\varphi_\ell\,\mathrm{d}x\mathrm{d}t = -\iint_E u \cdot \partial_t\varphi_\ell\,\mathrm{d}x\mathrm{d}t = \langle \partial_t u, \varphi_\ell\rangle\end{aligned}$$

in the limit $h \downarrow 0$, for a fixed $\ell \in \mathbb{N}$. This proves

$$\lim_{h\downarrow 0} \mathrm{I}_{h,\ell} = 0 \quad \text{for any } \ell \in \mathbb{N}. \tag{5.15}$$

Next, we note that

$$\mathrm{II}_{h,\ell} = \lim_{m\to\infty} \mathrm{II}_{h,\ell,m}, \qquad \text{where } \mathrm{II}_{h,\ell,m} := \iint_E \partial_t[u]_h \cdot \big(\varphi_m - \varphi_\ell\big)\,\mathrm{d}x\mathrm{d}t.$$

Here, we cannot argue as for the term $\mathrm{I}_{h,\ell}$, since we are not allowed to interchange the order of limits, i.e., we first have to pass to the limit $m \to \infty$ and subsequently let $h \downarrow 0$. For $m, \ell \in \mathbb{N}$ and $0 < h < \sigma \min\{1, \frac{1}{M}\}$, we compute

$$\begin{aligned}\mathrm{II}_{h,\ell,m} &= \iint_{\Omega\times(h,T-h)} \partial_t[u]_h \cdot \big(\varphi_m - \varphi_\ell\big)\,\mathrm{d}x\mathrm{d}t \\ &= \tfrac{1}{h}\int_h^{T-h}\int_\Omega \big[u(t+h) - u(t)\big] \cdot \big(\varphi_m(t) - \varphi_\ell(t)\big)\,\mathrm{d}x\mathrm{d}t \\ &= \tfrac{1}{h}\int_h^{T}\int_\Omega u(t) \cdot \big[(\varphi_m - \varphi_\ell)(t-h) - (\varphi_m - \varphi_\ell)(t)\big]\,\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where we used $\operatorname{spt}(\varphi_k) \subset \Omega \times (h, T-h)$ for every $k \in \mathbb{N}$. By Fubini's theorem, we further obtain

$$
\begin{aligned}
\mathrm{II}_{h,\ell,m} &= -\tfrac{1}{h}\int_h^T \int_\Omega u(t) \cdot \int_0^h \partial_t(\varphi_m - \varphi_\ell)(t-s)\,\mathrm{d}s\,\mathrm{d}x\mathrm{d}t \\
&= -\tfrac{1}{h}\int_0^h \Big[\iint_{\Omega\times(h,T)} u(t)\cdot \partial_t(\varphi_m - \varphi_\ell)(t-s)\,\mathrm{d}x\mathrm{d}t\Big]\mathrm{d}s \\
&= \tfrac{1}{h}\int_0^h \big\langle \partial_t u, (\varphi_m - \varphi_\ell)(\cdot - s)\big\rangle\,\mathrm{d}s. \qquad (5.16)
\end{aligned}
$$

In order to justify the last identity, we need to check that $(\varphi_m - \varphi_\ell)(\cdot - s) \in V_2^p(E)$ for $s \in [0,h]$. For this aim, consider $t \in [0,T)$, $s \in [0,h]$, and $x \in \Omega \setminus E^t$. If $x \in \Omega \setminus E^{t-s}$, we have $(\varphi_m - \varphi_\ell)(x, t-s) = 0$. In the case $x \in E^{t-s}$, we use assumption (2.12) in order to estimate

$$
\operatorname{dist}(x, \partial E^{t-s}) = \operatorname{dist}(x, \Omega \setminus E^{t-s}) \le \mathbf{e}^c(E^{t-s}, E^t) \le Ms \le Mh < \sigma.
$$

This means, however, that $x \in \Omega \setminus \overline{E^{t-s,\sigma}}$, from which we infer $(\varphi_m - \varphi_\ell)(x, t-s) = 0$ if $x \in \Omega \setminus E^t$. Consequently, we have $(\varphi_m - \varphi_\ell)(\cdot - s) \in V_2^p(E)$, which therefore justifies the last line in (5.16). Hence, we can estimate

$$
\begin{aligned}
\big|\mathrm{II}_{h,\ell,m}\big| &\le \tfrac{1}{h}\int_0^h \|\partial_t u\|_{(V_2^p(E))'} \|(\varphi_m - \varphi_\ell)(\cdot - s)\|_{V_2^p(E)}\,\mathrm{d}s \\
&= \|\partial_t u\|_{(V_2^p(E))'} \|\varphi_m - \varphi_\ell\|_{V_2^p(E)}.
\end{aligned}
$$

Here we pass to the limit $m \to \infty$, and deduce

$$
\big|\mathrm{II}_{h,\ell}\big| \le \|\partial_t u\|_{(V_2^p(E))'} \|\varphi_\ell - \zeta u\|_{V_2^p(E)} \quad \text{for any } h > 0 \text{ and } \ell \in \mathbb{N}. \qquad (5.17)
$$

Finally, for the last term in (5.14) we also have the estimate

$$
|\mathrm{III}_\ell| \le \|\partial_t u\|_{(V_2^p(E))'} \|\varphi_\ell - \zeta u\|_{V_2^p(E)} \quad \text{for any } \ell \in \mathbb{N}. \qquad (5.18)
$$

Now, we pass to the limit in (5.14). In view of (5.15), (5.17), and (5.18), we get

$$
\limsup_{h\downarrow 0}\left|\iint_E \zeta \partial_t [u]_h \cdot u\,\mathrm{d}x\mathrm{d}t - \langle \partial_t u, \zeta u\rangle\right| \le 2\|\partial_t u\|_{(V_2^p(E))'} \|\varphi_\ell - \zeta u\|_{V_2^p(E)}.
$$

Since the right-hand side can be made arbitrarily small by choosing $\ell \in \mathbb{N}$ large enough, this proves the claim (5.13).

Next, we have a closer look at the left-hand side in (5.13). Since ζ has compact support in $(0,T)$, we may choose $h > 0$ small enough to ensure $\operatorname{spt}(\zeta) \subset [h, T-h]$. For such $h > 0$, we compute

$$
\begin{aligned}
&\iint_E \zeta \partial_t [u]_h \cdot u\,\mathrm{d}x\mathrm{d}t - \tfrac{1}{2h}\iint_E \big(\zeta(t-h) - \zeta(t)\big)|u(t)|^2\,\mathrm{d}x\mathrm{d}t \\
&\quad = \tfrac{1}{h}\int_0^T \int_{E^t} \Big[\zeta(t)\big(u(t+h) - u(t)\big)\cdot u(t) - \tfrac{1}{2}\big(\zeta(t-h) - \zeta(t)\big)|u(t)|^2\Big]\,\mathrm{d}x\mathrm{d}t \\
&\quad = -\tfrac{1}{2h}\int_0^T \int_{E^t} \zeta(t)\big|u(t+h) - u(t)\big|^2\,\mathrm{d}x\mathrm{d}t \\
&\qquad + \tfrac{1}{2h}\int_0^T \int_{E^t} \zeta(t)|u(t+h)|^2\,\mathrm{d}x\mathrm{d}t - \tfrac{1}{2h}\int_0^T \int_{E^t} \zeta(t-h)|u(t)|^2\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

In the second integral of the right side we perform a transformation of variables and use the fact that $u(x,t) = 0$ for a.e. $x \in \Omega \setminus E^t$. Keeping in mind that $\zeta \geq 0$, this yields the estimate

$$
\begin{aligned}
\int_0^T \int_{E^t} \zeta(t)|u(t+h)|^2 \,\mathrm{d}x\mathrm{d}t &= \int_0^{T-h} \int_{E^t} \zeta(t)|u(t+h)|^2 \,\mathrm{d}x\mathrm{d}t \\
&= \int_h^T \int_{E^{t-h}} \zeta(t-h)|u(t)|^2 \,\mathrm{d}x\mathrm{d}t = \int_h^T \int_{E^{t-h}\cap E^t} \zeta(t-h)|u(t)|^2 \,\mathrm{d}x\mathrm{d}t \\
&\leq \int_0^T \int_{E^t} \zeta(t-h)|u(t)|^2 \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Therefore, the right-hand side of the penultimate identity is less than zero and can therefore be discarded. We have thus shown that

$$
\begin{aligned}
\iint_E \zeta \partial_t [u]_h \cdot u \,\mathrm{d}x\mathrm{d}t &\leq \tfrac{1}{2h} \int_0^T \int_{E^t} \big(\zeta(t-h) - \zeta(t)\big)|u(x,t)|^2 \,\mathrm{d}x\mathrm{d}t \\
&\to -\tfrac{1}{2} \iint_E \partial_t \zeta |u|^2 \,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

in the limit $h \downarrow 0$. Combining this inequality with (5.13) yields

$$
\langle \partial_t u, \zeta u \rangle \leq -\tfrac{1}{2} \iint_E \partial_t \zeta |u|^2 \,\mathrm{d}x\mathrm{d}t.
$$

This proves the assertion of Theorem 5.5 under the additional assumption that $u \in V^{p,2}_{cpt}(E)$. ∎

Our aim now is to get rid of the extra assumption $u \in V^{p,2}_{cpt}(E)$. The first step is the following lemma.

