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A General Coding Scheme for Signaling Gaussian Processes over Gaussian Decision Models

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Abstract—In this paper, we transform the \(n\)-finite transmission feedback information (FTFI) capacity of unstable Gaussian decision models with memory on past outputs, subject to an average cost constraint of quadratic form derived in [1], into controllers-encoders-decoders that control the output process, encode a Gaussian process, reconstruct the Gaussian process via a mean-square error (MSE) decoder, and achieve the \(n\)-FTFI capacity. For a Gaussian RV message \(X \sim N(0, \sigma_X^2)\) it is shown that the MSE decays according to \(\mathbb{E}[X - \hat{X}_n]^2 = \exp\{-2C_{0,n}(\kappa)\}\), \(\kappa \in (\kappa_{min}, \infty)\), where \(C_{0,n}(\kappa)\) is the \(n\)-FTFI capacity, and \(\kappa_{min}\) is the threshold on the power to ensure convergence.

Index Terms—Coding, \(n\)-finite transmission feedback information, Gaussian decision models, capacity.

I. INTRODUCTION

It has been recently shown [2] that randomized strategies in decision systems are operational, in the sense that not only they stabilize the system but also they encode information, which can be decoded at the output of the control system with arbitrary small probability of decoding error. In other words, the control system is used to communicate information, with an operational meaning as defined by Shannon’s capacity of communication channels.

Our main objective of this paper is to synthesize controller-encoder-decoder strategies for Gaussian recursive models (G-RMs), with input process \(\{A_t : t = 0, 1, \ldots, n\}\) that simultaneously control the output process \(\{Y_t : t = 0, 1, \ldots, n\}\), encode an information process \(\{X_t : t = 0, 1, \ldots, n\}\), and synthesize a decoder at the output \(\{Y_t : t = 0, 1, \ldots, n\}\) such that the process \(\{X_t : t = 0, 1, \ldots, n\}\) is transmitted reliably to the decoder.

If the G-RM is a control system model, then the system, as depicted in Figure 1, gives rise to several application. One application is that of tracking the dynamics of \(X_t\), \(t = 0, 1, \ldots, n\) at the output of the decoder. If \(\{X_t : t = 0, 1, \ldots, n\}\) is generated by a discrete deterministic recursion, then we show that it is possible for the decoder to track \(\{X_t : t = 0, 1, \ldots, n\}\) with arbitrary small MSE. This is contrary to standard tracking design systems in which the noise of the G-RM, i.e., \(\{V_t : t = 0, 1, \ldots, n\}\) imposes limitations on the minimum tracking error. Another application is that of signaling digital messages available to the controller, such as, values associated with actuating devices, for failure detection and monitoring applications.

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II. PROBLEM STATEMENT AND RELATED WORK

A. Notation

\(\langle \cdot, \cdot \rangle\) denotes inner product of elements of linear spaces, \(\mathcal{S}^q_{++}\) denotes the set of symmetric positive semi-definite matrices and \(\mathcal{S}^q_{++}\) the subset of positive definite matrices, with real entries.

B. Main Problem

We consider a multiple-input multiple output (MIMO) time-varying unstable G-RM given by

\[
Y_t = C_{i-1}Y_{t-1} + D_{i}A_{t} + V_{t}, \quad Y_{-1} = s, \quad i = 0, \ldots, n, \quad (1)
\]

subject to an average cost of quadratic form described by

\[
\frac{1}{n+1} \mathbb{E}_{s} \left\{ \sum_{i=0}^{n} \gamma_i(A_i, Y_{i-1}) \right\} \leq \kappa, \quad (2)
\]

\(A^n \triangleq \{A_0, A_1, \ldots, A_n\}\) is the input process, \(Y^n \triangleq \{Y_0, A_1, \ldots, Y_n\}\) is the output process, \(V^n \triangleq \{V_0, \ldots, V_n\}\) is an independent zero mean Gaussian noise process, denoted by \(V_i \sim N(0, K_{V_i}), K_{V_i} \succ 0, \quad i = 0, \ldots, n, \quad \gamma_i(a_i, y_{i-1}) \triangleq \langle a_i, R_i a_i \rangle + \langle y_{i-1}, Q_i y_{i-1} \rangle, \quad (C_{i-1}, D_i) \in \mathbb{R}^{p \times p} \times \mathbb{R}^{p \times q}, \quad (Q_i, R_i) \in \mathcal{S}^q_{++} \times \mathcal{S}^q_{++}.

