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Vector-valued Lambertian fields and their sources

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The electromagnetic field within an aperture in the wall of a blackbody cavity is a known example of a Lambertian source producing a far field which is unpolarized in all directions. In this work we show that in the electromagnetic context other Lambertian sources exist whose far fields, while obeying the cosine law for the radiant intensity, differ by their polarization states and degrees. For example, the far field may be azimuthally, radially, or circularly polarized, or the polarization state may vary depending on the direction. For specific Lambertian fields generated by quasihomogeneous sources it is possible to calculate explicitly the 3×3 electric cross-spectral density matrix of the nonevanescent part of the source. This enables one to assess the source's spatial coherence and partial polarization properties. In all cases, the coherence length turns out to be roughly half a wavelength, whereas the polarization characteristics of the sources may differ significantly. Our results could find uses, for instance, in radiometry and photometry, lighting applications, and remote sensing.

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I. INTRODUCTION

In recent years many results of classical coherence theory [1] have been extended from the approximate scalar representation of electromagnetic fields to a rigorous vectorial description (see, e.g., [2–4]). While the extension is straightforward in many cases, with expected results, there are nonetheless situations in which the transition from a one-component to a two- or even three-component description has led to interesting results and enforced reinterpretations and bifurcations of old concepts [5–8]. Here we shall examine the concept of a random, planar Lambertian source [9,10] in the true three-component vectorial setting. A Lambertian source is a radiator that gives rise to a field, whose radiant intensity in the observation half-space follows the cosine law, or equivalently, the radiance is constant, independent of the direction of propagation [11]. Lambertian sources are extremely important in many applications and they can, for instance, be used to model diffuse reflections successfully as is exemplified by the central role these sources enjoy in the assessment of surface roughness and scattering [12], the interpretation of satellite imagery [13,14], and in three-dimensional computer graphics [15]. There have not, however, been many investigations in which the far-field polarization state, i.e., the vectorial nature of the radiation, and its influence on the electromagnetic source correlation properties have been studied.

The archetypal Lambertian source, in both scalar and electromagnetic theory, is a planar aperture in the wall of a blackbody cavity [16–19]. The spatial coherence properties of the aperture field are explicitly known and are described by a 3×3 electric cross-spectral density matrix with a sinc-form trace. An analogous result in the scalar case states that the cosine law of the radiant intensity only occurs if the source correlations are of the sinc form [9]. We will show here that the situation is considerably more nuanced in the framework of vector-valued fields. Specifically, we demonstrate that the coherence properties of a Lambertian source can be far removed from those of a blackbody aperture. Indeed, the cosine-law behavior of the radiant intensity imposes no restrictions on the polarization properties of the radiation, implying that a large number of distinct electromagnetic Lambertian sources may exist. We specialize to quasihomogeneous electromagnetic source fields, which physically are characterized by rapid spatial correlation variations as compared to the intensity profiles, and analyze in detail a few important Lambertian special cases.

This paper is structured so that in Sec. II we recall, for later purposes, the formalism of propagating a nonparaxial electromagnetic field from a source plane to the far zone. Next in Sec. III we present the key quantities of vector-valued second-order coherence theory, including definitions for the degrees of spatial coherence and polarization. In Secs. IV and V we, respectively, introduce the electromagnetic quasihomogeneous sources and display the equations that identify and characterize Lambertian sources for vector-valued electromagnetic fields. Some interesting special cases of such sources, among them blackbody radiation from an aperture, are considered in Sec. VI, where we in particular compare the spatial coherence and polarization properties of the source fields. In Sec. VII we present a summary of our results and consider how they can be extended and built upon. The mathematical details of the derivations of some key expressions can be found in the two appendixes.

II. FAR FIELD PRODUCED BY A PLANAR, MONOCHROMATIC, ELECTROMAGNETIC SECONDARY SOURCE

The behavior of a monochromatic electromagnetic field with angular frequency ω is at a point $\mathbf{r}$ in vacuum fully

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described by the vector wave equation

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},\omega) - k^2\mathbf{E}(\mathbf{r},\omega) = \mathbf{0}, \tag{1}$$

where $\mathbf{E}(\mathbf{r},\omega)$ is the amplitude of the electric field and $k = \omega/c$ is the wave number with c being the vacuum speed of light. The corresponding magnetic field amplitude is given by

$$\mathbf{H}(\mathbf{r},\omega) = -\frac{i}{k}\nabla \times \mathbf{E}(\mathbf{r},\omega). \tag{2}$$

For our purposes it is useful to note that Eq. (1) is equivalent to the pair of equations consisting of the vector-valued Helmholtz equation

$$\nabla^2\mathbf{E}(\mathbf{r},\omega) + k^2\mathbf{E}(\mathbf{r},\omega) = \mathbf{0}, \tag{3}$$

and the divergence condition

$$\nabla \cdot \mathbf{E}(\mathbf{r},\omega) = 0, \tag{4}$$

which couples the components of the electric field vector. The Poynting vector that describes the energy flux of the electromagnetic field is defined by

$$\mathbf{P}(\mathbf{r},\omega) = \frac{c}{2}\mathrm{Re}\{\mathbf{E}(\mathbf{r},\omega) \times \mathbf{H}^*(\mathbf{r},\omega)\}, \tag{5}$$

where $*$ denotes complex conjugation.

We are interested in the electromagnetic field across a source plane (at $z = 0$) and throughout the far zone in the half-space $z > 0$. For a field propagating in vacuum, the scalar components of these two representations are connected by the Rayleigh diffraction formula [1], which extends to vectorial fields as

$$\mathbf{E}^{(\infty)}(\mathbf{r},\omega) = \int_{\mathcal{A}} G(\mathbf{r} - \boldsymbol{\rho},\omega)\mathbf{E}(\boldsymbol{\rho},\omega)\, d\boldsymbol{\rho}, \tag{6}$$

where the superscript (∞) identifies a far-field expression and the integration is over the planar source $\mathcal{A}$. We have adopted the convention that boldface Greek letters denote vectors in the plane $z = 0$ and that the corresponding integrals are two dimensional. We can remove the explicit dependence on $\mathcal{A}$ in Eq. (6) if we define $\mathbf{E}(\boldsymbol{\rho},\omega) = \mathbf{0}$ when $\boldsymbol{\rho} \notin \mathcal{A}$. We assume that this has been done and omit all explicit references to $\mathcal{A}$. On physical grounds the integrals and operators on the source are considered respectively convergent and permissible. We emphasize further that in obtaining the representation (6) it has implicitly been assumed that $\mathbf{E}(\boldsymbol{\rho},\omega)$ in the integrand is a true electric field, fully consistent with Maxwell's equations; $\mathbf{E}^{(\infty)}(\mathbf{r},\omega)$ is guaranteed to correspond to the correct far-zone electric field only when this assumption holds [20].

The Green's function in Eq. (6) is given by

$$\begin{aligned} G(\mathbf{r},\omega) &= -\frac{1}{2\pi}\partial_z\frac{\exp(ikr)}{r} = -i\frac{k}{2\pi}\partial_z h_0^{(1)}(kr) \\ &= i\frac{k^2}{2\pi}(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})h_1^{(1)}(kr), \end{aligned} \tag{7}$$

where $r = |\mathbf{r}|$, $\hat{\mathbf{r}} = \mathbf{r}/r$, $\hat{\mathbf{z}}$ is the unit vector in the z direction (perpendicular to the source plane), and $h_n^{(1)}(z)$ denotes the spherical Hankel function of type 1 and order n. In the far zone ($r \to \infty$) we may use the approximation

$$|\mathbf{r} - \boldsymbol{\rho}| \approx r - \hat{\mathbf{r}}\cdot\boldsymbol{\rho}, \tag{8}$$

whereby Eq. (7) together with the asymptotic properties of the spherical Hankel function [$h_1^{(1)}(z) \sim -\exp(iz)/z$] yield the far-field expression

$$G(\mathbf{r} - \boldsymbol{\rho},\omega) \sim -i\frac{k}{2\pi}(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\frac{\exp(ikr)}{r}\exp(-ik\hat{\mathbf{r}}\cdot\boldsymbol{\rho}). \tag{9}$$