LEMMA 5.8. *Assume that $p > 1$, that E satisfies* (2.12), *and that $u \in V^p_2(E)$ satisfies $\partial_t u \in \big(V^p_2(E)\big)'$. Moreover, let $\eta \in C^{0,1}(E)$ be given so that $\eta u \in V^{p,2}_{cpt}(E)$. Then we have*

$$
\langle \partial_t u, \zeta \eta^2 u \rangle \leq -\tfrac{1}{2} \iint_E \partial_t \zeta \eta^2 |u|^2 \,\mathrm{d}x\mathrm{d}t - \iint_E \zeta \eta \partial_t \eta |u|^2 \,\mathrm{d}x\mathrm{d}t
$$

for every cutoff function in time $\zeta \in C^{0,1}_0(0,T)$ with $\zeta \geq 0$ on $(0,T)$.

Proof. Lemma 5.3 provides us with a sequence $\varphi_\ell \in C^\infty_0(E,\mathbb{R}^N)$, $\ell \in \mathbb{N}$, that satisfies $\varphi_\ell \to \zeta u$ in $V^p_2(E)$, in the limit $\ell \to \infty$. Since $\eta \in C^{0,1}(E)$ and $u \in L^2(\Omega_T,\mathbb{R}^N)$, we have $\partial_t(\eta u) = \eta \partial_t u + u \partial_t \eta \in \big(V^p_2(E)\big)' + L^2(\Omega_T) = \big(V^p_2(E)\big)'$. Hence, we can calculate

$$
\begin{aligned}
\langle \partial_t u, \eta^2 \varphi_\ell \rangle &= -\iint_E u \cdot \partial_t(\eta^2 \varphi_\ell) \,\mathrm{d}x\mathrm{d}t \\
&= \langle \partial_t(\eta u), \eta \varphi_\ell \rangle - \iint_E \eta \partial_t \eta \, u \cdot \varphi_\ell \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{5.19}
$$

In the limit $\ell \to \infty$, the convergence $\varphi_\ell \to \zeta u$ in $V^p_2(E)$ implies

$$
\begin{cases}
\eta^2 \varphi_\ell \to \eta^2 \zeta u & \text{in } V^p_2(E), \\
\eta \varphi_\ell \to \eta \zeta u & \text{in } V^p_2(E), \\
\varphi_\ell \to \zeta u & \text{in } L^2(E,\mathbb{R}^N).
\end{cases}
$$

We can use this to pass to the limit $\ell \to \infty$ in (5.19), with the result that

$$\langle \partial_t u, \zeta\eta^2 u\rangle = \langle \partial_t(\eta u), \zeta\eta u\rangle - \iint_E \zeta\eta\partial_t\eta|u|^2 \,\mathrm{d}x\mathrm{d}t \tag{5.20}$$

holds true. At this point we apply the integration by parts inequality from Lemma 5.7 to $\eta u \in V^{p,2}_{cpt}(E)$. This yields the claimed inequality. ∎

Now, for $\sigma > 0$ we again use the functions η_σ defined in (5.9). For a parameter $h \in (0, \frac{\sigma}{2M})$ we define backward-in-time Steklov averages:

$$\eta_{\sigma,h}(x,t) := \tfrac{1}{h}\int_{t-h}^{t} \eta_\sigma(x,s)\,\mathrm{d}s. \tag{5.21}$$

In the following lemma we compile some useful properties of the functions $\eta_{\sigma,h}$.

LEMMA 5.9. *Assume that the domain E satisfies* (2.12). *Then for every $\sigma > 0$ and $h \in (0, \frac{\sigma}{2M})$, the functions $\eta_{\sigma,h}$ defined in* (5.21) *have the following properties:*

(i) *$\eta_{\sigma,h} \in C^{0,1}(E)$;*
(ii) *for a.e. $(x,t) \in E$ with $x \in E^{t,2\sigma}$ we have $\partial_t\eta_{\sigma,h}(x,t) \geq 0$;*
(iii) *$\partial_t\eta_{\sigma,h} \geq -\frac{M}{\sigma}$ a.e. on E;*
(iv) *for $t \in [0,T)$ and $x \in E^t \setminus E^{t,\frac{\sigma}{2}}$ we have $\eta_{\sigma,h}(x,t) = 0$.*

Proof. Since $\tilde\eta$ and $x \mapsto \operatorname{dist}(x, \partial E^t)$ are Lipschitz continuous, the maps $\eta_{\sigma,h}(x,t)$ depend Lipschitz continuously on the spatial variable x for every $t \in [0,T)$. Moreover, we compute

$$\partial_t\eta_{\sigma,h}(x,t) = \tfrac{1}{h}\big[\eta_\sigma(x,t) - \eta_\sigma(x,t-h)\big] \quad \text{for a.e. } (x,t) \in E, \tag{5.22}$$

which implies $|\partial_t\eta_{\sigma,h}| \leq \frac{1}{h}$ a.e. on E. Therefore, the maps $\eta_{\sigma,h}$ are also Lipschitz continuous in time. This proves (i).

The assertion (ii) follows from (5.22) since $\eta_\sigma(x,t) = 1$ for $x \in E^{t,2\sigma}$ and $\eta_\sigma \leq 1$ holds everywhere on E.

For the proof of (iii), we claim that for $x \in \mathbb{R}^n$ and $-h \leq s \leq t < T$ we have

$$\eta_\sigma(x,t) - \eta_\sigma(x,s) \geq -\tfrac{M}{\sigma}(t-s). \tag{5.23}$$

If $s \in [-h, 0)$ or if $s \in [0,T)$ and $x \notin E^s$, this is clear since in both cases, we have $\eta_\sigma(x,s) = 0$ and $\eta_\sigma(x,t) \geq 0$. In the remaining case $s \in [0,T)$ and $x \in E^s$, we choose a point $y \in \partial E^t$ with $|x-y| = \operatorname{dist}(x, \Omega \setminus E^t)$. Using the facts $x \in E^s$, $y \in \partial E^t \subset \Omega \setminus E^t$ and the definition of $\mathbf{e}^c$ from (2.11), we estimate

$$\begin{aligned} \operatorname{dist}(x, \Omega \setminus E^s) &\leq |x-y| + \operatorname{dist}(y, \Omega \setminus E^s) \\ &\leq \operatorname{dist}(x, \Omega \setminus E^t) + \mathbf{e}^c(E^s, E^t) \\ &\leq \operatorname{dist}(x, \Omega \setminus E^t) + M(t-s). \end{aligned} \tag{5.24}$$

The last estimate is a consequence of the choice of y and assumption (2.12). Since $\tilde\eta$ is nondecreasing and has Lipschitz constant 1, the preceding estimate implies

$$\begin{aligned} \eta_\sigma(x,t) - \eta_\sigma(x,s) &\geq \tilde\eta\left(\frac{\operatorname{dist}(x, \Omega \setminus E^t)}{\sigma}\right) - \tilde\eta\left(\frac{\operatorname{dist}(x, \Omega \setminus E^t)}{\sigma} + \tfrac{M}{\sigma}(t-s)\right) \\ &\geq -\tfrac{M}{\sigma}(t-s). \end{aligned}$$

Hence, in any case we have shown the claim (5.23). Combining (5.23) with (5.22), we arrive at

$$\partial_t\eta_{\sigma,h}(x,t) = \tfrac{1}{h}\left[\eta_\sigma(x,t) - \eta_\sigma(x,t-h)\right] \geq -\tfrac{M}{\sigma}$$

for a.e. $(x,t) \in E$, which establishes assertion (iii).

It remains to prove (iv). We fix $t \in [0,T)$ and $x \in E^t \setminus E^{t,\frac{\sigma}{2}}$. We wish to show

$$\eta_\sigma(x,s) = 0 \qquad \text{for any } s \in [t-h,t]. \tag{5.25}$$

For the proof, we proceed similarly as for the proof of (5.23). In the case $s \in [-h,0)$ and in the case $s \in [0,T)$ and $x \notin E^s$, the claim is clear from the construction of η_σ. For $s \in [0,T)$ and $x \in E^s$, we can argue as in (5.24), with the result

$$\operatorname{dist}(x,\Omega \setminus E^s) \leq \operatorname{dist}(x,\Omega\setminus E^t) + M(t-s) \leq \tfrac{\sigma}{2} + Mh < \sigma.$$

The last two inequalities follow from $x \notin E^{t,\frac{\sigma}{2}}$ and $t-s \leq h < \frac{\sigma}{2M}$. By definition of η_σ, this implies (5.25) in the remaining case. Having established (5.25), the last assertion (iv) is clear from the definition (5.21) of $\eta_{\sigma,h}$. □

Now we have all the ingredients at our disposal to prove the main result of this subsection.