The G-RM may correspond to a control system or a communication channel, as shown in Figure 1. The distribution of the G-RM is \(P_{Y_t \mid A, Y_{t-1}, s} = Q_t(dy_t | y_{t-1}, a_t), \quad i = 1, \ldots, n, \quad \text{and for } i = 0, \text{ the distribution is } Q_0(dy_0 | s, a_0).

By [1], the characterization of the \(n\)-finite transmission feedback information (FTFI) capacity is

\[
C_{0,n}(\kappa) = J_{A^n} - Y^n |_s (\pi^*_1, \kappa) \triangleq \sup_{\mathcal{P}_{[0,n]}(\kappa)} \left\{ \mathbb{E}_S \left\{ \sum_{i=0}^{n} \log \left( \frac{Q_i(dy_i | Y_{i-1}, A_i)}{P_{\pi}^*(dy_i | Y_{i-1})} \right) \right\} \right\} \quad (3)
\]

\[
P_{\pi}^*(dy_i | y_{i-1}) = \int_{A_i} Q_i(dy_i | y_{i-1}, a_i) \otimes \pi_i(da_i | y_{i-1}), \quad (4)
\]

for \(i = 1, \ldots, n, \quad \text{and} \quad P_{\pi}^*(dy_0 | s) = \int_{a_0} Q_0(dy_0 | s, a_0) \otimes \pi_0(da_0 | s), \quad (5)
\]

for \(i = 0. \text{ The set of randomized strategies } \mathcal{P}_{[0,n]}(\kappa) \text{ are conditionally Gaussian and Markov defined by} \)

\[
P_{[0,n]}(\kappa) \triangleq \left\{ \pi_i(da_i | y_{i-1}), \quad i = 0, \ldots, n : \frac{1}{n+1} \mathbb{E}_s \left\{ \sum_{i=0}^{n} \gamma_i(A_i, Y_{i-1}) \right\} \leq \kappa \right\} \subset \mathcal{P}_{[0,n]}(\kappa).
\]
The corresponding joint process \( \{A_0, Y_0, \ldots, A_n, Y_n\} \) and output process \( \{Y_0, \ldots, Y_n\} \), for fixed \( S = s \), are Gaussian. Further, under the standard detectability and stabilizability conditions [1, Theorem 4.1] the feedback capacity is
\[
C(\kappa) \triangleq \lim_{n \to \infty} \frac{1}{n+1} J_{A^n \to Y^n}(\pi^n, \kappa), \quad \kappa \in [\kappa_{\min}, \infty)..
\]
(6)

The main objective is to transform the distribution \( \pi_i(da_i|y_{i-1}), i = 0, \ldots, n \) that achieves the \( n \)–FCCI capacity and capacity \( C_{0,n}(\kappa) \), into a controller-encoder and to construct a decoder, such that

1. the controller-encoder operates at \( C_{0,n}(\kappa) \),
2. the decoder is MSE optimal,

for Gaussian, Markov process, described by the recursion
\[
X_{i+1} = F_i X_i + G_i W_i, \quad X_0 = x \in X_i \triangleq \mathbb{R}^q \quad (7)
\]
where \( W_i \sim N(0, K_{W_i}), i = 0, \ldots, n-1 \) are \( \mathbb{W}_i = \mathbb{R}^k \)–valued zero mean Gaussian, independent of \( V_i, i = 0, 1, \ldots, n \), and \( X_0 \sim N(0, K_{X_0}) \).

It should be mentioned that controllers-encoders have contradictory goals. The controller aims at stabilization, while the encoder aims at communicating new information. A low-cost control strategy would want the state process to be kept near the origin, with as little randomness as feasible injected by the coding part, while a communication strategy requires informative deviations.