With this we may rewrite the representation (6) as [1,21]

$$\mathbf{E}^{(\infty)}(\mathbf{r},\omega) = -i(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})^{1/2}\frac{\exp(ikr)}{r}\mathbf{a}(\hat{\mathbf{r}},\omega), \tag{10}$$

where $\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} > 0$. In addition, $\mathbf{a}(\hat{\mathbf{r}},\omega)$ is equal to

$$\mathbf{a}(\hat{\mathbf{k}},\omega) = \frac{k}{2\pi}(\hat{\mathbf{z}}\cdot\hat{\mathbf{k}})^{1/2}\int \mathbf{E}(\boldsymbol{\rho},\omega)\exp(-ik\hat{\mathbf{k}}\cdot\boldsymbol{\rho})\, d\boldsymbol{\rho}. \tag{11}$$

Here we have replaced $\hat{\mathbf{r}}$ with $\hat{\mathbf{k}}$, since $\mathbf{a}(\hat{\mathbf{k}},\omega)$ directly relates to the angular spectrum of the field $\mathbf{E}(\boldsymbol{\rho},\omega)$.

On using Eq. (6) in Eq. (2) and exchanging the order of integration and differentiation, we get the expression

$$\mathbf{H}^{(\infty)}(\mathbf{r},\omega) = -\frac{i}{k}\int [\nabla G(\mathbf{r} - \boldsymbol{\rho},\omega)] \times \mathbf{E}(\boldsymbol{\rho},\omega)\, d\boldsymbol{\rho} \tag{12}$$

for the magnetic field. From Eqs. (7) and (8), we find

$$\nabla G(\mathbf{r} - \boldsymbol{\rho},\omega) \sim \frac{k^2}{2\pi}(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\frac{\exp(ikr)}{r}\exp(-ik\hat{\mathbf{r}}\cdot\boldsymbol{\rho})\hat{\mathbf{r}}, \tag{13}$$

so that Eq. (12) yields the representation

$$\mathbf{H}^{(\infty)}(\mathbf{r},\omega) = -i(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})^{1/2}\frac{\exp(ikr)}{r}\hat{\mathbf{r}} \times \mathbf{a}(\hat{\mathbf{r}},\omega), \tag{14}$$

where we have also made use of Eq. (11). Starting from the Rayleigh integral (6) and using the asymptotic expression (13), we completely analogously arrive at

$$\nabla \cdot \mathbf{E}^{(\infty)}(\mathbf{r},\omega) = k(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})^{1/2}\frac{\exp(ikr)}{r}\hat{\mathbf{r}}\cdot\mathbf{a}(\hat{\mathbf{r}},\omega), \tag{15}$$

which together with the divergence condition (4), which here holds for $\mathbf{r}$ such that $\hat{\mathbf{z}}\cdot\mathbf{r} > 0$, implies that

$$\hat{\mathbf{k}}\cdot\mathbf{a}(\hat{\mathbf{k}},\omega) = 0, \tag{16}$$

for all $\hat{\mathbf{k}}$ with $\hat{\mathbf{z}}\cdot\hat{\mathbf{k}} > 0$.

Projection $\boldsymbol{\kappa}$ of vector $\hat{\mathbf{k}}$ onto the source plane $z = 0$ can be expressed as

$$\boldsymbol{\kappa} = (\mathbf{U} - \hat{\mathbf{z}}\hat{\mathbf{z}})\cdot\hat{\mathbf{k}}, \tag{17}$$

where $\mathbf{U}$ is the 3×3 unit matrix. Then, for $\hat{\mathbf{z}}\cdot\hat{\mathbf{k}} > 0$, we have

$$\hat{\mathbf{k}} = \frac{\kappa}{k}\hat{\boldsymbol{\kappa}} + \sqrt{1 - \left(\frac{\kappa}{k}\right)^2}\hat{\mathbf{z}}, \tag{18}$$

where $\kappa = |\boldsymbol{\kappa}|$ and, in particular,

$$k\hat{\mathbf{k}}\cdot\boldsymbol{\rho} = \boldsymbol{\kappa}\cdot\boldsymbol{\rho}, \tag{19}$$

since $\hat{\mathbf{z}}\cdot\boldsymbol{\rho} = 0$. We can thus rewrite Eq. (11) as

$$\mathbf{a}(\hat{\mathbf{k}},\omega) = \frac{k}{2\pi}(\hat{\mathbf{z}}\cdot\hat{\mathbf{k}})^{1/2}\int \mathbf{E}(\boldsymbol{\rho},\omega)\exp(-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho})\, d\boldsymbol{\rho}, \tag{20}$$

which represents a two-dimensional Fourier transform. Since $|\hat{\mathbf{k}}| = 1$, Eq. (17) implies that $\kappa \leqslant 1$, and consequently from the far field we can only recover a filtered version of $\mathbf{E}(\boldsymbol{\rho},\omega)$.

Specifically, we can obtain the propagating or nonevanescent (NE) component of the field in the source plane, given by

$$\mathbf{E}_{\mathrm{NE}}(\boldsymbol{\rho},\omega) = \frac{1}{2\pi k}\int_{\kappa\leqslant 1}(\hat{\mathbf{z}}\cdot\hat{\mathbf{k}})^{-1/2}\mathbf{a}(\hat{\mathbf{k}},\omega)\exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\rho})\,d\boldsymbol{\kappa}, \\ = \frac{k}{2\pi}\int_{\hat{\mathbf{z}}\cdot\hat{\mathbf{k}}>0}(\hat{\mathbf{z}}\cdot\hat{\mathbf{k}})^{1/2}\mathbf{a}(\hat{\mathbf{k}},\omega)\exp(ik\hat{\mathbf{k}}\cdot\boldsymbol{\rho})\,d\hat{\mathbf{k}}, \tag{21}$$

where the second equality follows from the relations (17)–(19). The first integral above is over the unit circle while the second is over a solid angle 2π in the positive z direction. Inverting Eq. (21) completes Eq. (20) by making explicit that $\mathbf{a}(\hat{\mathbf{k}},\omega)$ only depends on the nonevanescent part of the field, i.e.,

$$\mathbf{a}(\hat{\mathbf{k}},\omega) = \frac{k}{2\pi}(\hat{\mathbf{z}}\cdot\hat{\mathbf{k}})^{1/2}\int \mathbf{E}_{\mathrm{NE}}(\boldsymbol{\rho},\omega)\exp(-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho})\,d\boldsymbol{\rho}. \tag{22}$$

This relation connects the source and the far field it produces and it will be used in the later sections.

III. ELECTROMAGNETIC SPATIAL COHERENCE AND POLARIZATION

The second-order (classical) coherence properties of an electromagnetic field are described by the correlation operator (cross-spectral density matrix) of its electric field vector [1,22]

$$\mathbf{W}(\mathbf{r},\mathbf{r}',\omega) = \langle \mathbf{E}(\mathbf{r},\omega)\mathbf{E}^{\dagger}(\mathbf{r}',\omega)\rangle, \tag{23}$$

where $\langle\cdots\rangle$ denotes an ensemble average over monochromatic field realizations (as considered in the previous section) and $\dagger$ stands for the Hermitian transpose. This definition is the complex conjugate of what is typically employed in coherence theory, but it is more natural in a functional analytic context. The degree of coherence for electromagnetic fields is, in the space-frequency domain, defined as [4,22–24]

$$\mu_{\mathrm{EM}}(\mathbf{r},\mathbf{r}',\omega) = \frac{\|\mathbf{W}(\mathbf{r},\mathbf{r}',\omega)\|_{\mathrm{F}}}{\{\mathrm{tr}[\mathbf{W}(\mathbf{r},\mathbf{r},\omega)]\}^{1/2}\{\mathrm{tr}[\mathbf{W}(\mathbf{r}',\mathbf{r}'\omega)]\}^{1/2}}, \tag{24}$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the matrix Frobenius norm and tr represents the matrix trace. The degree is bounded as $0\leqslant \mu_{\mathrm{EM}}(\mathbf{r},\mathbf{r}',\omega)\leqslant 1$ with the lower and upper limit corresponding to complete incoherence and full coherence, respectively, at points $\mathbf{r}$ and $\mathbf{r}'$ and at frequency ω. The 3×3 polarization matrix characterizing the polarization properties of the field is obtained by setting $\mathbf{r}=\mathbf{r}'$ in Eq. (23), i.e.,

$$\Phi(\mathbf{r},\omega) = \mathbf{W}(\mathbf{r},\mathbf{r},\omega), \tag{25}$$

and the related degree of polarization is [5]

$$P_3(\mathbf{r},\omega) = \left\{\frac{3}{2}\left[\frac{\mathrm{tr}\,\Phi^2(\mathbf{r},\omega)}{\mathrm{tr}^2\,\Phi(\mathbf{r},\omega)} - \frac{1}{3}\right]\right\}^{1/2}. \tag{26}$$