Proof of Theorem 5.5. We apply Lemma 5.8 with the functions $\eta_{\sigma,h}$ which is possible since Lemma 5.9 (i), (iv) implies $\eta_{\sigma,h} \in C^{0,1}(E)$ and $\operatorname{spt}(\eta_{\sigma,h}(\cdot,t)) \subset \overline{E^{t,\frac{\sigma}{2}}}$ for every $t \in [0,T)$. For this reason we know that

$$\langle \partial_t u, \zeta\eta_{\sigma,h}^2 u\rangle \leq -\tfrac12 \iint_E \partial_t\zeta\eta_{\sigma,h}^2|u|^2\,\mathrm{d}x\mathrm{d}t - \iint_E \zeta\eta_{\sigma,h}\,\partial_t\eta_{\sigma,h}|u|^2\,\mathrm{d}x\mathrm{d}t \tag{5.26}$$

holds true for every cutoff function in time $\zeta \in C_0^{0,1}(0,T)$ with $\zeta \geq 0$, any $\sigma > 0$, and any $h \in (0,\frac{\sigma}{2M})$. Since $\eta_{\sigma,h} \geq 0$ on E and $\partial_t\eta_{\sigma,h}(t) \geq 0$ on $E^{t,2\sigma}$ by Lemma 5.9 (ii), we can bound the last integral by

$$\begin{aligned}
-\iint_E \zeta\eta_{\sigma,h}\,\partial_t\eta_{\sigma,h}|u|^2\,\mathrm{d}x\mathrm{d}t &\leq -\int_0^T\int_{E^t\setminus E^{t,2\sigma}} \zeta\eta_{\sigma,h}\,\partial_t\eta_{\sigma,h}|u|^2\,\mathrm{d}x\mathrm{d}t \\
&\leq \tfrac{M}{\sigma}\|\zeta\|_{L^\infty}\int_0^T\int_{E^t\setminus E^{t,2\sigma}} |u|^2\,\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{5.27}$$

where we used Lemma 5.9 (iii) for the last estimate. At this point, we shall use the convergences

$$\begin{cases} \zeta\eta_{\sigma,h}^2 u \to \zeta\eta_\sigma^2 u & \text{in } L^p\big(0,T;W_0^{1,p}(\Omega)\big) \text{ and in } L^2(\Omega_T), \\ \eta_{\sigma,h}u \to \eta_\sigma u & \text{in } L^2(E) \end{cases} \tag{5.28}$$

in the limit $h \downarrow 0$. Both convergences follow from the fact $u \in V_2^p(E)$ and the dominated convergence theorem. For the L^p–$W^{1,p}$-convergence, we additionally use that $D\eta_{\sigma,h} \to D\eta_\sigma$ a.e. in Ω_T in the limit $h \downarrow 0$ and $\|D\eta_{\sigma,h}\|_{L^\infty} \leq \|D\eta_\sigma\|_{L^\infty} \leq 1$ for any $h > 0$. Combining (5.27) and (5.28) with (5.26) and keeping in mind that $\partial_t u \in \big(V_2^p(E)\big)'$, we deduce by letting $h \downarrow 0$ that

$$\langle \partial_t u, \zeta\eta_\sigma^2 u\rangle \leq -\tfrac12 \iint_E \partial_t\zeta\eta_\sigma^2|u|^2\,\mathrm{d}x\mathrm{d}t + \tfrac{M}{\sigma}\|\zeta\|_{L^\infty}\int_0^T\int_{E^t\setminus E^{t,2\sigma}}|u|^2\,\mathrm{d}x\mathrm{d}t. \tag{5.29}$$

On the domain of integration $E^t \setminus E^{t,2\sigma}$ of the last integral, we have $\operatorname{dist}(x,\Omega\setminus E^t) \leq 2\sigma$. Hence, in the case $p \geq 2$, we can estimate

$$
\begin{aligned}
(5.30)\qquad \tfrac{1}{\sigma}\int_0^T \int_{E^t\setminus E^{t,2\sigma}} |u|^2 \,\mathrm{d}x\mathrm{d}t &\le 4\sigma \int_0^T \int_{E^t} \left(\frac{|u(x,t)|}{\operatorname{dist}(x,\partial E^t)}\right)^2 \mathrm{d}x\mathrm{d}t \\
&\le c(n,p,\delta)\sigma|\Omega_T|^{1-\frac{2}{p}} \left(\iint_E |Du|^p \,\mathrm{d}x\mathrm{d}t\right)^{\frac{2}{p}} \to 0
\end{aligned}
$$

in the limit $\sigma \downarrow 0$. In the last estimate we used Hölder's inequality and Hardy's inequality (3.2) from Lemma 3.2. In the case $p \in (\frac{2(n+1)}{n+2}, 2)$, we estimate the same integral by an application of the Hardy–Sobolev inequality (3.3), with the result

$$
\begin{aligned}
&\tfrac{1}{\sigma}\int_0^T \int_{E^t\setminus E^{t,2\sigma}} |u|^2 \,\mathrm{d}x\mathrm{d}t \\
&\quad\le 4\sigma^{p+\frac{np}{2}-n-1}\|u\|_{L^\infty\text{-}L^2}^{2-p} \int_0^T \left[\int_{E^t} \left(\frac{|u(x,t)|}{[\operatorname{dist}(x,\partial E^t)]^{1-\frac{n}{p}+\frac{n}{2}}}\right)^2 \mathrm{d}x\right]^{\frac{p}{2}} \mathrm{d}t \\
&\quad\le c(n,p,\delta)\sigma^{\frac{p}{2}(n+2)-(n+1)}\|u\|_{L^\infty\text{-}L^2}^{2-p} \int_0^T \int_{E^t} |Du|^p \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Due to our additional assumption $u \in L^\infty\big(0,T; L^2(\Omega,\mathbb{R}^N)\big)$ in the case $p < 2$, the right-hand side is finite. Moreover, since $p > \frac{2(n+1)}{n+2}$, the exponent of σ is positive. Consequently, we also showed in the case $p \in (\frac{2(n+1)}{n+2}, 2)$, that

$$
\lim_{\sigma\downarrow 0} \tfrac{1}{\sigma}\int_0^T \int_{E^t\setminus E^{t,2\sigma}} |u|^2 \,\mathrm{d}x\mathrm{d}t = 0. \tag{5.31}
$$

Furthermore, we rely on the convergences

$$
\begin{cases} \zeta\eta_\sigma^2 u \rightharpoonup \zeta u & \text{weakly in } V_2^p(E), \\ \eta_\sigma u \to u & \text{in } L^2(E) \end{cases} \tag{5.32}
$$

as $\sigma \downarrow 0$. The first convergence is a consequence of Lemma 5.2, applied to $\zeta u \in V_2^p(E)$, while the second convergence follows from the dominated convergence theorem. Since $\partial_t u \in \big(V_2^p(E)\big)'$, the convergences (5.30), (5.31), and (5.32) are sufficient to pass to the limit $\sigma \downarrow 0$ in (5.29) and to deduce

$$
\begin{aligned}
\langle \partial_t u, \zeta u\rangle &= \lim_{\sigma\downarrow 0}\langle \partial_t u, \zeta\eta_\sigma^2 u\rangle \\
&\le \lim_{\sigma\downarrow 0}\left[-\tfrac{1}{2}\iint_E \eta_\sigma^2\partial_t\zeta|u|^2\,\mathrm{d}x\mathrm{d}t + \tfrac{M}{\sigma}\|\zeta\|_{L^\infty}\int_0^T\int_{E^t\setminus E^{t,2\sigma}}|u|^2\,\mathrm{d}x\mathrm{d}t\right] \\
&= -\tfrac{1}{2}\iint_E \partial_t\zeta|u|^2\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

This completes the proof of Theorem 5.5. ∎

5.3.3. Left-sided continuity in time. In the following, we choose a representative of $u \in L^\infty\big(0,T; L^2(\Omega,\mathbb{R}^N)\big)$ that for every $\tau \in (0,T)$ satisfies

$$
u(x,\tau) = \liminf_{h\downarrow 0} \tfrac{1}{h}\int_{\tau-h}^{\tau} u(x,s)\,\mathrm{d}s \quad \text{for a.e. } x \in \Omega. \tag{5.33}
$$

As a consequence of Theorem 5.5, we obtain the following.

LEMMA 5.10. *We assume that E satisfies* (2.10) *and* (2.12) *and that* $p > \frac{2(n+1)}{n+2}$. *We consider a map* $u \in V_2^p(E) \cap L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big)$ *with* $\partial_t u \in \big(V_2^p(E)\big)'$. *Then, for any representative of u that satisfies* (5.33), *we have*

$$\lim_{h\downarrow 0} \tfrac{1}{h} \iint_{\Omega\times(\tau_o-h,\tau_o)} |u|^2\,\mathrm{d}x\mathrm{d}t = \int_\Omega |u(x,\tau_o)|^2\,\mathrm{d}x \tag{5.34}$$

for any $\tau_o \in (0,T)$.