C. Related Work

The Shannon coding capacity For the memoryless additive Gaussian noise (AGN) channel
\[
Y_i = A_i + V_i, \quad i = 0, \ldots, n, \quad \frac{1}{n+1} \mathbb{E} \{ \sum_{i=0}^{n} |A_i|^2 \} \leq \kappa \quad (8)
\]
with or without feedback, the capacity is given by [3]
\[
C^{\text{Sh}}(\kappa) \triangleq \frac{1}{2} \log \left( 1 + \frac{\kappa}{K_V} \right). \quad (9)
\]
The input process that achieves it is independent and identically distributed (IID) Gaussian \( A_i \sim P_A(da) \sim N(0, \sigma^2_A) \), \( i = 0, 1, \ldots \).

Elias Coding Scheme of a Gaussian Message. Elias [4] introduced a coding scheme to communicate a Gaussian RV \( X \sim N(0, \sigma^2_X) \) reliably over the memoryless AGN channel (8), that achieves \( C^{\text{Sh}}(\kappa) \), given by
\[
A_i = \left[ \frac{\kappa}{\mathbb{E} \{ X - \mathbb{E} \{ X|Y^{i-1}\} \}^2} \right] \left( X - \mathbb{E} \{ X|Y^{i-1}\} \right). \quad (10)
\]

**Fig. 1.** Depicts Shannon’s communication block diagram and its analogy to stochastic control systems.

### III. The \( n \)–FCCI Capacity and Controller-Encoder-Decoder

By [1] the \( n \)–FCCI capacity of the G-RM is achieved by the input process
\[
A_i = e_i(Y_{i-1}, Z_i) = U_i + Z_i = \Gamma_i Y_{i-1} + Z_i \quad (12)
\]

(i) \( U_i \triangleq \Gamma_i Y_{i-1}, i = 0, \ldots, n \) is the control process,
(ii) \( Z_i \) independent of \( (A^{i-1}, Y^{i-1}) \), \( i = 0, \ldots, n \),
(iii) \( Z^i \) independent of \( V^i \), for \( i = 0, \ldots, n \).
(iv) \( \{ Z_i \sim N(0, K_{Z_i}) \} : i = 0, \ldots, n \) an independent Gaussian process.

The corresponding output process is

\[ Y_i = \left( C_{i-1} + D_i \Gamma_i \right) Y_{i-1} + D_i Z_i + V_i, \quad Y_{-1} = s. \quad (13) \]

Furthermore, the optimal control and innovations parts of the strategy are found in [1, Section IV]. We include them below for completeness.

(a) The optimal control part of the strategy \( \{ U_i^* : i = 0, \ldots, n \} \) is given by

\[ U_i^* = \Gamma_i^* Y_{i-1}, \quad i = 0, \ldots, n, \quad (14) \]

\[ \Gamma_i^* = -\left( D_i^T P(i+1) D_i + R_i \right)^{-1} D_i^T P(i+1) C_{i-1}, \quad (15) \]

where \( \Gamma^*_n = 0, \{ P(i) : i = 0, \ldots, n \} \) is a solution of the Riccati difference matrix equation

\[ P(i) = C_{i-1}^T P(i+1) C_{i-1} + Q_{i-1} - C_{i-1}^T P(i+1) D_i \left( D_i^T P(i+1) D_i + R_i \right)^{-1} \left( C_{i-1}^T P(i+1) D_i \right)^T, \quad P(n) = Q_{n-1} \quad (16) \]

(b) The optimal innovations part of the strategy \( \{ K_{Z_i}^* \geq 0 : i = 0, \ldots, n \} \) is the solution of the following problem.