This quantity obeys $0\leqslant P_3(\mathbf{r},\omega)\leqslant 1$ with the two limits reflecting a fully unpolarized and a completely polarized field at point $\mathbf{r}$, at frequency ω.

The radiant intensity is defined as the power radiated by the source per unit solid angle into the far zone [11]. For random electromagnetic fields the radiant intensity in the direction $\hat{\mathbf{r}}$ can be expressed as [25]

$$J(\hat{\mathbf{r}},\omega) = \lim_{r\to\infty}[r^2|\langle \mathbf{P}(\mathbf{r},\omega)\rangle|], \tag{27}$$

where $\mathbf{P}(\mathbf{r},\omega)$ is the Poynting vector given in Eq. (5).

IV. PLANAR, QUASIHOMOGENEOUS, ELECTROMAGNETIC SECONDARY SOURCES AND THEIR FAR FIELDS

Let us return to the electromagnetic source field in the plane $z=0$ and assume that it is quasihomogeneous. The general representation of vector-valued quasihomogeneous source distributions can be written as

$$\mathbf{W}(\boldsymbol{\Sigma},\boldsymbol{\Delta},\omega) = \mathbf{S}^{1/2}(\boldsymbol{\Sigma},\omega)\mathbf{M}(\boldsymbol{\Delta},\omega)\mathbf{S}^{1/2}(\boldsymbol{\Sigma},\omega), \tag{28}$$

where we have introduced the average and difference vectors $\boldsymbol{\Sigma} = (\boldsymbol{\rho}+\boldsymbol{\rho}')/2$ and $\boldsymbol{\Delta} = \boldsymbol{\rho}-\boldsymbol{\rho}'$, respectively, and the superscript $1/2$ denotes the positive matrix square root. The matrix $\mathbf{S}(\boldsymbol{\Sigma},\omega)$ is diagonal with the spectral densities of the Cartesian components of the electromagnetic field on its diagonal, whereas the elements of $\mathbf{M}(\boldsymbol{\Delta},\omega)$ are the correlation coefficients of the Cartesian field components. The matrix $\mathbf{S}(\boldsymbol{\Sigma},\omega)$ varies much more slowly with $\boldsymbol{\Sigma}$ than $\mathbf{M}(\boldsymbol{\Delta},\omega)$ does with $\boldsymbol{\Delta}$, which is a characteristic of a quasihomogeneous source. The representation (28) is consistent with (and an extension into three-component electric fields of) the quasihomogeneous beam-field sources presented, for instance, in [2,26–30]. To simplify the notation, we write Eq. (28) in the form

$$\mathrm{vec}[\mathbf{W}(\boldsymbol{\Sigma},\boldsymbol{\Delta},\omega)] = \mathfrak{S}(\boldsymbol{\Sigma},\omega)\,\mathrm{vec}[\mathbf{M}(\boldsymbol{\Delta},\omega)], \tag{29}$$

where vec[$\mathbf{F}$] denotes the 9×1 column vector whose elements are the elements of a 3×3 matrix $\mathbf{F}$ taken in column-first order [31], and

$$\mathfrak{S}(\boldsymbol{\Sigma},\omega) = \mathbf{S}^{1/2}(\boldsymbol{\Sigma},\omega)\otimes\mathbf{S}^{1/2}(\boldsymbol{\Sigma},\omega) \tag{30}$$

is a 9×9 matrix, where $\otimes$ denotes the matrix Kronecker product [31]. The diagonality of $\mathbf{S}(\boldsymbol{\Sigma},\omega)$ is inherited by $\mathfrak{S}(\boldsymbol{\Sigma},\omega)$.

Owing to the assumed quasihomogeneity of the source, whereby $\mathfrak{S}(\boldsymbol{\Sigma},\omega)$ is a slowly varying function as compared to $\mathbf{M}(\boldsymbol{\Delta},\omega)$, the spatial Fourier transform of $\mathfrak{S}(\boldsymbol{\Sigma},\omega)$ is sharply peaked whereas the Fourier transform of $\mathbf{M}(\boldsymbol{\Delta},\omega)$ is broad and slowly changing. Apart from possible pathological cases the mixing of the evanescent components of $\mathbf{M}(\boldsymbol{\Delta},\omega)$ into the nonevanescent part of $\mathbf{W}(\boldsymbol{\Sigma},\boldsymbol{\Delta},\omega)$, as caused by the Fourier-plane convolution corresponding to Eq. (29), is hence negligible and to a high degree of accuracy we have

$$\mathrm{vec}[\mathbf{W}_{\mathrm{NE}}(\boldsymbol{\Sigma},\boldsymbol{\Delta},\omega)] = \mathrm{vec}[\langle \mathbf{E}_{\mathrm{NE}}(\boldsymbol{\rho},\omega)\mathbf{E}^{\dagger}_{\mathrm{NE}}(\boldsymbol{\rho}',\omega)\rangle], \\ = \mathfrak{S}(\boldsymbol{\Sigma},\omega)\,\mathrm{vec}[\mathbf{M}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega)]. \tag{31}$$

Next we introduce the far-field correlation matrix

$$\mathbf{A}(\hat{\mathbf{k}},\omega) = \langle \mathbf{a}(\hat{\mathbf{k}},\omega)\mathbf{a}^{\dagger}(\hat{\mathbf{k}},\omega)\rangle, \tag{32}$$

with $\mathbf{a}(\hat{\mathbf{k}},\omega)$ given in Eq. (22). This matrix specifies the intensity and the polarization state of the far field in the direction $\hat{\mathbf{k}}$. Employing Eqs. (22) and (31) we obtain for quasihomogeneous electromagnetic sources that

$$\begin{aligned}\mathrm{vec}[\mathbf{A}(\hat{\mathbf{k}},\omega)] &= \left(\frac{k}{2\pi}\right)^2 (\hat{\mathbf{z}}\cdot\hat{\mathbf{k}}) \iint \mathrm{vec}[\langle \mathbf{E}_{\mathrm{NE}}(\boldsymbol{\rho},\omega)\mathbf{E}^{\dagger}_{\mathrm{NE}}(\boldsymbol{\rho}',\omega)\rangle] \\ &\quad\times \exp[-i\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')]\,d\boldsymbol{\rho}\,d\boldsymbol{\rho}' \\ &= \left(\frac{k}{2\pi}\right)^2 (\hat{\mathbf{z}}\cdot\hat{\mathbf{k}}) \iint \mathfrak{S}(\boldsymbol{\Sigma},\omega)\,\mathrm{vec}[\mathbf{M}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega)] \\ &\quad\times \exp(-i\boldsymbol{\kappa}\cdot\boldsymbol{\Delta})\,d\boldsymbol{\Sigma}\,d\boldsymbol{\Delta} \\ &= \left(\frac{k}{2\pi}\right)^2 (\hat{\mathbf{z}}\cdot\hat{\mathbf{k}})\,\tilde{\mathfrak{S}}(\mathbf{0},\omega) \int \mathrm{vec}[\mathbf{M}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega)] \\ &\quad\times \exp(-i\boldsymbol{\kappa}\cdot\boldsymbol{\Delta})\,d\boldsymbol{\Delta}, \end{aligned} \tag{33}$$

where

$$\tilde{\mathfrak{S}}(\boldsymbol{\xi},\omega) = \int \mathfrak{S}(\boldsymbol{\Sigma},\omega)\exp(-i\boldsymbol{\xi}\cdot\boldsymbol{\Sigma})\,d\boldsymbol{\Sigma}. \tag{34}$$

