Alestalo, Pekka; Trotsenko, Dmitry A.

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СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

RADIAL EXTENSIONS OF BILIPSCHITZ MAPS BETWEEN UNIT SPHERES

P. ALESTALO, D.A. TROTSENKO

ABSTRACT. Let E_1 and E_2 be real inner product spaces, and let S_1 and S_2 be the corresponding unit spheres. We consider different proofs showing that the radial extension of an L-bilipschitz map $f\colon S_1 \to S_2$ is L-bilipschitz with the same constant L. We also consider certain other sets having this kind of an extension property.

Keywords: bilipschitz map, unit sphere.

1. INTRODUCTION

Extending an L-bilipschitz map so that the extension has the same bilipschitz constant L may seem almost impossible, but it can be achieved in certain restricted cases. These are the optimal cases of stability, where one requires that the bilipschitz constant L' of the extension tends to 1 as the bilipschitz constant L of the original map tends to 1.

We start with a definition.

Definition 1. *Let (X, d) and (Y, d') be metric spaces, and let $L \geq 1$. A mapping $f\colon X \to Y$ is L-bilipschitz if*

$$d(x,y)/L \leq d'(f(x), f(y)) \leq Ld(x,y)$$

for all $x, y \in X$.

The following result is from [10, Theorem 4.5].

Theorem 1. *Every L-bilipschitz map $f\colon \mathbf{R}^k \to \mathbf{R}^k$ extends to an L-bilipschitz map $F\colon \mathbf{R}^n \to \mathbf{R}^n$ for $n > k$.*

This extension is easily obtained by the formula

$$F(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (f(x_1, \ldots, x_k), x_{k+1}, \ldots, x_n),$$

which also generalizes to the Hilbert space ℓ^2.

The following 1-dimensional result of [9] was proved by the second author many years before its publication. However, the main result [9, Theorem 5.1] gives a sharp geometric characterization for the set A that guarantees every L-bilipschitz map $f\colon A \to \mathbf{R}$ to be monotone, and thus extendable.

Theorem 2. *Let $A \subset \mathbf{R}$ and let $f\colon A \to \mathbf{R}$ be a monotone L-bilipschitz map. Then it has an L-bilipschitz extension $F\colon \mathbf{R} \to \mathbf{R}$.*

The proof is based on linear interpolation in the components of $\mathbf{R} \setminus A$.

2. Radial extensions of maps between unit spheres

A mapping defined on the unit sphere of a normed space has a natural radial extension. Here we need mainly a special case where the range is also a unit sphere, but the same radial extension formula can be applied to all mappings of the unit sphere into a normed space.

Definition 2. *Let E_1 and E_2 be real normed spaces with the corresponding unit spheres S_1 and S_2. For a mapping $f\colon S_1 \to S_2$ we define its radial extension $F\colon E_1 \to E_2$ by the formula*

$$F(x) = \begin{cases} \|x\| f\big(x/\|x\|\big), & \text{if } x \neq \overline{0}, \\ \overline{0}, & \text{if } x = \overline{0}. \end{cases}$$

Our main example of bilipschitz extension with the same constant is the theorem below. A 2-dimensional version appeared already in [4, Corollary 2.4] and similar results were obtained in [5, Section 3], and [6, Theorems 4.8 and 4.14]. These authors, as well as [8, Lemma 2.4], deal with a more general extension problem, where the image of f is not a circle or a sphere but a star-shaped Jordan curve or surface.

Some of the results mentioned above are sharp in the case where the Jordan curve is a circle, but the general setting makes the proofs more complicated. Therefore, we present here two elementary proofs. We also note that yet another proof was given by J. Väisälä (also in inner product spaces, pers. comm. in 2013).

Theorem 3. *Let E_1 and E_2 be real inner product spaces and let $f\colon S_1 \to S_2$ be an L-bilipschitz map between the unit spheres. Then the radial extension $F\colon E_1 \to E_2$ of f is L-bilipschitz.*

Proof. (Calculus version) Since $F^{-1}(x) = \|x\| f^{-1}(x/\|x\|)$ for $x \neq \overline{0}$, it is enough to show that F is L-Lipschitz.