Proof. First, we note that (5.33), Jensen's inequality and Fatou's lemma imply

$$\begin{aligned}\int_\Omega |u(x,\tau_o)|^2\,\mathrm{d}x &\le \int_\Omega \bigg[\liminf_{h\downarrow 0} \tfrac{1}{h}\int_{\tau_o-h}^{\tau_o} |u(x,t)|^2\,\mathrm{d}t\bigg]\mathrm{d}x \\ &\le \liminf_{h\downarrow 0} \tfrac{1}{h} \iint_{\Omega\times(\tau_o-h,\tau_o)} |u(x,t)|^2\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{5.35}$$

Now, we turn our attention to the opposite inequality. For $h \in (0,\tau_o)$, we define

$$\eta_h(x) := \eta_h(x,\tau_o) \equiv \tilde\eta\left(\frac{\mathrm{dist}(x,\Omega\setminus E^{\tau_o})}{h}\right) \quad \text{for } x\in\mathbb{R}^n, \tag{5.36}$$

where $\tilde\eta \in C^{0,1}(\mathbb{R})$ is the same function as in (5.9). We subdivide the integral on the right-hand side of (5.35) into two parts:

$$\begin{aligned}&\tfrac{1}{h}\int_{\tau_o-h}^{\tau_o}\int_\Omega |u|^2\,\mathrm{d}x\mathrm{d}t \\ &\quad = \tfrac{1}{h}\int_{\tau_o-h}^{\tau_o}\int_{E^t\setminus E^{\tau_o,2h}} (1-\eta_h^2)|u|^2\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{h}\int_{\tau_o-h}^{\tau_o}\int_\Omega \eta_h^2|u|^2\,\mathrm{d}x\mathrm{d}t \\ &\quad =: \mathrm{I}_h + \mathrm{II}_h,\end{aligned} \tag{5.37}$$

with the obvious labeling of I_h and II_h. For any $x \in E^t \setminus E^{\tau_o,2h}$, we can choose $y \in \Omega\setminus E^{\tau_o}$ with $|x-y| = \mathrm{dist}(x,\Omega\setminus E^{\tau_o}) \le 2h$. Using (2.12) for $t\in(\tau_o-h,\tau_o)$, we estimate

$$\begin{aligned}\mathrm{dist}(x,\Omega\setminus E^t) &\le \mathrm{dist}(y,\Omega\setminus E^t) + |x-y| \\ &\le \mathbf{e}^c(E^t,E^{\tau_o}) + |x-y| \\ &\le M(\tau_o-t) + |x-y| \le (M+2)h.\end{aligned} \tag{5.38}$$

This shows that $E^t\setminus E^{\tau_o,2h} \subset E^t\setminus E^{t,(M+2)h}$. Hence, enlarging the domain of integration in I_h, we deduce

$$\mathrm{I}_h \le \tfrac{1}{h}\int_{\tau_o-h}^{\tau_o}\int_{E^t\setminus E^{t,(M+2)h}} |u|^2\,\mathrm{d}x\mathrm{d}t.$$

We note that $\mathrm{dist}(x,\partial E^t) \le (M+2)h$ on the domain of integration. For the further estimate of the above integral, we distinguish between the cases $p\ge 2$ and $p\in(\frac{2(n+1)}{n+2},2)$. In the first case, we apply Hölder's inequality and Hardy's inequality (3.2) and deduce

$$\begin{aligned}\mathrm{I}_h &\le (M+2)^2 h \int_{\tau_o-h}^{\tau_o}\int_{E^t}\left(\frac{|u|}{\mathrm{dist}(x,\partial E^t)}\right)^2\mathrm{d}x\mathrm{d}t \\ &\le ch|\Omega|^{1-\frac{2}{p}}\int_{\tau_o-h}^{\tau_o}\left(\int_\Omega |Du|^p\,\mathrm{d}x\right)^{\frac{2}{p}}\mathrm{d}t \to 0\end{aligned}$$

in the limit $h \downarrow 0$. In the case $p \in (\frac{2(n+1)}{n+2}, 2)$, however, we use the Hardy–Sobolev inequality (3.3) for the estimate

$$\begin{aligned}\mathrm{I}_h &\le (M+2)^2 h^{p+\frac{np}{2}-n-1}\|u\|^{2-p}_{L^\infty\text{-}L^2}\int_{\tau_0-h}^{\tau_0}\left[\int_{E^t}\left(\frac{|u|}{[\operatorname{dist}(x,\partial E^t)]^{1-\frac{n}{p}+\frac{n}{2}}}\right)^2 \mathrm{d}x\right]^{\frac{p}{2}}\mathrm{d}t\\ &\le (M+2)^2 h^{p+\frac{np}{2}-n-1}\|u\|^{2-p}_{L^\infty\text{-}L^2}\int_{\tau_0-h}^{\tau_0}\int_{E^t}|Du|^p\,\mathrm{d}x\mathrm{d}t \to 0\end{aligned}$$

in the limit $h \downarrow 0$. Note that the exponent of h is positive since $p > \frac{2(n+1)}{n+2}$. In any case, we have therefore shown

$$\lim_{h\downarrow 0}\mathrm{I}_h = 0. \tag{5.39}$$

For the estimate of II_h, we choose an $\varepsilon_\mathrm{o} \in (0,h)$ with $E^{\tau_\mathrm{o},h}\times(\tau_\mathrm{o}-2\varepsilon_\mathrm{o},\tau_\mathrm{o}+2\varepsilon_\mathrm{o}) \subset E$, which is possible since E is open. Then, for any $\varepsilon \in (0,\varepsilon_0)$ we define $\zeta \in C_0^{0,1}(0,T)$ by

$$\zeta(t) := \begin{cases} \frac{1}{h}(t-\tau_\mathrm{o}+h) & \text{for } \tau_\mathrm{o}-h \le t \le \tau_\mathrm{o}-\varepsilon,\\ (\frac{1}{\varepsilon}-\frac{1}{h})(\tau_\mathrm{o}-t) & \text{for } \tau_\mathrm{o}-\varepsilon < t \le \tau_\mathrm{o},\\ 0 & \text{otherwise.}\end{cases}$$

We apply Theorem 5.5 with ζ and $\eta_h u \in V_2^p(E)$ in place of u, which yields

$$\begin{aligned}\langle\partial_t u, \zeta\eta_h^2 u\rangle = \langle\partial_t(\eta_h u), \zeta\eta_h u\rangle &\le -\tfrac{1}{2}\iint_E \partial_t\zeta\eta_h^2|u|^2\,\mathrm{d}x\mathrm{d}t\\ &= \tfrac{1}{2\varepsilon}\int_{\tau_\mathrm{o}-\varepsilon}^{\tau_\mathrm{o}}\int_\Omega \eta_h^2|u|^2\,\mathrm{d}x\mathrm{d}t - \tfrac{1}{2h}\int_{\tau_\mathrm{o}-h}^{\tau_\mathrm{o}}\int_\Omega \eta_h^2|u|^2\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

We use this for the estimate

$$\mathrm{II}_h \le \tfrac{1}{\varepsilon}\int_{\tau_\mathrm{o}-\varepsilon}^{\tau_\mathrm{o}}\int_\Omega \eta_h^2|u|^2\,\mathrm{d}x\mathrm{d}t + 2\|\partial_t u\|_{(V_2^p(E))'}\big\|\zeta\eta_h^2 u\big\|_{V_2^p(E)}. \tag{5.40}$$

Because of $E^{\tau_\mathrm{o},h}\times(\tau_\mathrm{o}-\varepsilon_\mathrm{o},\tau_\mathrm{o}) \subset E$, the facts $u \in V_2^p(E)$, $\partial_t u \in \big(V_2^p(E)\big)'$, and $\operatorname{spt}\eta_h \subset E^{\tau_\mathrm{o},h}$ imply

$$\begin{cases} \eta_h u \in L^p\big(\tau_\mathrm{o}-\varepsilon,\tau_\mathrm{o}; W_0^{1,p}(E^{\tau_\mathrm{o},h},\mathbb{R}^N)\big)\cap L^2\big(\Omega_T,\mathbb{R}^N\big),\\ \partial_t(\eta_h u) \in L^{p'}\big(\tau_\mathrm{o}-\varepsilon,\tau_\mathrm{o}; W^{-1,p'}(E^{\tau_\mathrm{o},h},\mathbb{R}^N)\big) + L^2\big(\Omega_T,\mathbb{R}^N\big).\end{cases} \tag{5.41}$$

We claim that this implies

$$\eta_h u \in C^0\big([\tau_\mathrm{o}-\varepsilon,\tau_\mathrm{o}]; L^2(E^{\tau_\mathrm{o},h},\mathbb{R}^N)\big). \tag{5.42}$$

In fact, in the case $p \ge 2$, in which the space $L^2(\Omega_T,\mathbb{R}^N)$ can be omitted in (5.41), this follows from a standard application of [39, Prop. III.1.2]. In the remaining case, we give a slight modification of this argument that yields the same result. To this end, we choose a smooth cutoff function $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ on $[\tau_\mathrm{o}-\varepsilon,\tau_\mathrm{o}]$ and $\psi = 0$ outside of $[\tau_\mathrm{o}-2\varepsilon,\tau_\mathrm{o}+\varepsilon]$. Then, we consider the Steklov averages

$$u_\delta(x,t) := \fint_{t-\delta}^t \psi(s)\eta_h(x)u(x,s)\,\mathrm{d}s \quad \text{for } (x,t)\in E \text{ and } \delta\in(0,\varepsilon).$$