By inverting the transform in Eq. (33) we find

$$\begin{aligned}\mathrm{vec}[\mathbf{M}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega)] &= \frac{1}{k^2}[\tilde{\mathfrak{S}}(\mathbf{0},\omega)]^{\sharp} \int_{\kappa\leqslant 1} \frac{1}{(\hat{\mathbf{z}}\cdot\hat{\mathbf{k}})}\,\mathrm{vec}[\mathbf{A}(\hat{\mathbf{k}},\omega)]\exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\Delta})\,d\boldsymbol{\kappa} \\ &= [\tilde{\mathfrak{S}}(\mathbf{0},\omega)]^{\sharp}\,\mathrm{vec}\left[\int_{\hat{\mathbf{z}}\cdot\hat{\mathbf{k}}>0} \mathbf{A}(\hat{\mathbf{k}},\omega)\exp(ik\hat{\mathbf{k}}\cdot\boldsymbol{\Delta})\,d\hat{\mathbf{k}}\right], \end{aligned} \tag{35}$$

where $\sharp$ denotes the matrix pseudoinverse and the last equality follows as in Eq. (21). We will not here delve into the intricacies associated with a singular $\tilde{\mathfrak{S}}(\mathbf{0},\omega)$, so for the present purposes $\sharp$ can safely be interpreted simply as the ordinary matrix inverse. On introducing the representation (35) into Eq. (31), we finally obtain

$$\begin{aligned}\mathrm{vec}[\mathbf{W}_{\mathrm{NE}}(\boldsymbol{\Sigma},\boldsymbol{\Delta},\omega)] &= \mathfrak{S}(\boldsymbol{\Sigma},\omega)[\tilde{\mathfrak{S}}(\mathbf{0},\omega)]^{\sharp}\,\mathrm{vec}\left[\int_{\hat{\mathbf{z}}\cdot\hat{\mathbf{k}}>0} \mathbf{A}(\hat{\mathbf{k}},\omega)\exp(ik\hat{\mathbf{k}}\cdot\boldsymbol{\Delta})\,d\hat{\mathbf{k}}\right]. \end{aligned} \tag{36}$$

This formula gives the nonevanescent part of the correlation function of a planar, quasihomogeneous, electromagnetic secondary source in terms of the matrix $\mathbf{A}(\hat{\mathbf{k}},\omega)$ which specifies the far-field intensity and polarization state distributions.

V. ELECTROMAGNETIC LAMBERTIAN SOURCES

Let us now study the far-field correlation matrix defined in Eq. (33) and its relation to the properties of the source distribution in more detail. To begin with we note that the definition (5), the expressions (10) and (14), and the (divergence) condition (16) allow us to express the radiant intensity of Eq. (27) in terms of $\mathbf{a}(\hat{\mathbf{r}},\omega)$ and, ultimately, in terms of $\mathbf{A}(\hat{\mathbf{k}},\omega)$ as

$$J(\hat{\mathbf{r}},\omega) = \frac{c}{2}(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\langle|\mathbf{a}(\hat{\mathbf{r}},\omega)|^2\rangle = \frac{c}{2}(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\,\mathrm{tr}[\mathbf{A}(\hat{\mathbf{r}},\omega)], \tag{37}$$

where the last step follows from the definition (32). For scalar fields a Lambertian source is defined as a source whose radiant intensity obeys Lambert's law

$$J(\hat{\mathbf{r}},\omega) = (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})J(\omega). \tag{38}$$

We may directly extend this definition to electromagnetic fields, whereby Eq. (37) immediately implies that a vector-valued source is Lambertian if, and only if, it satisfies the condition

$$\mathrm{tr}[\mathbf{A}(\hat{\mathbf{k}},\omega)] = A(\omega), \tag{39}$$

for some constant $A(\omega) > 0$ and all $\hat{\mathbf{k}}$ with $\hat{\mathbf{z}}\cdot\hat{\mathbf{k}} > 0$.

However, besides the condition (39) amounting to the Lambertian property, it follows from the definition (32) and the condition (16) that $\mathbf{A}(\hat{\mathbf{k}},\omega)$, representing a far-field (or angular) correlation matrix, also has to satisfy the relations

$$\hat{\mathbf{k}}\cdot\mathbf{A}(\hat{\mathbf{k}},\omega) = \mathbf{0}, \quad \mathbf{A}(\hat{\mathbf{k}},\omega)\cdot\hat{\mathbf{k}} = \mathbf{0}, \tag{40}$$

i.e., $\mathbf{A}(\hat{\mathbf{k}},\omega)$ is a 2×2 matrix in the space orthogonal to $\hat{\mathbf{k}}$. As the matrix $\mathbf{A}(\hat{\mathbf{k}},\omega)$ is Hermitian and non-negative definite, the most general form that the far-field correlation matrix of a Lambertian source can have is, in view of the restrictions (39) and (40), thus given by

$$\mathbf{A}(\hat{\mathbf{k}},\omega) = A(\omega)\mathbf{B}^{\dagger}(\hat{\mathbf{k}},\omega)\mathbf{B}(\hat{\mathbf{k}},\omega), \tag{41}$$

where

$$\begin{aligned}\mathbf{B}(\hat{\mathbf{k}},\omega) &= \beta_{ss}(\hat{\mathbf{k}},\omega)\hat{\mathbf{s}}\hat{\mathbf{s}} + \beta_{sp}(\hat{\mathbf{k}},\omega)\hat{\mathbf{s}}\hat{\mathbf{p}} \\ &\quad + \beta_{ps}(\hat{\mathbf{k}},\omega)\hat{\mathbf{p}}\hat{\mathbf{s}} + \beta_{pp}(\hat{\mathbf{k}},\omega)\hat{\mathbf{p}}\hat{\mathbf{p}},\end{aligned} \tag{42}$$

and $\beta_{uv}(\hat{\mathbf{k}},\omega)$, with $u,v\in\{s,p\}$, are arbitrary complex functions that satisfy the normalization condition

$$\sum_{u,v}|\beta_{uv}(\hat{\mathbf{k}},\omega)|^2 = 1. \tag{43}$$

In addition, the unit vectors $\hat{\mathbf{s}}$ and $\hat{\mathbf{p}}$ in Eq. (42), corresponding to s- and p-polarized fields are defined in terms of the unit vector $\hat{\mathbf{k}}$, so that these three vectors form an orthonormal basis for $\mathbb{R}^3$. Specifically, if we represent the vector $\hat{\mathbf{k}}$ in the spherical coordinates (r,θ,ϕ) as

$$\hat{\mathbf{k}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}, \tag{44}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit vectors along the x and y axes, we explicitly have the representations

$$\hat{\mathbf{s}} = \frac{\hat{\mathbf{k}}\times\hat{\mathbf{z}}}{|\hat{\mathbf{k}}\times\hat{\mathbf{z}}|} = \sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}}, \tag{45}$$

and

$$\hat{\mathbf{p}} = \hat{\mathbf{k}}\times\hat{\mathbf{s}} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}. \tag{46}$$

We observe that the latter two vectors are well defined only when $\hat{\mathbf{k}}$ is not parallel to $\hat{\mathbf{z}}$, but since this special case corresponds to a set of measure zero (the z axis), it can safely be ignored.