\[ J_{A^n \rightarrow Y^n} |_{s} (\pi^*, \kappa) = C_{0,n}(\kappa_0^*, \ldots, \kappa_n^*) = \sum_{i=0}^{n} C_i(\kappa_i^*) \quad (17) \]

\[ \Delta \sup_{K_{Z_0} \geq 0, i = 0, \ldots, n} \sum_{i=0}^{n} C_i(\kappa_i^*) \quad (18) \]

\[ C_i(\kappa_i) \equiv \frac{1}{2} \log \left| \frac{D_i K_{Z_i} D_i^T + K_{V_i}}{|K_{V_i}|} \right|, \quad i = 0, \ldots, n \quad (19) \]

\[ \kappa_i \equiv \kappa_i(K_{Z_i}) \quad (20) \]

\[ \Delta \begin{cases} 
\text{tr} \left( R_{n} K_{Z_n} \right), & i = n \\
\text{tr} \left( P(i+1) \left[ D_i K_{Z_i} D_i^T + K_{V_i} \right] \right) + R_{i} K_{Z_i}, & i = 1, \ldots, n - 1 \\
\text{tr} \left( P(1) \left[ D_0 K_{Z_0} D_0^T + K_{V_0} \right] + R_0 K_{Z_0} \right) + (s, P(0)s), & i = 0.
\end{cases} \]

**Example 3.1:** Let \( p = q = 1 \). Then the solution of the Riccati difference matrix equation (16) does not depend on the covariance \( K_{Z_i}, i = 0, \ldots, n \), and hence this simplifies the computation of the optimal \( K_{Z_i}^* \) in the optimization problem (18). By the Kuhn-Tucker conditions we obtain

\[ K_{Z_i}^* = \left\{ \frac{1}{2 \lambda R_{n} } - \frac{K_{V_i}}{D_n^2} \right\}^+, \quad \{ x \}^+ \equiv \max \{ 0, x \}, \quad (21) \]

\[ K_{Z_i}^* = \left\{ \frac{1}{2 \lambda (P(i+1) D_i^2 + R_i)} - \frac{K_{V_i}}{D_i^2} \right\}^+, \quad (22) \]

for \( i = n - 1, n - 2, \ldots, 0 \), where \( \lambda = \lambda_n(\kappa) \geq 0 \) is chosen to satisfy the average constraint with equality given by

\[ \sum_{i=0}^{n-1} \left\{ \frac{1}{2 \lambda} \left( P(i+1) D_i^2 + R_i \right) K_{V_i} \right\}^+ + P(i+1) K_{V_i} \]

\[ + \left\{ \frac{1}{2 \lambda} \left( R_n K_{V_n} D_n^2 \right) \right\}^+ + s^2 P(0) = \kappa(n + 1). \quad (23) \]

Upon substituting into (19) then we obtain

\[ C_{0,n}(\kappa) \equiv J_{A^n \rightarrow Y^n} |_{s} (\pi^*, \kappa) = \frac{1}{2} \sum_{i=0}^{n} \log \left| \frac{D_i^2 K_{Z_i}^* + K_{V_i}}{|K_{V_i}|} \right| \quad (24) \]

Clearly, in general, for each \( i, C_i(\kappa_i^*) > 0 \) provided \( \kappa_i^* \in (\kappa_{min,i}, \infty) \) and these critical values depend on whether the coefficients of the G-RM (13) lie outside or inside the unit circle, i.e., \( |C_{i-1}| \geq 1 \) or \( |C_{i-1}| < 1 \), for \( i = 0, \ldots, n \) (see [1]). Expressions (21), (22) are known as water-filling in information theory. Clearly, the following hold.

If \( \kappa \equiv \kappa_{min} \in [0, \infty) \) is such that

\[ \lambda_n(\kappa_{min}) > \frac{D_n^2}{2 K_{V_n} R_n} \], \quad (25) \]

\[ \lambda_i(\kappa_{min}) > \frac{D_i^2}{2 K_{V_i} (P(i+1) D_i^2 + R_i)}, \quad i = 0, \ldots, n - 1, \quad (26) \]

then \( C_{0,n}(\kappa) = 0 \), and

\[ \kappa_{min} \equiv \kappa_{0,n}(K_{Z_i}^*) \bigg|_{K_{Z_n} = 0} = \sum_{i=0}^{n-1} P(i+1) K_{V_i} + s^2 P(0). \quad (27) \]

\[ \kappa_{min} \equiv \kappa_{0,n}(K_{Z_i}^*) \bigg|_{K_{Z_n} = 0} = \sum_{i=0}^{n-1} P(i+1) K_{V_i} + s^2 P(0). \quad (28) \]

Hence, \( \kappa_{min} \) is the minimum power above which a non-negative information rate occurs, i.e., \( C_{0,n}(\kappa) > 0, \forall \kappa \in (\kappa_{min}, \infty) \). \( \kappa_{min} \) is the solution of the Linear-Quadratic Gaussian stochastic optimal control problem of minimizing the average power, when \( A_i = g_{i}(Y_{i-1}, s), i = 0, \ldots, n \), i.e., non-random; see the numerical example in Fig 2.