Since $E^{\tau_\mathrm{o},h}\times(\tau_\mathrm{o}-2\varepsilon,\tau_\mathrm{o}+2\varepsilon) \subset E$, the construction implies $u_\delta \in V_2^p(E)$ for any $\delta \in (0,\varepsilon)$. From standard properties of the Steklov averages, we infer

$$u_\delta \to \psi\eta_h u \quad \text{in } V_2^p(E) \text{ as } \delta\downarrow 0$$

as well as

$$\big\|\partial_t u_\delta\big\|_{(V_2^p(E))'} \le \|\partial_t(\psi\eta_h u)\|_{(V_2^p(E))'} \quad \text{for any } \delta\in(0,\varepsilon). \tag{5.43}$$

We note that the right-hand side is finite since

$$\partial_t(\psi\eta_h u) = \eta_h\psi\partial_t u + \eta_h\partial_t\psi\, u \in \big(V_2^p(E)\big)' + L^2\big(\Omega_T,\mathbb{R}^N\big) = \big(V_2^p(E)\big)'.$$

Moreover, it is clear from the construction that $u_\delta(\tau_o - 2\varepsilon) = 0$ for every $\delta\in(0,\varepsilon)$. From these facts, we deduce, for any $\tau\in[\tau_o-\varepsilon,\tau_o]$,

$$\begin{aligned}\|u_\delta(\tau)-u_\gamma(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)} &= 2\int_{\tau_o-2\varepsilon}^{\tau}\int_\Omega \partial_t(u_\delta-u_\gamma)\cdot(u_\delta-u_\gamma)\,\mathrm{d}x\mathrm{d}t\\ &\le 2\big\|\partial_t(u_\delta-u_\gamma)\big\|_{(V_2^p(E))'}\|u_\delta-u_\gamma\|_{V_2^p(E)}\end{aligned}$$

for any $\delta,\gamma\in(0,\varepsilon)$. Taking the supremum over $\tau\in[\tau_o-\varepsilon,\tau_o]$ and making use of (5.43), we deduce

$$\sup_{\tau\in[\tau_o-\varepsilon,\tau_o]}\big\|u_\delta(\tau)-u_\gamma(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} \le 4\big\|\partial_t(\psi\eta_h u)\big\|_{(V_2^p(E))'}\|u_\delta-u_\gamma\|_{V_2^p(E)} \to 0$$

in the limit $\delta,\gamma\downarrow 0$. This proves that the mollifications u_δ converge in $C^0\big([\tau_o-\varepsilon,\tau_o];L^2(E^{\tau_o,h},\mathbb{R}^N)\big)$. In particular, we infer that the limit map $\eta_h u$ satisfies (5.42). This, however, implies the convergence

$$\lim_{\varepsilon\downarrow 0}\tfrac{1}{\varepsilon}\int_{\tau_o-\varepsilon}^{\tau_o}\int_\Omega \eta_h^2|u|^2\,\mathrm{d}x\mathrm{d}t = \int_\Omega \eta_h^2|u(x,\tau_o)|^2\,\mathrm{d}x,$$

where $u(x,\tau_o)$ is defined via (5.33). This we use to pass to the limit $\varepsilon\downarrow 0$ in (5.40). We obtain

$$\mathrm{II}_h \le \int_\Omega \eta_h^2|u(\tau_o)|^2\,\mathrm{d}x + 2\|\partial_t u\|_{(V_2^p(E))'}\big\|\chi_{\Omega\times(\tau_o-h,\tau_o)}\eta_h^2 u\big\|_{V_2^p(E)}. \tag{5.44}$$

Next, using the properties of ζ and η_h, we estimate

$$\int_{\tau_o-h}^{\tau_o}\int_{E^t}|D(\eta_h^2u)|^p\,\mathrm{d}x\mathrm{d}t \le \frac{c}{h^p}\int_{\tau_o-h}^{\tau_o}\int_{E^t\setminus E^{\tau_o,2h}}|u|^p\,\mathrm{d}x\mathrm{d}t + c\int_{\tau_o-h}^{\tau_o}\int_\Omega |Du|^p\,\mathrm{d}x\mathrm{d}t.$$

Since we know $E^t\setminus E^{\tau_o,2h}\subset E^t\setminus E^{t,(M+2)h}$ from (5.38), we can apply Hardy's inequality from Lemma 3.2, with the result

$$\begin{aligned}\frac{1}{h^p}\int_{\tau_o-h}^{\tau_o}\int_{E^t\setminus E^{\tau_o,2h}}|u|^p\,\mathrm{d}x\mathrm{d}t &\le (M+2)^p\int_{\tau_o-h}^{\tau_o}\int_{E^t}\left(\frac{|u|}{\operatorname{dist}(x,\partial E^t)}\right)^p\mathrm{d}x\mathrm{d}t\\ &\le c(n,p,\delta)\int_{\tau_o-h}^{\tau_o}\int_\Omega|Du|^p\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Inserting this above we obtain

$$\lim_{h\downarrow 0}\int_{\tau_o-h}^{\tau_o}\int_{E^t}|D(\eta_h^2u)|^p\,\mathrm{d}x\mathrm{d}t = 0.$$

Furthermore, we have

$$\lim_{h\downarrow 0}\int_{\tau_o-h}^{\tau_o}\int_{E^t}\eta_h^4|u|^2\,\mathrm{d}x\mathrm{d}t=0.$$

These identities, however, imply $\|\chi_{\Omega\times(\tau_o-h,\tau_o)}\eta_h^2 u\|_{V_2^p(E)}\to 0$ as $h\downarrow 0$. Hence, (5.44) implies

$$\limsup_{h\downarrow 0}\mathrm{II}_h\le\int_\Omega|u(x,\tau_o)|^2\,\mathrm{d}x. \tag{5.45}$$

Joining (5.37), (5.39), and (5.45), we arrive at

$$\limsup_{h\downarrow 0}\tfrac{1}{h}\int_{\tau_o-h}^{\tau_o}\int_\Omega|u|^2\,\mathrm{d}x\mathrm{d}t\le\int_\Omega|u(x,\tau_o)|^2\,\mathrm{d}x.$$

In view of (5.35), this proves the claimed convergence (5.34). ∎

Lemma 5.11. *Suppose that E satisfies* (2.10) *and* (2.12) *and that $p>\frac{2(n+1)}{n+2}$. Let $u\in V_2^p(E)\cap L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big)$ be given with $\partial_t u\in\big(V_2^p(E)\big)'$. Then, for every representative of u that satisfies* (5.33)*, we have*

$$\lim_{\tau\uparrow\tau_0}\big\|u(\tau)-u(\tau_o)\big\|_{L^2(\Omega,\mathbb{R}^N)}=0 \tag{5.46}$$

for any $\tau_o\in(0,T)$.

Proof. We consider the cutoff functions $\eta_h=\eta_h(x)$ defined in (5.36), for any $h>0$, and estimate

$$\begin{aligned}\big\|u(\tau)-u(\tau_o)\big\|_{L^2(\Omega,\mathbb{R}^N)}&\\ \le\big\|\eta_h u(\tau)-\eta_h u(\tau_o)\big\|_{L^2(\Omega,\mathbb{R}^N)}&+\big\|(1-\eta_h)u(\tau)\big\|_{L^2(\Omega,\mathbb{R}^N)}\\ +\big\|(1-\eta_h)u(\tau_o)\big\|_{L^2(\Omega,\mathbb{R}^N)}&\\ =:\mathrm{I}_h(\tau)+\mathrm{II}_h(\tau)+\mathrm{III}_h&\end{aligned} \tag{5.47}$$

for $0<\tau<\tau_o$. We choose $\varepsilon=\varepsilon(h)>0$ so small that $E^{\tau_o,h}\times[\tau_o-\varepsilon,\tau_o]\subset E$, which is possible since E is relatively open. As in (5.42), the facts $\partial_t u\in\big(V_2^p(E)\big)'$, $u\in V_2^p(E)$, and $\operatorname{spt}\eta_h\subset E^{\tau_o,h}$ imply $\eta_h u\in C^0\big([\tau_o-\varepsilon,\tau_o];L^2(E^{\tau_o,h},\mathbb{R}^N)\big)$, so that

$$\lim_{\tau\uparrow\tau_o}\mathrm{I}_h(\tau)=0\quad\text{for every } h>0. \tag{5.48}$$

In order to estimate $\mathrm{II}_h(\tau)$, we apply Theorem 5.5 with the function u replaced by $(1-\eta_h)u\in V_2^p(E)$ and the cutoff function $\zeta_\delta\in C_0^{0,1}(0,T)$ defined by

$$\zeta_\delta(s):=\begin{cases}\frac{1}{\delta}(s-\tau+\delta) & \text{for } \tau-\delta\le s<\tau,\\ 1 & \text{for } \tau\le s<\tau_o-\delta,\\ \frac{1}{\delta}(\tau_o-s) & \text{for } \tau_o-\delta\le s\le\tau_o,\\ 0 & \text{otherwise}\end{cases}$$

for any $0<\delta<\min\{\tau,\tau_o-\tau\}$. This implies

$$\begin{aligned}&\tfrac{1}{\delta}\int_{\tau-\delta}^{\tau}\int_\Omega\big|(1-\eta_h)u\big|^2\,dx\mathrm{d}t\\ &\quad\le\tfrac{1}{\delta}\int_{\tau_o-\delta}^{\tau_o}\int_\Omega\big|(1-\eta_h)u\big|^2\,dx\mathrm{d}t+2\big|\big\langle\partial_t[(1-\eta_h)u],\zeta_\delta(1-\eta_h)u\big\rangle\big|.\end{aligned} \tag{5.49}$$