VI. EXAMPLES OF VECTOR-VALUED LAMBERTIAN SOURCES

We next consider specific examples of the far-field correlation matrices $\mathbf{A}(\hat{\mathbf{k}},\omega)$ of the form (41) and determine the nonevanescent parts of the corresponding source distributions. For simplicity we only consider sources with

$$\mathfrak{S}(\boldsymbol{\Sigma},\omega)[\tilde{\mathfrak{S}}(\mathbf{0},\omega)]^{\sharp} \approx \sigma\mathfrak{U}, \tag{47}$$

for all $\boldsymbol{\Sigma}$ of interest. Here $\sigma > 0$ and $\mathfrak{U}$ is the 9×9 unit matrix. Physically the restriction (47) can be regarded to mean that we analyze planar electromagnetic secondary sources for which the spectral densities of the Cartesian components are approximately constant over the region of interest. For our purposes here, this is sufficient.

The condition (47) allows us to convert the representation (36) into the form

$$\mathbf{W}_{\mathrm{NE}}(\boldsymbol{\Sigma},\boldsymbol{\Delta},\omega) \approx \mathbf{W}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega) = \sigma \int_{\hat{\mathbf{z}}\cdot\hat{\mathbf{k}}>0} \mathbf{A}(\hat{\mathbf{k}},\omega)\exp(ik\hat{\mathbf{k}}\cdot\boldsymbol{\Delta})\,d\hat{\mathbf{k}}. \quad (48)$$

Making use of Eq. (39) we find, for any quasihomogeneous Lambertian source that obeys Eq. (47), that

$$\mathrm{tr}[\mathbf{W}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega)] = \sigma A(\omega)\int_{\hat{\mathbf{z}}\cdot\hat{\mathbf{k}}>0} \exp(ik\hat{\mathbf{k}}\cdot\boldsymbol{\Delta})\,d\hat{\mathbf{k}} = 2\pi\sigma A(\omega)\frac{\sin(k\Delta)}{k\Delta}, \quad (49)$$

indicating that the trace of the correlation function of the nonevanescent part of the source is of a sinc form, and also

$$\mathrm{tr}[\mathbf{W}_{\mathrm{NE}}(\mathbf{0},\omega)] = 2\pi\sigma A(\omega). \quad (50)$$

In what follows, the far-field distributions have $A(\omega) = 1$ so that without losing generality we can take $\sigma = (2\pi)^{-1}$, whereby the definition (24) gives the expression

$$\mu_{\mathrm{EM}}(\boldsymbol{\Delta},\omega) = \|\mathbf{W}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega)\|_{\mathrm{F}} \quad (51)$$

for the electromagnetic degree of coherence of the field.

A. Lambertian field I: Blackbody radiation

The far-field correlation matrix of blackbody radiation emanating from an aperture in the cavity wall is given by [16–18]

$$\mathbf{A}^{(\mathrm{bb})}(\hat{\mathbf{k}},\omega) = \tfrac{1}{2}(\hat{\mathbf{s}}\hat{\mathbf{s}} + \hat{\mathbf{p}}\hat{\mathbf{p}}). \quad (52)$$

This form corresponds to a far field which in every direction is composed of uncorrelated, equal-intensity, s-polarized and p-polarized components, i.e., locally the field is an unpolarized plane wave. The related cross-spectral density matrix of the source, which is obtained by introducing the expression (52) into the representation (48), has been determined elsewhere [up to the multiplicative factor $(2\pi)^{-1}$] and is given by [18]

$$\mathbf{W}_{\mathrm{NE}}^{(\mathrm{bb})}(\boldsymbol{\Delta},\omega) = \frac{j_1(k\Delta)}{k\Delta}\hat{\boldsymbol{\Delta}}\hat{\boldsymbol{\Delta}} + \frac{1}{2}\left[j_0(k\Delta) - \frac{j_1(k\Delta)}{k\Delta}\right] \times [(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}})(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}}) + \hat{\mathbf{z}}\hat{\mathbf{z}}] - \frac{i}{2}\frac{J_2(k\Delta)}{k\Delta}(\hat{\boldsymbol{\Delta}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\boldsymbol{\Delta}}), \quad (53)$$

where $J_n(z)$ and $j_n(z)$ denote the Bessel function and spherical Bessel function of order n, respectively, and $\Delta = |\boldsymbol{\Delta}|$. For the degree of coherence of this source field, we then find from Eq. (51) the expression [18]

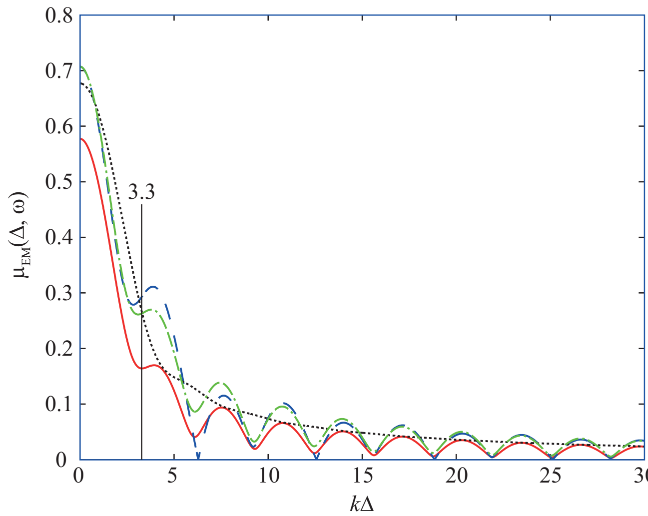

FIG. 1. Electromagnetic degree of coherence of the source for various Lambertian fields: blackbody radiation (red solid), s-polarized far field (blue dashed), p-polarized far field (green dash-dotted), and circularly polarized far field (black dotted). Vertical line at $k\Delta = 3.3$ marks the coherence length.

$$\mu_{\mathrm{EM}}^{(\mathrm{bb})}(\Delta,\omega) = \left\{\left[\frac{j_1(k\Delta)}{k\Delta}\right]^2 + \frac{1}{2}\left[j_0(k\Delta) - \frac{j_1(k\Delta)}{k\Delta}\right]^2 + \frac{1}{2}\left[\frac{J_2(k\Delta)}{k\Delta}\right]^2\right\}^{1/2}. \quad (54)$$

The behavior of $\mu_{\mathrm{EM}}^{(\mathrm{bb})}(\Delta,\omega)$ as a function of Δ is shown in Fig. 1 by the red solid curve. The first minimum is approximately at $k\Delta \approx 3.3$, corresponding to $\Delta \approx \lambda/2$. We may consider this distance to represent a coherence length of the source. The polarization matrix of Eq. (25) becomes

$$\Phi_{\mathrm{NE}}^{(\mathrm{bb})}(\mathbf{r},\omega) = \tfrac{1}{3}\mathbf{U}, \quad (55)$$

while for the degree of polarization across the source region Eq. (26) implies that

$$P_3^{(\mathrm{bb})}(\mathbf{r},\omega) = 0. \quad (56)$$

Thus the source, which is genuinely a three-component electric field, is completely unpolarized.