![Fig. 2. Example in which the rate of a scalar system is shown for different values of power level, \( \kappa \), for \( n = 100 \) and \( n = 1000 \). Note that \( C_i \) is chosen such that \( |C_i| > 1 \).](image-url)
A. Optimal Controllers-Encoders

Now, we use the calculation of the $n$–FTFI capacity to show that linear controller-encoder strategies in $(x_i, y_i^{-1})$, denoted by $\{\mu^e_i(x_i, y_i^{-1}) : i = 0, \ldots, n\}$, achieve the $n$–FTFI, and that among such linear controller-encoders, then the conditional mean decoder minimizes the MSE.

**Theorem 3.2:** In the class of linear controller-encoders that encode the Gaussian Markov process $X^n$ defined by (7) and operate at the $n$–FTFI capacity, the optimal controller-encoder exists, the conditional mean decoder minimizes the MSE, and these are given below.

Let $\{(\Gamma_i^*, K_{Z_i}^*) : i = 0, \ldots, n\}$ be the optimal strategy corresponding to (15) and (18), and joint process $\{\{A_i^*, Y_i^*, Z_i^*\} : i = 0, \ldots, n\}$.

Define the filter estimates and conditional covariances

$$\tilde{X}_{i+1|i} \triangleq E_i^* \{X_i | Y_{i}^{*,-1}\}, \quad \tilde{X}_{i|i} \triangleq E_i^* \{X_i | Y_{i}^{*,-1}\},$$

$$\Sigma_{i+1|i} \triangleq E_i^* \{\left(X_i - \tilde{X}_{i+1|i}\right)\left(X_i - \tilde{X}_{i+1|i}\right)^T | Y_{i}^{*,-1}\},$$

$$\Sigma_{i|i} \triangleq E_i^* \{\left(X_i - \tilde{X}_{i|i}\right)\left(X_i - \tilde{X}_{i|i}\right)^T | Y_{i}^{*,-1}\}, \quad i = 0, \ldots, n.$$

(a) Controller-Encoder. The mutual information between $X^n$ and $Y^n$ for fixed $S = s$, denoted by $I^{\mu^e}(X^n; Y^n|s)$ of the controller-encoder strategy

$$A_i^* = \mu_i^L(X_i, Y_i^{*,i-1}) = \Gamma_i^* Y_i^{*,i-1} + \Theta_i^* \{X_i - \tilde{X}_{i|i-1}\},$$

$$\Theta_i = K_{Z_i}^*, \Sigma_{i|i-1} \triangleq \Sigma_i, \quad \Theta_i^* \triangleright 0, \quad i = 0, \ldots, n,$$

$$Y_i^* = \{C_{i-1} + D_i \Gamma_i^*\} Y_i^{*,i-1} + D_i \Theta_i^* \{X_i - \tilde{X}_{i|i-1}\} + V_i,$$

operates at the $n$–FTFI capacity of the G-RM, i.e.,

$$I^{\mu^e}(X^n; Y^n|s) = J_{A^n}^{\mu^e} \mu^e_i(X_i, Y_i^{*,i-1}|s, \sigma^n, \kappa).$$

Moreover, the following hold.