Since η_h is independent of time, the last term can be estimated by

$$\begin{aligned}\big|\big\langle \partial_t[(1-\eta_h)u], \zeta_\delta(1-\eta_h)u\big\rangle\big| &= \big|\big\langle \partial_t u, \zeta_\delta(1-\eta_h)^2 u\big\rangle\big| \\ &\le \|\partial_t u\|_{(V_2^p(E))'}\big\|\zeta_\delta(1-\eta_h)^2 u\big\|_{V_2^p(E)}.\end{aligned}$$

Moreover, Lemma 5.10, applied with $(1-\eta_h)u$ in place of u, implies

$$\lim_{\delta\downarrow 0}\tfrac{1}{\delta}\int_{\tau_o-\delta}^{\tau_o}\int_\Omega \big|(1-\eta_h)u\big|^2\,dx\mathrm{d}t = \int_\Omega |(1-\eta_h)u(x,\tau_o)|^2\,dx = \mathrm{III}_h,$$

and the same identity holds with τ instead of τ_o. Consequently, from (5.49) we infer by letting $\delta\downarrow 0$ that

$$\mathrm{II}_h(\tau) \le \mathrm{III}_h + 2\big\|\partial_t u\big\|_{(V_2^p(E))'}\big\|\chi_{\Omega\times(\tau,\tau_o)}(1-\eta_h)^2 u\big\|_{V_2^p(E)}$$

holds true for every $\tau\in(0,\tau_o)$. For a fixed value of $h>0$, the last term vanishes in the limit $\tau\uparrow\tau_o$. This implies

$$\limsup_{\tau\uparrow\tau_o}\mathrm{II}_h(\tau)\le \mathrm{III}_h \tag{5.50}$$

for every $h>0$. Joining (5.48) and (5.50) with (5.47), we deduce

$$\limsup_{\tau\uparrow\tau_o}\big\|u(\tau)-u(\tau_o)\big\|_{L^2(\Omega,\mathbb{R}^N)} \le 2\mathrm{III}_h = 2\big\|(1-\eta_h)u(\tau_o)\big\|_{L^2(\Omega,\mathbb{R}^N)}$$

for any $h>0$. Since the right-hand side can be made arbitrarily small by choosing $h>0$ small enough, this establishes the asserted convergence (5.46). ◻

5.3.4. Localization in time. In this section we establish a version of the variational inequality (2.6) that is localized to noncylindrical subdomains $E\cap\Omega\times(\tau_o,\tau)$.

LEMMA 5.12. *We assume that the integrand satisfies* (2.13)*, and that the noncylindrical domain fulfills* (2.10) *and* (2.12)*. If u is a solution of the variational inequality* (2.6) *and satisfies $\partial_t u\in\big(V^p(E)\big)'$, then it also satisfies the localized variational inequality*

$$\begin{aligned}\iint_{E\cap\Omega\times(\tau_o,\tau)} & f(x,u,Du)\mathrm{d}x\mathrm{d}t \\ &\le \iint_{E\cap\Omega\times(\tau_o,\tau)}\Big[\partial_t v\cdot(v-u)+f(x,v,Dv)\Big]\mathrm{d}x\mathrm{d}t \\ &\quad - \tfrac12\big\|(v-u)(\tau)\big\|^2_{L^2(E^\tau,\mathbb{R}^N)} + \tfrac12\|(v-u)(\tau_o)\|^2_{L^2(E^{\tau_o},\mathbb{R}^N)}\end{aligned}\tag{5.51}$$

for every $v\in V_2^p(E)$ with $\partial_t v\in L^2\big(\Omega_T,\mathbb{R}^N\big)$ and any $0\le\tau_o<\tau<T$.

Proof. We consider $0<\tau_o<\tau<T$ and $w\in V_2^p(E)$ with $\partial_t w\in L^2(\Omega_T,\mathbb{R}^N)$. Note that this implies $\partial_t w\in\big(V_2^p(E)\big)'$. For $\varepsilon\in(0,\frac12\tau_o)$ we define

$$\zeta_\varepsilon(t) := \begin{cases} \frac{t}{\varepsilon} & \text{for } t\in[0,\varepsilon),\\ 1 & \text{for } t\in[\varepsilon,\tau_o-\varepsilon],\\ \frac{\tau_o-t}{\varepsilon} & \text{for } t\in(\tau_o-\varepsilon,\tau_o),\\ 0 & \text{for } t\in[\tau_o,T]\end{cases}\tag{5.52}$$

and obtain due to the integration by parts formula from Theorem 5.5 that

$$\begin{aligned}\iint_{E\cap\Omega_\tau} &\partial_t w\cdot(w-u)\mathrm{d}x\mathrm{d}t\\ &= \langle \partial_t w, \zeta_\varepsilon(w-u)\rangle + \iint_{E\cap\Omega_\tau}(1-\zeta_\varepsilon)\partial_t w\cdot(w-u)\mathrm{d}x\mathrm{d}t\\ &= \langle \partial_t u, \zeta_\varepsilon(w-u)\rangle + \langle \partial_t (w-u), \zeta_\varepsilon(w-u)\rangle + \iint_{E\cap\Omega_\tau}(1-\zeta_\varepsilon)\partial_t w\cdot(w-u)\mathrm{d}x\mathrm{d}t\\ &\le \langle \partial_t u, \zeta_\varepsilon(w-u)\rangle - \tfrac12\iint_E \partial_t\zeta_\varepsilon|w-u|^2\mathrm{d}x\mathrm{d}t + \iint_{E\cap\Omega_\tau}(1-\zeta_\varepsilon)\partial_t w\cdot(w-u)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Passing to the limit $\varepsilon\downarrow 0$ and taking into account that $\partial_t u\in\big(V_2^p(E)\big)'$ and that $u(0)=u_{\mathrm{o}}$ in the L^2-sense by Lemma 3.3 yields that

$$\begin{aligned}\iint_{E\cap\Omega_\tau}\partial_t w\cdot(w-u)\mathrm{d}x\mathrm{d}t &\le \langle \partial_t u, \chi_{\Omega_{\tau_{\mathrm{o}}}}(w-u)\rangle + \iint_{E\cap\Omega\times(\tau_{\mathrm{o}},\tau)}\partial_t w\cdot(w-u)\mathrm{d}x\mathrm{d}t\\ &\quad + \tfrac12\big\|(w-u)(\tau_{\mathrm{o}})\big\|^2_{L^2(\Omega,\mathbb{R}^N)} - \tfrac12\big\|w(0)-u_{\mathrm{o}}\big\|^2_{L^2(\Omega,\mathbb{R}^N)}\end{aligned}$$

holds true for a.e. $\tau_{\mathrm{o}}\in[0,\tau)$. We use the preceding inequality in the variational inequality (2.6), and obtain for a.e. τ_{o},τ with $0<\tau_{\mathrm{o}}<\tau\le T$ that

$$\begin{aligned}\iint_{E\cap\Omega_\tau} &f(x,u,Du)\mathrm{d}x\mathrm{d}t + \tfrac12\big\|(w-u)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)}\\ &\le \iint_{E\cap\Omega_\tau} f(x,w,Dw)\mathrm{d}x\mathrm{d}t + \tfrac12\|(w-u)(\tau_{\mathrm{o}})\|^2_{L^2(\Omega,\mathbb{R}^N)}\\ &\quad + \iint_{E\cap\Omega\times(\tau_{\mathrm{o}},\tau)}\partial_t w\cdot(w-u)\mathrm{d}x\mathrm{d}t + \langle\partial_t u, \chi_{\Omega_{\tau_{\mathrm{o}}}}(w-u)\rangle.\end{aligned}\tag{5.53}$$