B. Lambertian field II: s-polarized far field

The far-field correlation matrix corresponding to a completely s-polarized (or azimuthally polarized) Lambertian field is given by

$$\mathbf{A}^{(\mathrm{s})}(\hat{\mathbf{k}},\omega) = \hat{\mathbf{s}}\hat{\mathbf{s}}. \quad (57)$$

Unlike in the previous blackbody radiation example the far field is now fully polarized in every direction. In Appendix A, we derive for the corresponding source distribution the expression

$$\mathbf{W}_{\mathrm{NE}}^{(\mathrm{s})}(\boldsymbol{\Delta},\omega) = \left[j_0(k\Delta) - \frac{\mathfrak{h}_0(k\Delta)}{k\Delta}\right] \times (\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}})(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}}) + \frac{\mathfrak{h}_0(k\Delta)}{k\Delta}\hat{\boldsymbol{\Delta}}\hat{\boldsymbol{\Delta}}. \quad (58)$$

Here $\mathfrak{h}_n(z)$ denotes the spherical Struve function of order n, which we define in terms of the Struve function $\mathbb{H}_{n+1/2}(z)$ as

$$\mathfrak{h}_n(z) = \sqrt{\frac{\pi}{2z}}\mathbb{H}_{n+1/2}(z). \quad (59)$$

Inserting Eq. (58) into Eq. (51), we readily obtain

$$\mu_{\mathrm{EM}}^{(\mathrm{s})}(\Delta,\omega) = \left\{\left[j_0(k\Delta) - \frac{\mathfrak{h}_0(k\Delta)}{k\Delta}\right]^2 + \left[\frac{\mathfrak{h}_0(k\Delta)}{k\Delta}\right]^2\right\}^{1/2}. \quad (60)$$

This degree of coherence is illustrated in Fig. 1 by the blue dashed curve. As for a blackbody radiator, the coherence length across the planar source is about $\lambda/2$. Using Eqs. (25) and (26), the polarization matrix and the degree of polarization of the source take on, respectively, the forms

$$\Phi_{\mathrm{NE}}^{(\mathrm{s})}(\mathbf{r},\omega) = \tfrac{1}{2}(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}}), \quad (61)$$

$$P_3^{(\mathrm{s})}(\mathbf{r},\omega) = \tfrac{1}{2}. \quad (62)$$

These equations show that the source has no z component and it corresponds to an electric field which is unpolarized in the sense of the traditional degree of polarization of two-component fields [1].

C. Lambertian field III: p-polarized far field

The far-field matrix corresponding to a completely p-polarized (or radially polarized) Lambertian field can be expressed as

$$\mathbf{A}^{(\mathrm{p})}(\hat{\mathbf{k}},\omega) = \hat{\mathbf{p}}\hat{\mathbf{p}} = 2\mathbf{A}^{(\mathrm{bb})}(\hat{\mathbf{k}},\omega) - \mathbf{A}^{(\mathrm{s})}(\hat{\mathbf{k}},\omega), \quad (63)$$

with $\mathbf{A}^{(\mathrm{bb})}(\hat{\mathbf{k}},\omega)$ and $\mathbf{A}^{(\mathrm{s})}(\hat{\mathbf{k}},\omega)$ given in Eqs. (52) and (57), respectively. The corresponding cross-spectral density matrix of this source is computed as $\mathbf{W}_{\mathrm{NE}}^{(\mathrm{p})}(\hat{\mathbf{k}},\omega) = 2\mathbf{W}_{\mathrm{NE}}^{(\mathrm{bb})}(\hat{\mathbf{k}},\omega) - \mathbf{W}_{\mathrm{NE}}^{(\mathrm{s})}(\hat{\mathbf{k}},\omega)$, where the two cross-spectral density matrices are found from Eqs. (53) and (58), leading to

$$\begin{aligned}\mathbf{W}_{\mathrm{NE}}^{(\mathrm{p})}(\boldsymbol{\Delta},\omega) = &\left[2\frac{j_1(k\Delta)}{k\Delta} - \frac{\mathfrak{h}_0(k\Delta)}{k\Delta}\right]\hat{\boldsymbol{\Delta}}\hat{\boldsymbol{\Delta}} \\ &+\left[-\frac{j_1(k\Delta)}{k\Delta} + \frac{\mathfrak{h}_0(k\Delta)}{k\Delta}\right](\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}})(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}}) \\ &+\left[j_0(k\Delta) - \frac{j_1(k\Delta)}{k\Delta}\right]\hat{\mathbf{z}}\hat{\mathbf{z}} - i\frac{J_2(k\Delta)}{k\Delta}(\hat{\boldsymbol{\Delta}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\boldsymbol{\Delta}}).\end{aligned} \quad (64)$$

In view of Eq. (51), the electromagnetic degree of coherence of the source assumes the form

$$\begin{aligned}\mu_{\mathrm{EM}}^{(\mathrm{p})}(\Delta,\omega) = &\left\{\left[2\frac{j_1(k\Delta)}{k\Delta} - \frac{\mathfrak{h}_0(k\Delta)}{k\Delta}\right]^2 + \left[-\frac{j_1(k\Delta)}{k\Delta} + \frac{\mathfrak{h}_0(k\Delta)}{k\Delta}\right]^2\right. \\ &\left.+\left[j_0(k\Delta) - \frac{j_1(k\Delta)}{k\Delta}\right]^2 + 2\left[\frac{J_2(k\Delta)}{k\Delta}\right]^2\right\}^{1/2}.\end{aligned} \quad (65)$$

The degree of coherence is depicted in Fig. 1 with the green dash-dotted line implying that, again, the coherence length is roughly $\lambda/2$. The polarization quantities of Eqs. (25) and (26) are found to be

$$\Phi_{\mathrm{NE}}^{(\mathrm{p})}(\mathbf{r},\omega) = \tfrac{1}{6}\mathbf{U} + \tfrac{1}{2}\hat{\mathbf{z}}\hat{\mathbf{z}}, \quad (66)$$

$$P_3^{(\mathrm{p})}(\mathbf{r},\omega) = \tfrac{1}{2}. \quad (67)$$

The polarization matrix is diagonal indicating that the orthogonal field components are uncorrelated. The intensities of the x and y field components are the same and, in contrast to the s-polarized Lambertian field, the source has a z component with an intensity higher than that of the transverse components.

D. Lambertian field IV: Circularly polarized far field

The far-field matrix corresponding to a circularly polarized Lambertian field is given by

$$\begin{aligned}\mathbf{A}^{(\pm)}(\hat{\mathbf{k}},\omega) &= \frac{1}{2}[\hat{\mathbf{s}}\hat{\mathbf{s}} + \hat{\mathbf{p}}\hat{\mathbf{p}} \pm i(\hat{\mathbf{s}}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{s}})] \\ &= \mathbf{A}^{(\mathrm{bb})}(\hat{\mathbf{k}},\omega) \pm \frac{i}{2}(\hat{\mathbf{s}}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{s}}),\end{aligned} \quad (68)$$

where the two signs refer to handedness of the circular polarization. The source correlation matrix, derived in Appendix B, can be written as

$$\begin{aligned}\mathbf{W}_{\mathrm{NE}}^{(\pm)}(\boldsymbol{\Delta},\omega) = &\frac{j_1(k\Delta)}{k\Delta}\hat{\boldsymbol{\Delta}}\hat{\boldsymbol{\Delta}} + \frac{1}{2}\left[j_0(k\Delta) - \frac{j_1(k\Delta)}{k\Delta}\right] \\ &\times[(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}})(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}}) + \hat{\mathbf{z}}\hat{\mathbf{z}}] - \frac{i}{2}\frac{J_2(k\Delta)}{k\Delta} \\ &\times(\hat{\boldsymbol{\Delta}}\hat{\mathbf{z}}+\hat{\mathbf{z}}\hat{\boldsymbol{\Delta}})\pm\frac{1}{2}j_1(k\Delta)[(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}})\hat{\mathbf{z}}-\hat{\mathbf{z}}(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}})] \\ &\pm\frac{i}{2}\frac{J_1(k\Delta)}{k\Delta}[\hat{\boldsymbol{\Delta}}(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}}) - (\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}})\hat{\boldsymbol{\Delta}}].\end{aligned} \quad (69)$$

The electromagnetic degree of coherence function is now obtained from Eq. (51) as

$$\begin{aligned}\mu_{\mathrm{EM}}^{(\pm)}(\Delta,\omega) = &\left\{\left[\frac{j_1(k\Delta)}{k\Delta}\right]^2 + \frac{1}{2}\left[j_0(k\Delta) - \frac{j_1(k\Delta)}{k\Delta}\right]^2\right. \\ &+\frac{1}{2}\left[\frac{J_2(k\Delta)}{k\Delta}\right]^2 + \frac{1}{2}[j_1(k\Delta)]^2 \\ &\left.+\frac{1}{2}\left[\frac{J_1(k\Delta)}{k\Delta}\right]^2\right\}^{1/2},\end{aligned} \quad (70)$$

which is the same for both left-hand and right-hand circular polarizations. The degree of coherence as a function of Δ is shown in Fig. 1 by the black dotted curve demonstrating that the coherence length is about $\lambda/2$ in this case as well. From Eqs. (25) and (26) we find

$$\Phi_{\rm NE}^{(\pm)}(\mathbf{r},\omega) = \frac{1}{3}\mathbf{U} \pm \frac{i}{4}(\hat{\mathbf{x}}\hat{\mathbf{y}} - \hat{\mathbf{y}}\hat{\mathbf{x}}), \tag{71}$$

$$P_3^{(\pm)}(\mathbf{r},\omega) = \frac{\sqrt{3}}{4}, \tag{72}$$

and we see that the source-field polarization matrix contains a component which corresponds to a (fully) circularly polarized field confined to the xy plane.