Let $x, y \in E_1$. Choose 2-dimensional subspaces containing $\overline{0}$, x and y in E_1, and $\overline{0}$, $f(x)$ and $f(y)$ in E_2. In this way we can reduce the proof to the case where $F\colon \mathbf{R}^2 \to \mathbf{R}^2$.

Let thus $x, y \in \mathbf{R}^2$ and let $x' = F(x)$, $y' = F(y)$. We may assume that $0 < \|y\| \leq \|x\|$. Since $\|x'\| = \|x\|$ and $\|y'\| = \|y\|$, we can again rotate to obtain a

further simplification $x_2' = x_2 = 0$ and $x_1' = x_1$, so that $x' = x$. Let α and α' be the angles between x and y, or x' and y', respectively. Also, let

$$d = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|, \ d' = \left\| \frac{x'}{\|x'\|} - \frac{y'}{\|y'\|} \right\| = \left\| f\!\left(x/\|x\|\right) - f\!\left(y/\|y\|\right) \right\|.$$
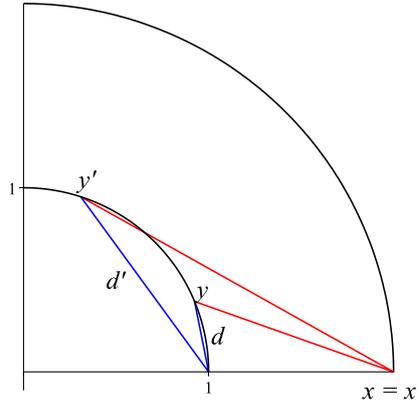

FIG. 1. The special case $\|y\| = \|y'\| = 1$ is shown. Here $x/\|x\| = x'/\|x'\| = 1$ on the horizontal axis after normalization. The angles α and α' are preserved under radial projection to the unit sphere.

The Lipschitz continuity of f implies that $d' \le Ld$. By the Cosine Theorem, we have

$$\begin{cases} \|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos \alpha, \\ \|x' - y'\|^2 = \|x'\|^2 + \|y'\|^2 - 2\|x'\|\|y'\| \cos \alpha'. \end{cases}$$

Since $\sin(\alpha/2) = d/2$, we have $\cos \alpha = 1 - d^2/2$. In a similar way,

$$\cos \alpha' = 1 - \frac{d'^2}{2} \ge 1 - \frac{L^2 d^2}{2}.$$

Writing $t = \|y\|/\|x\| = \|y'\|/\|x'\| \le 1$ and combining the results above, we obtain

$$\begin{aligned} \frac{\|x' - y'\|^2}{\|x - y\|^2} &= \frac{t^2 - 2t \cos \alpha' + 1}{t^2 - 2t \cos \alpha + 1} \le \frac{t^2 - 2t(1 - L^2 d^2/2) + 1}{t^2 - 2t(1 - d^2/2) + 1} \\ &= \frac{t^2 - 2t + 1 + L^2 d^2 t}{t^2 - 2t + 1 + d^2} \equiv h(d, t). \end{aligned}$$

Since

$$\frac{\partial h}{\partial t} = \frac{d^2(L^2 - 1)(1 - t^2)}{(t^2 - 2t + 1 + d^2 t)^2} \ge 0,$$

the maximum of $h(d, t)$ for $0 \le t \le 1$ is $h(d, 1) = L^2$. This implies that $\|x' - y'\| \le L\|x - y\|$, and the proof is complete. $\qquad\square$

Proof. (Real analysis version; proof in $\mathbf{R}^n$) By Theorem 4 below, the radial extension F is $(2L + 1)$-bilipschitz, so there are no serious regularity problems. We show that the bilipschitz constant is actually at most L. We calculate the derivative of the radial extension in spherical coordinates. By Rademacher's theorem [2, 3.1.2]

the Lipschitz map f is differentiable almost everywhere in the unit sphere, and $\|Df\|_\infty \leq L$ by [3, Remark 4.2]. Since

$$DF = \begin{bmatrix} Df & 0 \\ 0 & 1 \end{bmatrix},$$

we have $\|DF\|_\infty \leq L$ as well. By [3, 4.2], this implies that F is L-bilipschitz. The same reasoning applies to F^{-1} and the claim follows. $\square$

Remark 1. (i) In infinite-dimensional cases the original mapping f need not be surjective. Then the inverse appearing in the proof must be suitably interpreted; c.f. the following item (ii).