Since according to Lemma 5.11 the L^2-boundary terms depend left-continuously on τ (respectively, τ_{o}), we know that the above inequality holds for every τ,τ_{o} with $0<\tau_{\mathrm{o}}\le\tau<T$ if we choose a suitable representative of u. We now consider a general testing function $v\in V_2^p(E)$ with $\partial_t v\in L^2\big(\Omega_T,\mathbb{R}^N\big)$. For $\varepsilon\in(0,\tau_{\mathrm{o}})$ we consider

$$w_\varepsilon := \zeta_\varepsilon u + (1-\zeta_\varepsilon)v,$$

where ζ_ε is the cutoff function in time defined in (5.52). Lemma 5.3 ensures the existence of some sequence $(\varphi_\ell)_{\ell\in\mathbb{N}}\subset C_0^\infty(E,\mathbb{R}^n)$ of smooth functions such that $\varphi_\ell\to u$ as $\ell\to\infty$ in the norm topology of $V_2^p(E)$. The maps

$$w_{\varepsilon,\ell} := \zeta_\varepsilon\varphi_\ell + (1-\zeta_\varepsilon)v$$

satisfy $w_{\varepsilon,\ell}\in V_2^p(E)$ and $\partial_t w_{\varepsilon,\ell}\in L^2\big(\Omega_T,\mathbb{R}^N\big)$. Hence, they are admissible as comparison functions in the above variational inequality. Using the convergence $w_{\varepsilon,\ell}\to w_\varepsilon$ in $V_2^p(E)$ and the growth condition (2.13), we pass to the limit $\ell\to\infty$ in the variational inequality, which yields

$$\begin{aligned}\iint_{E\cap\Omega_\tau} &f(x,u,Du)\mathrm{d}x\mathrm{d}t + \tfrac12\big\|(v-u)(\tau)\big\|^2_{L^2(E^\tau,\mathbb{R}^N)}\\ &\le \iint_{E\cap\Omega_\tau} f(x,w_\varepsilon,Dw_\varepsilon)\mathrm{d}x\mathrm{d}t + \tfrac12\big\|(v-u)(\tau_{\mathrm{o}})\big\|^2_{L^2(E^{\tau_{\mathrm{o}}},\mathbb{R}^N)}\\ &\quad + \iint_{E\cap\Omega\times(\tau_{\mathrm{o}},\tau)}\partial_t v\cdot(v-u)\mathrm{d}x\mathrm{d}t + \langle\partial_t u, \chi_{\Omega_{\tau_{\mathrm{o}}}}(w_\varepsilon-u)\rangle\end{aligned}$$

for every τ, τ_o with $0 < \tau_o \le \tau < T$. Finally, we observe that $\chi_{\Omega_{\tau_o}}(w_\varepsilon - u) \to 0$ in $V_2^p(E)$ as $\varepsilon \downarrow 0$ and, using dominated convergence and the growth condition (2.13), we deduce

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \iint_{E\cap\Omega_\tau} & f(x, w_\varepsilon, Dw_\varepsilon)\mathrm{d}x\mathrm{d}t \\ &= \iint_{E\cap\Omega_{\tau_o}} f(x,u,Du)\mathrm{d}x\mathrm{d}t + \iint_{E\cap\Omega\times(\tau_o,\tau)} f(x,v,Dv)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Therefore, we arrive in the limit $\varepsilon \downarrow 0$ at the following variational inequality:

$$\begin{aligned}\iint_{E\cap\Omega\times(\tau_o,\tau)} & f(x,u,Du)\mathrm{d}x\mathrm{d}t \\ &\le \iint_{E\cap\Omega\times(\tau_o,\tau)} \Big[\partial_t v\cdot(v-u) + f(x,v,Dv)\Big]\mathrm{d}x\mathrm{d}t \\ &\quad - \tfrac12\big\|(v-u)(\tau)\big\|^2_{L^2(E^\tau,\mathbb{R}^N)} + \tfrac12\|(v-u)(\tau_o)\|^2_{L^2(E^{\tau_o},\mathbb{R}^N)}\end{aligned}$$

which holds true whenever $v \in V_2^p(E)$ with $\partial_t v \in L^2\big(\Omega_T,\mathbb{R}^N\big)$. This completes the proof of the lemma. ∎

5.3.5. Right-sided continuity in time. Having established the localized version of the variational inequality, we can argue similarly as in the proof of Lemma 3.3 to obtain continuity backward in time. In view of Lemma 5.11, this provides the remaining part of the proof of Theorem 2.8.

LEMMA 5.13. *Suppose that $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn} \to [0,\infty]$ is a variational integrand satisfying* (2.13)*, that $u_o \in L^2\big(E^0,\mathbb{R}^N\big)$, and that the noncylindrical domain E satisfies the conditions* (2.10) *and* (2.12)*. Let $u \in V_2^p(E)$ be a variational solution in the sense of Definition* 2.1*. Then the solution is right-sided continuous with respect to the L^2-norm, i.e.,*

$$\lim_{\tau\downarrow\tau_0} \|u(\tau) - u(\tau_o)\|_{L^2(\Omega,\mathbb{R}^N)} = 0 \tag{5.54}$$

for every $\tau_o \in (0,T)$ and u attains the initial values $u(0) = u_o$ in the sense

$$\lim_{t\downarrow 0} \|u(t) - u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} = 0. \tag{5.55}$$

Proof. The proof of (5.55) is almost literally the same as the one of (5.54), which is why we restrict ourselves to the proof of (5.54). For $\tau_o \in (0,T)$ and $\varepsilon > 0$ we can find $\delta > 0$ with

$$E^{\tau_o,\varepsilon} \times (\tau_o - \delta, \tau_o + 2\delta) \subset E. \tag{5.56}$$

This is due to the fact that E is a relatively open set and $E^{\tau_o,\varepsilon} \times \{\tau_o\} \Subset E$. Next, we choose a standard mollifier $\phi \in C_0^\infty(B_1, \mathbb{R}_{\ge 0})$ and let $\phi_\varepsilon(x) := \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$, so that $\phi_\varepsilon \in C_0^\infty(B_\varepsilon, \mathbb{R}_{\ge 0})$. Then, the function

$$u_{\tau_o}^{(\varepsilon)} := \big(u(\tau_o)\chi_{E^{\tau_o,2\varepsilon}}\big) * \phi_\varepsilon$$

satisfies $u_{\tau_o}^{(\varepsilon)} \in C_0^\infty\big(E^{\tau_o,\varepsilon},\mathbb{R}^N\big)$ and $u_{\tau_o}^{(\varepsilon)} \to u(\tau_o)$ in $L^2\big(\Omega,\mathbb{R}^N\big)$ as $\varepsilon \downarrow 0$. Moreover, according to (5.56) we have that $\operatorname{spt} u_{\tau_o}^{(\varepsilon)} \subset E^t$ for any $t \in (\tau_o - \delta, \tau_o + 2\delta)$. Furthermore, since $u_{\tau_o}^{(\varepsilon)}$ and $Du_{\tau_o}^{(\varepsilon)}$ are bounded, hypothesis (2.13) ensures that

$$\int_\Omega f\big(x, u_{\tau_o}^{(\varepsilon)}, Du_{\tau_o}^{(\varepsilon)}\big)\,\mathrm{d}x < \infty$$

for any $\varepsilon > 0$. We may choose $v(x,t) := \zeta(t)u_{\tau_o}^{(\varepsilon)}(x)$ as comparison function in the localized variational inequality (5.51), for any cutoff function $\zeta \in C_0^{0,1}(\tau_o - \delta, \tau_o + 2\delta)$ with $\zeta \geq 0$ and $\zeta \equiv 1$ on $[\tau_o, \tau_o + \delta]$. Hence, we obtain

$$\begin{aligned}\iint_{E\cap\Omega\times(\tau_o,\tau)} & f(x,u,Du)\,\mathrm{d}x\mathrm{d}t + \tfrac12\big\|u_{\tau_o}^{(\varepsilon)} - u(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} \\ &\leq \iint_{E\cap\Omega\times(\tau_o,\tau)} f\big(x, u_{\tau_o}^{(\varepsilon)}, Du_{\tau_o}^{(\varepsilon)}\big)\,\mathrm{d}x\mathrm{d}t + \tfrac12\|u_{\tau_o}^{(\varepsilon)} - u(\tau_o)\|^2_{L^2(\Omega,\mathbb{R}^N)}\end{aligned}$$

for every $\tau \in (\tau_o, \tau_o + \delta)$. Discarding the nonnegative first term on the left-hand side and applying the triangle inequality, we deduce

$$\begin{aligned}\big\|u(\tau_o) - u(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} & \\ &\leq 4(\tau - \tau_o)\int_\Omega f\big(x, u_{\tau_o}^{(\varepsilon)}, Du_{\tau_o}^{(\varepsilon)}\big)\,\mathrm{d}x + 4\big\|u_{\tau_o}^{(\varepsilon)} - u(\tau_o)\big\|^2_{L^2(\Omega,\mathbb{R}^N)}\end{aligned}$$

for every $\tau \in (\tau_o, \tau_o + \delta)$, where $\delta = \delta(\varepsilon)$. Now, for a given $\kappa > 0$ we first choose $\varepsilon > 0$ small enough to ensure

$$4\big\|u_{\tau_o}^{(\varepsilon)} - u(\tau_o)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} < \tfrac12\kappa^2.$$

This fixes $\delta > 0$ in dependence on κ. Next, we choose $\tau \in (\tau_o, \tau_o + \delta)$ sufficiently close to τ_o to guarantee

$$4(\tau - \tau_o)\int_\Omega f\big(x, u_{\tau_o}^{(\varepsilon)}, Du_{\tau_o}^{(\varepsilon)}\big)\,\mathrm{d}x < \tfrac12\kappa^2.$$

Collecting the estimates, we arrive at

$$\big\|u(\tau_o) - u(\tau)\big\|_{L^2(\Omega,\mathbb{R}^N)} < \kappa$$

if we choose $\tau > \tau_o$ close enough to τ_o. This yields the claim (5.54) and concludes the proof of the lemma. ∎

In view of Lemmas 5.11 and 5.13, the proof of Theorem 2.8 is complete.