E. Lambertian field V: s- to p-polarization gradient

Next we consider a Lambertian far-field matrix of the form

$$\begin{aligned}
&\mathbf{A}^{(\mathrm{s}\to\mathrm{p})}(\hat{\mathbf{k}},\omega)\\
&\quad= \cos^2\theta\hat{\mathbf{s}}\hat{\mathbf{s}} + \sin^2\theta\hat{\mathbf{p}}\hat{\mathbf{p}} = \hat{\mathbf{s}}\hat{\mathbf{s}} + \sin^2\theta(\hat{\mathbf{p}}\hat{\mathbf{p}} - \hat{\mathbf{s}}\hat{\mathbf{s}})\\
&\quad= \mathbf{A}^{(\mathrm{s})}(\hat{\mathbf{k}},\omega) + [1-(\hat{\mathbf{z}}\cdot\hat{\mathbf{k}})^2][\mathbf{A}^{(\mathrm{p})}(\hat{\mathbf{k}},\omega) - \mathbf{A}^{(\mathrm{s})}(\hat{\mathbf{k}},\omega)],
\end{aligned} \tag{73}$$

where $\cos\theta = \hat{\mathbf{z}}\cdot\hat{\mathbf{k}}$. Thus the far field is s-polarized when θ is small and it changes gradually to p-polarized when θ increases. The linearity of the representation (48), the finiteness of the integration range, and the argument in the exponential within the integral imply that the cross-spectral density matrix of the source corresponding to the far-field matrix (73) can be written as

$$\begin{aligned}
&\mathbf{W}_{\rm NE}^{(\mathrm{s}\to\mathrm{p})}(\mathbf{\Delta},\omega)\\
&\quad= \mathbf{W}_{\rm NE}^{(\mathrm{s})}(\mathbf{\Delta},\omega) - \frac{1}{k^2}\left(\partial_\xi^2 + \partial_\eta^2\right)\left[\mathbf{W}_{\rm NE}^{(\mathrm{p})}(\mathbf{\Delta},\omega) - \mathbf{W}_{\rm NE}^{(\mathrm{s})}(\mathbf{\Delta},\omega)\right]\\
&\quad= \left[\frac{\mathfrak{h}_0(k\Delta)}{k\Delta} - 4\frac{j_2(k\Delta)}{(k\Delta)^2}\right]\hat{\mathbf{\Delta}}\hat{\mathbf{\Delta}}\\
&\qquad+ \left[2\frac{j_1(k\Delta)}{k\Delta} - \frac{\mathfrak{h}_0(k\Delta)}{k\Delta} + \frac{j_2(k\Delta)}{(k\Delta)^2}\right](\hat{\mathbf{z}}\times\hat{\mathbf{\Delta}})(\hat{\mathbf{z}}\times\hat{\mathbf{\Delta}})\\
&\qquad+ \left[j_0(k\Delta) - 2\frac{j_1(k\Delta)}{k\Delta} + 3\frac{j_2(k\Delta)}{(k\Delta)^2}\right]\hat{\mathbf{z}}\hat{\mathbf{z}}\\
&\qquad+ i\left[-\frac{J_2(k\Delta)}{k\Delta} + 2\frac{J_3(k\Delta)}{(k\Delta)^2}\right](\hat{\mathbf{\Delta}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{\Delta}}),
\end{aligned} \tag{74}$$

where the differentiations are straightforward but tedious and we have written $\mathbf{\Delta} = \xi\hat{\mathbf{x}} + \eta\hat{\mathbf{y}}$, and used Eqs. (58) and (64). The electromagnetic degree of coherence introduced in Eq. (51) takes on the form

$$\begin{aligned}
\mu_{\rm EM}^{(\mathrm{s}\to\mathrm{p})}(\Delta,\omega) = \Bigg\{&\left[\frac{\mathfrak{h}_0(k\Delta)}{k\Delta} - 4\frac{j_2(k\Delta)}{(k\Delta)^2}\right]^2\\
&+ \left[2\frac{j_1(k\Delta)}{k\Delta} - \frac{\mathfrak{h}_0(k\Delta)}{k\Delta} + \frac{j_2(k\Delta)}{(k\Delta)^2}\right]^2\\
&+ \left[j_0(k\Delta) - 2\frac{j_1(k\Delta)}{k\Delta} + 3\frac{j_2(k\Delta)}{(k\Delta)^2}\right]^2\\
&+ \left[-\frac{J_2(k\Delta)}{k\Delta} + 2\frac{J_3(k\Delta)}{(k\Delta)^2}\right]^2\Bigg\}^{1/2}.
\end{aligned} \tag{75}$$

Although not explicitly shown, the coherence length is again about $\lambda/2$. Using Eqs. (25) and (26), the polarization matrix and the degree of polarization become

$$\Phi_{\rm NE}^{(\mathrm{s}\to\mathrm{p})}(\omega) = \tfrac{7}{30}\hat{\mathbf{x}}\hat{\mathbf{x}} + \tfrac{7}{30}\hat{\mathbf{y}}\hat{\mathbf{y}} + \tfrac{16}{30}\hat{\mathbf{z}}\hat{\mathbf{z}}, \tag{76}$$

$$P_3^{(\mathrm{s}\to\mathrm{p})}(\mathbf{r},\omega) = \tfrac{3}{10}. \tag{77}$$

At a single point the source field has three uncorrelated electric field components of which the x and y components have the same intensity while the intensity of the z component is about twice as large. The degree of polarization indicates that the field is weakly polarized.

F. Lambertian field VI: p- to s-polarization gradient

Symmetrically with the previous example we then consider a Lambertian far-field matrix of the form

$$\begin{aligned}
\mathbf{A}^{(\mathrm{p}\to\mathrm{s})}(\hat{\mathbf{k}},\omega) &= \sin^2\theta\hat{\mathbf{s}}\hat{\mathbf{s}} + \cos^2\theta\hat{\mathbf{p}}\hat{\mathbf{p}}\\
&= (\hat{\mathbf{s}}\hat{\mathbf{s}} + \hat{\mathbf{p}}\hat{\mathbf{p}}) - (\cos^2\theta\hat{\mathbf{s}}\hat{\mathbf{s}} + \sin^2\theta\hat{\mathbf{p}}\hat{\mathbf{p}})\\
&= 2\mathbf{A}^{(\mathrm{bb})}(\hat{\mathbf{k}},\omega) - \mathbf{A}^{(\mathrm{s}\to\mathrm{p})}(\hat{\mathbf{k}},\omega),
\end{aligned} \tag{78}$$

with the corresponding cross-spectral density matrix of the source given by

$$\begin{aligned}
\mathbf{W}_{\rm NE}^{(\mathrm{p}\to\mathrm{s})}(\mathbf{\Delta},\omega) &= 2\mathbf{W}_{\rm NE}^{(\mathrm{bb})}(\mathbf{\Delta},\omega) - \mathbf{W}_{\rm NE}^{(\mathrm{s}\to\mathrm{p})}(\mathbf{\Delta},\omega)\\
&= \left[2\frac{j_1(k\Delta)}{k\Delta} - \frac{\mathfrak{h}_0(k\Delta)}{k\Delta} + 4\frac{j_2(k\Delta)}{(k\Delta)^2}\right]\hat{\mathbf{\Delta}}\hat{\mathbf{\Delta}}\\
&\quad+ \left[j_0(k\Delta) - 3\frac{j_1(k\Delta)}{k\Delta} + \frac{\mathfrak{h}_0(k\Delta)}{k\Delta} - \frac{j_2(k\Delta)}{(k\Delta)^2}\right]\\
&\quad\times(\hat{\mathbf{z}}\times\hat{\mathbf{\Delta}})(\hat{\mathbf{z}}\times\hat{\mathbf{\Delta}}) + \left[\frac{j_1(k\Delta)}{k\Delta} - 3\frac{j_2(k\Delta)}{(k\Delta)^2}\right]\\
&\quad\times\hat{\mathbf{z}}\hat{\mathbf{z}} - i2\frac{J_3(k\Delta)}{(k\Delta)^2}(\hat{\mathbf{\Delta}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{\Delta}}).
\end{aligned} \tag{79}$$