(ii) The first proof also applies to the case, where f is defined only in a subset $A \subset S_1$. In this case the extension F will be defined in the cone $\mathbf{R}_+ A$; c.f. [1, Section 4].

(iii) Radial extension with the same bilipschitz constant is no longer true if $f: S \to S$ and $S \subset \mathbf{R}^n$ is a smooth convex surface different from a sphere. Here is a sketch of the proof for $n = 2$: Let $x_0 \in S$ be a point where the tangent line T of S is not orthogonal to the ray R from the origin passing through x_0. For every $L > 1$, we can easily construct a smooth L-bilipschitz map $f: S \to S$ such that $\|Df(x_0)\| = L$. As in the proof above, the radial part of DF is an isometry, but since T is not orthogonal to R, the maximal stretching $\|DF(x_0)\|$ of the radial extension F at x_0 will be strictly greater than L.

As a corollary, we obtain a similar extension result for mappings of a product type between cylinders.

Corollary 1. *Let $S \subset \mathbf{R}^m$ be the unit sphere and let $f: \mathbf{R}^k \times S \to \mathbf{R}^k \times S$ be an L-bilipschitz map of the form $f(x, y) = (f_1(x), f_x(y))$, where $f_1: \mathbf{R}^k \to \mathbf{R}^k$ and $f_x: S \to S$ for every $x \in \mathbf{R}^k$. Then f has an L-bilipschitz extension $F: \mathbf{R}^n \to \mathbf{R}^n$ for all $n \geq k + m$. Moreover, the extension F can be chosen to satisfy the similar product form $F(x, y) = (F_1(x), F_x(y))$ as the original map f.*

Proof. The claim follows by combining Theorems 1 and 3. $\square$

3. Radial extensions in normed spaces

The following result was proved in [1] with a slightly larger constant $3L$ instead of $2L + 1$. The smaller constant follows, at least implicitly, from [7, 2.12], and a 2-dimensional version can be found in [4, 2.2]. For completeness, we give here the proof in the general case.

Theorem 4. *Let E_1 and E_2 be real normed spaces and let $f: S_1 \to S_2$ be an L-bilipschitz map between the unit spheres. Then the radial extension $F: E_1 \to E_2$ of f is $(2L + 1)$-bilipschitz.*

Proof. Let $x, y \in E_1 \setminus \{\overline{0}\}$. We have

$$
\begin{aligned}
\|x\|\|y\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| &= \left\| \|y\|x - \|x\|y \right\| \\
&\leq \left\| \|y\|x - \|y\|y \right\| + \left\| \|y\|y - \|x\|y \right\| \\
&\leq \|y\|\|x - y\| + \|x - y\|\|y\| = 2\|y\|\|x - y\|,
\end{aligned}
$$

and thus
$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\|}.$$

This implies that

$$
\begin{aligned}
\|F(x) - F(y)\| &= \left| \|x\|f(x/\|x\|) - \|y\|f(y/\|y\|) \right| \\
&\leq \left| \|x\|f(x/\|x\|) - \|x\|f(y/\|y\|) \right| \\
&\quad + \left| \|x\|f(y/\|y\|) - \|y\|f(y/\|y\|) \right| \\
&\leq \|x\| \cdot L \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| + \left| \|x\| - \|y\| \right| \underbrace{\|f(y/\|y\|)\|}_{=1} \\
&\leq (2L+1)\|x-y\|.
\end{aligned}
$$

Since F^{-1} is the radial extension of f^{-1}, the converse part is also true. This completes the proof. $\qquad\square$

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PEKKA ALESTALO
DEPARTMENT OF MATHEMATICS AND SYSTEMS ANALYSIS,
AALTO UNIVERSITY, PL 11100 AALTO,
HELSINKI, FINLAND
E-mail address: pekka.alestalo@aalto.fi

DMITRY ALEXANDROVICH TROTSENKO
SOBOLEV INSTITUTE OF MATHEMATICS,
PR. KOPTYUGA, 4,
630090, NOVOSIBIRSK, RUSSIA
E-mail address: trotsenk@yandex.ru