5.4. Uniqueness. In this section, we give the proof of Theorem 2.9. We begin with a reformulation of the variational inequality.

LEMMA 5.14. *Under the growth condition* (2.13) *on the integrand and the assumptions* (2.10) *and* (2.12) *on the domain, consider a variational solution u of* (2.6) *in the sense of Definition* 2.1 *with $\partial_t u \in \big(V^p(E)\big)'$. Then u also satisfies the strong formulation of the variational inequality, i.e.,*

$$\iint_{E\cap\Omega_\tau} f(x,u,Du)\,\mathrm{d}x\mathrm{d}t \leq \langle \partial_t u, \chi_{\Omega_\tau}(v-u)\rangle + \iint_{E\cap\Omega_\tau} f(x,v,Dv)\,\mathrm{d}x\mathrm{d}t \tag{5.57}$$

for every $v \in V_2^p(E)$ and every $\tau \in (0,T)$.

Proof. We recall the variational inequality (5.53) from the proof of Lemma 5.12, which holds true under the present assumptions. According to the remark after (5.53), this inequality holds true for all τ_o, τ with $0 < \tau_o \le \tau < T$. In particular, we may choose $\tau_o = \tau \in (0, T)$, which yields the inequality

$$\iint_{E\cap\Omega_\tau} f(x,u,Du)\,\mathrm{d}x\mathrm{d}t \le \langle \partial_t u, \chi_{\Omega_\tau}(w-u)\rangle + \iint_{E\cap\Omega_\tau} f(x,w,Dw)\,\mathrm{d}x\mathrm{d}t \tag{5.58}$$

for any $w \in V_2^p(E)$ with $\partial_t w \in L^2(\Omega_T, \mathbb{R}^N)$. Given $v \in V_2^p(E)$, Lemma 5.1 provides us with a sequence $w_i \in C_0^\infty\big(E, \mathbb{R}^N\big)$, $i \in \mathbb{N}$, with $w_i \to v$ in the norm topology of $V_2^p(E)$. Since $w_i \in C_0^\infty\big(E, \mathbb{R}^N\big)$, these maps are admissible in (5.58). By virtue of $\partial_t u \in \big(V^p(E)\big)'$ and the growth assumption (2.13), the convergence in $V_2^p(E) \subset V^p(E)$ is sufficient to pass to the limit in (5.58), which yields the asserted strong formulation (5.57) of the variational inequality. ∎

As an immediate corollary, we record the following.

COROLLARY 5.15. *Suppose that $f\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to [0,\infty]$ is a variational integrand with* $(2.4)_1$ *and* (2.13) *for which the partial maps $(u,\xi) \mapsto f(x,u,\xi)$ are of class C^1 for a.e. $x \in \Omega$. For the noncylindrical domain E, we assume* (2.10) *and* (2.12). *Then every variational solution u of* (2.6) *with $\partial_t u \in \big(V^p(E)\big)'$ is a weak solution of the differential equation*

$$\partial_t u - \operatorname{div}\big(D_\xi f(x,u,Du)\big) = -D_u f(x,u,Du) \quad \textit{in } E. \tag{5.59}$$

Proof. For any $\varphi \in C_0^\infty\big(E, \mathbb{R}^N\big)$ and $s > 0$, the maps $v = u + s\varphi$ are admissible as comparison maps in the strong formulation of the variational inequality (5.57). Dividing the resulting inequality by s and rearranging terms, we arrive at

$$\begin{aligned}\iint_E u\cdot\partial_t\varphi\,\mathrm{d}x\mathrm{d}t = -\langle\partial_t u,\varphi\rangle &\le \tfrac{1}{s}\iint_E \Big[f(x,u+s\varphi,Du+sD\varphi) - f(x,u,Du)\Big]\mathrm{d}x\mathrm{d}t\\ &\to \iint_E \Big[D_\xi f(x,u,Du)\cdot D\varphi + D_u f(x,u,Du)\cdot\varphi\Big]\mathrm{d}x\mathrm{d}t,\end{aligned}$$

in the limit $s \to 0$. For the passage to the limit, we rely on the dominated convergence theorem and the Lipschitz condition (4.13), which is valid under our assumptions on f. Since we may replace φ by $-\varphi$ in the preceding argument, we also get the opposite inequality. Hence, we infer the equation

$$\iint_E u\cdot\partial_t\varphi\,\mathrm{d}x\mathrm{d}t = \iint_E \Big[D_\xi f(x,u,Du)\cdot D\varphi + D_u f(x,u,Du)\varphi\Big]\,\mathrm{d}x\mathrm{d}t$$

for any $\varphi \in C_0^\infty(E, \mathbb{R}^N)$, which is the weak formulation of (5.59). ∎

Now we proceed to the main result of this section.

Proof of Theorem 2.9. We recall that the assumptions of Theorem 2.9 include the conditions $(2.4)_1$ and (2.13) on the integrand and the properties (2.10) and (2.14) on the domain E. Let us assume that there are two solutions u_1, u_2 of the variational inequality (2.6) in the sense of Definition 2.1 that satisfy $\partial_t u_1, \partial_t u_2 \in \big(V^p(E)\big)'$. Lemma 5.14 implies that the variational inequalities can be rewritten in the form

$$\iint_{E\cap\Omega_\tau} f(x,u_i,Du_i)\,\mathrm{d}x\mathrm{d}t \le \langle\partial_t u_i, \chi_{\Omega_\tau}(v-u_i)\rangle + \iint_{E\cap\Omega_\tau} f(x,v,Dv)\,\mathrm{d}x\mathrm{d}t$$

for every $v \in V_2^p(E)$, every $\tau \in (0,T)$, and $i = 1, 2$. In particular, we may choose $v = \frac{1}{2}(u_1+u_2)$ in both inequalities for u_1 and for u_2. Adding the resulting inequalities, we deduce

$$\begin{aligned}
&\iint_{E\cap\Omega_\tau} f(x, u_1, Du_1)\,\mathrm{d}x\mathrm{d}t + \iint_{E\cap\Omega_\tau} f(x, u_2, Du_2)\,\mathrm{d}x\mathrm{d}t \\
&\quad\le \tfrac{1}{2}\langle \partial_t(u_1 - u_2), \chi_{\Omega_\tau}(u_2 - u_1)\rangle + 2\iint_{E\cap\Omega_\tau} f\big(x, \tfrac{1}{2}(u_1+u_2), \tfrac{1}{2}(Du_1+Du_2)\big)\,\mathrm{d}x\mathrm{d}t \\
&\quad\le \tfrac{1}{2}\langle \partial_t(u_1 - u_2), \chi_{\Omega_\tau}(u_2 - u_1)\rangle \\
&\qquad\quad + \iint_{E\cap\Omega_\tau} f(x, u_1, Du_1)\,\mathrm{d}x\mathrm{d}t + \iint_{E\cap\Omega_\tau} f(x, u_2, Du_2)\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

for every $\tau \in (0,T)$. In the last step we used the convexity assumption $(2.4)_1$. This implies

$$\langle \partial_t(u_1 - u_2), \chi_{\Omega_\tau}(u_1 - u_2)\rangle \le 0. \tag{5.60}$$

At this stage, we exploit our assumption (2.14). This assumption is equivalent to the conditions (2.12) and (5.11), which, respectively, are constraints on the decreasing and the increasing of the domain in time. This means that both Theorem 5.5 and Corollary 5.6 are applicable in the present situation, which implies

$$\langle \partial_t(u_1 - u_2), \zeta(u_1 - u_2)\rangle = -\tfrac{1}{2}\iint_E \partial_t\zeta |u_1 - u_2|^2\,\mathrm{d}x\mathrm{d}t$$

for any cutoff function in time $\zeta \in C_0^{0,1}(0,T)$. We use this identity with the functions ζ_ε defined in (5.52) with τ instead of τ_o and pass to the limit $\varepsilon \downarrow 0$. Since $u_1, u_2 \in C^0\big([0,T), L^2(\Omega,\mathbb{R}^N)\big)$ and $u_1(0) = u_o = u_2(0)$, which we already established in Theorem 2.8, we infer

$$\begin{aligned}
\langle \partial_t(u_1 - u_2), \chi_{\Omega_\tau}(u_1 - u_2)\rangle &= \tfrac{1}{2}\int_{\Omega\times\{\tau\}} |u_1 - u_2|^2\,\mathrm{d}x - \tfrac{1}{2}\int_{\Omega\times\{0\}} |u_1 - u_2|^2\,\mathrm{d}x \\
&= \tfrac{1}{2}\int_{\Omega\times\{\tau\}} |u_1 - u_2|^2\,\mathrm{d}x
\end{aligned}$$

for every $\tau \in (0,T)$. Combining this with (5.60), we deduce

$$\int_{\Omega\times\{\tau\}} |u_1 - u_2|^2\,\mathrm{d}x \le 0$$

for every $\tau \in (0,T)$, which yields the desired uniqueness result $u_1 = u_2$. ◻

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