(b) Filter Estimates. Define the innovations process by $\{\nu_i^e \triangleq Y_i^* - E_i^* \{Y_i^* | Y_{i}^{*,i-1}\} : i = 0, \ldots, n\}$. Then the optimal filter estimates satisfy the following recursions

$$\tilde{X}_{i+1|i} = F_i \tilde{X}_{i|i-1} + \Psi_{i|i-1} \{Y_i^* - \{C_{i-1} + D_i \Gamma_i^*\} Y_{i}^{*,i-1}\},$$

$$\tilde{X}_{i+1|i+1} = F_i \tilde{X}_{i|i} + \Psi_{i|i} \nu_i^e,$$

$$\Sigma_{i|i+1} = F_i \Sigma_{i|i} F_i^T + G_i K_{W_i} G_i^T - F_i \Sigma_{i|i-1} (D_i \Theta_i^*)^T \right\}{\left[D_i K_{Z_i}^* D_i^T + K_{V_i}\right]}^{-1} \left[D_i \Theta_i^* \right] \Sigma_{i|i-1} F_i^T,$$

$$\Sigma_{i|i-1} = E_i \{\left(X_i - \tilde{X}_{0|i}\right)\left(X_i - \tilde{X}_{0|i}\right)^T \},$$

$$\Sigma_{i|i} = \Sigma_{i|i-1} - \Psi_{i|i-1} \left[D_i \Theta_i^* \right] \Sigma_{i|i-1},$$

$$\Sigma_{i+1|i} = F_i \Sigma_{i|i} F_i^T + G_i K_{W_i} G_i^T.$$

where the filter gains are defined by

$$\Psi_{i|i-1} \triangleq F_i \tilde{X}_{i|i-1}, \quad \Sigma_{i|i-1} \triangleq \Sigma_{i|i-1} \left[D_i \Theta_i^* \right] T \left[D_i K_{Z_i}^* D_i^T + K_{V_i}\right]^{-1}.$$
B. Asymptotic Properties

The asymptotic properties of the controller-encoder-decoder are obtained by analyzing (6), under the following assumptions (see [14] on Linear Quadratic stochastic optimal control theory with complete information).

**Assumption 3.4:** (Time-invariant policies) Suppose the G-RM is time-invariant with parameters \((C, D, K_Y, R, Q)\) and

1. the pair \((C, D)\) is stabilizable;
2. the pair \((G, C)\) is detectable, \(Q \overset{\Delta}{=} G^T G, \ G \in \mathbb{S}_+^{p \times p};\)
3. the policies are restricted to time-invariant policies

\[
\pi_i (d_{a_i} | y_{i-1}) = \pi^\infty (d_{a_i} | y_{i-1}), \ i = 0, 1, \ldots
\]

Under Assumptions 3.4, then \(C(\kappa), \) defined by (6), is

\[
C(\kappa) = \sup_{K_Z \in \mathbb{S}_+^{m \times m}} \log \frac{||DK_ZD^T + K_Y||}{||K_Y||}, \tag{45}
\]

\[
\text{tr} (RK_Z) + \text{tr} (PDK_ZD^T + K_Y) \leq \kappa \tag{46}
\]

for \(\kappa \in [\kappa_{\text{min}}, \infty),\) and the optimal policy \(\Gamma^{\infty,*}\) is given by

\[
\Gamma^* = -\left( D^T PD + R \right)^{-1} D^T PC, \tag{47}
\]

\[
P = -C^T PD \left( D^T PD + R \right)^{-1} \left( C^T PD \right)^T + C^T PC + Q, \quad \text{spec}(C + D\Gamma^*) \subset \mathbb{D}_o. \tag{48}
\]

Here, \(\text{spec}(A) \subset \mathbb{C}\) denotes the set of all eigenvalues of \(A,\) and \(\mathbb{D}_o \overset{\Delta}{=} \{ c \in \mathbb{C} : |c| < 1 \}\) denotes the open unit disc of the space of complex number \(\mathbb{C}.\) Similarly, we can also identify conditions for the asymptotic properties of optimal controllers-encoders-decoders of Section III-A.

Further, by [1, Theorem 4.1], then \(C(\kappa)\) is the feedback capacity, irrespectively of whether matrix \(C\) of the G-RM is stable or unstable.

**Numerical example.** The asymptotic case can be easily solved using (45)-(48), for values satisfying Assumption 3.4.

**IV. Conclusions**

A constructive procedure is developed to synthesize \{controller-encoder-decoder\} strategies, that encode Gaussian Markov processes, communicate them over unstable Gaussian recursive models to the decoder. Examples illustrate the convergence of the MSE to zero, as the number of transmissions tends to infinity.

**References**