Using this representation it is straightforward to write the degree-of-coherence function as

$$\begin{aligned}
&\mu_{\rm EM}^{(\mathrm{p}\to\mathrm{s})}(\Delta,\omega)\\
&\quad= \Bigg\{\left[2\frac{j_1(k\Delta)}{k\Delta} - \frac{\mathfrak{h}_0(k\Delta)}{k\Delta} + 4\frac{j_2(k\Delta)}{(k\Delta)^2}\right]^2\\
&\qquad+ \left[j_0(k\Delta) - 3\frac{j_1(k\Delta)}{k\Delta} + \frac{\mathfrak{h}_0(k\Delta)}{k\Delta} - \frac{j_2(k\Delta)}{(k\Delta)^2}\right]^2\\
&\qquad+ \left[\frac{j_1(k\Delta)}{k\Delta} - 3\frac{j_2(k\Delta)}{(k\Delta)^2}\right]^2 + \left[2\frac{J_3(k\Delta)}{(k\Delta)^2}\right]^2\Bigg\}^{1/2}.
\end{aligned} \tag{80}$$

The polarization matrix is

$$\Phi_{\rm NE}^{(\mathrm{s}\to\mathrm{p})}(\mathbf{r},\omega) = \tfrac{13}{30}\hat{\mathbf{x}}\hat{\mathbf{x}} + \tfrac{13}{30}\hat{\mathbf{y}}\hat{\mathbf{y}} + \tfrac{4}{30}\hat{\mathbf{z}}\hat{\mathbf{z}}, \tag{81}$$

which yields

$$P_3^{(\mathrm{p}\to\mathrm{s})}(\mathbf{r},\omega) = \tfrac{3}{10} \tag{82}$$

for the degree of polarization. The coherence length is again $\lambda/2$ (not explicitly shown). As in the previous example the source has three uncorrelated components but, in this case, the intensity of the transverse components is higher than that of the z component. However, the degree of polarization is as in the previous case.

VII. CONCLUSIONS

In this work we have theoretically investigated electromagnetic Lambertian sources and demonstrated that in addition to the known example of the field in an opening of a blackbody cavity, other Lambertian sources exist whose far fields differ by their polarization properties. In particular, we considered quasihomogeneous Lambertian sources whose far fields are in every direction azimuthally, radially, or circularly polarized or whose far-field polarization varies in different directions, changing gradually from azimuthal polarization to radial polarization or vice versa. In all cases we were able to compute the source correlation matrix pertaining to the nonevanescent electric field components, and consequently, also the electromagnetic degree of coherence, the polarization matrix, and the degree of polarization. In every example that we considered, the coherence length of the source was found to be on the order of half a wavelength while the polarization characteristics of the sources were different.

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APPENDIX A: DERIVATION OF EQ. (58)

On introducing the far-field matrix (57) corresponding to a completely s-polarized field into the expression (48) and using Eq. (45), we obtain for the cross-spectral density matrix of the source the representation

$$\begin{aligned}
\mathbf{W}^{(s)}_{\mathrm{NE}}(\mathbf{\Delta},\omega) &= \frac{1}{2\pi}\int_0^{\pi/2}\int_0^{2\pi}\sin\theta\begin{bmatrix}\sin^2\phi & -\sin\phi\cos\phi & 0\\ -\sin\phi\cos\phi & \cos^2\phi & 0\\ 0 & 0 & 0\end{bmatrix}\exp\left[ik\sin\theta(\xi\cos\phi+\eta\sin\phi)\right]d\phi\,d\theta\\
&= \frac{1}{2\pi}\int_0^{\pi/2}\begin{bmatrix}-(k^2\sin\theta)^{-1}\partial_\eta^2 & (k^2\sin\theta)^{-1}\partial_\xi\partial_\eta & 0\\ (k^2\sin\theta)^{-1}\partial_\xi\partial_\eta & -(k^2\sin\theta)^{-1}\partial_\xi^2 & 0\\ 0 & 0 & 0\end{bmatrix}\int_0^{2\pi}\exp\left[ik\Delta\sin\theta\cos(\phi_\Delta-\phi)\right]d\phi\,d\theta\\
&= \int_0^{\pi/2}\begin{bmatrix}-(k^2\sin\theta)^{-1}\partial_\eta^2 & (k^2\sin\theta)^{-1}\partial_\xi\partial_\eta & 0\\ (k^2\sin\theta)^{-1}\partial_\xi\partial_\eta & -(k^2\sin\theta)^{-1}\partial_\xi^2 & 0\\ 0 & 0 & 0\end{bmatrix}J_0(k\Delta\sin\theta)\,d\theta\\
&= \int_0^{\pi/2}\begin{bmatrix}\partial_\eta\left[\eta(k\Delta)^{-1}J_1(k\Delta\sin\theta)\right] & -\partial_\xi\left[\eta(k\Delta)^{-1}J_1(k\Delta\sin\theta)\right] & 0\\ -\partial_\eta\left[\xi(k\Delta)^{-1}J_1(k\Delta\sin\theta)\right] & \partial_\xi\left[\xi(k\Delta)^{-1}J_1(k\Delta\sin\theta)\right] & 0\\ 0 & 0 & 0\end{bmatrix}d\theta\\
&= \begin{bmatrix}\partial_\eta\left[\eta(k\Delta)^{-1}\mathfrak{h}_0(k\Delta)\right] & -\partial_\xi\left[\eta(k\Delta)^{-1}\mathfrak{h}_0(k\Delta)\right] & 0\\ -\partial_\eta\left[\xi(k\Delta)^{-1}\mathfrak{h}_0(k\Delta)\right] & \partial_\xi\left[\xi(k\Delta)^{-1}\mathfrak{h}_0(k\Delta)\right] & 0\\ 0 & 0 & 0\end{bmatrix}\\
&= \left[j_0(k\Delta)-\frac{\mathfrak{h}_0(k\Delta)}{k\Delta}\right](\hat{\mathbf{z}}\times\hat{\mathbf{\Delta}})(\hat{\mathbf{z}}\times\hat{\mathbf{\Delta}})+\frac{\mathfrak{h}_0(k\Delta)}{k\Delta}\hat{\mathbf{\Delta}}\hat{\mathbf{\Delta}},
\end{aligned}\tag{A1}$$

where the last equality is Eq. (58). In the derivation we have denoted $\mathbf{\Delta} = \xi\hat{\mathbf{x}} + \eta\hat{\mathbf{y}}$ and ϕ_Δ is the angle that the vector $\mathbf{\Delta}$ makes with respect to the x axis. Further, we have used the relations

$$\partial_z J_0(z) = -J_1(z), \tag{A2}$$

$$\int_0^{2\pi}\exp[iz\cos(\phi_\Delta-\phi)]\,d\phi = 2\pi J_0(z), \tag{A3}$$

$$\int_0^{\pi/2} J_1(z\sin\theta)\,d\theta = \frac{1-\cos z}{z} = \mathfrak{h}_0(z), \tag{A4}$$

where $\mathfrak{h}_0(z)$ is a spherical Struve function defined in Eq. (59).

APPENDIX B: DERIVATION OF EQ. (69)

When we consider the far-field matrix corresponding to a circularly polarized field as given by Eq. (68), we observe that the source cross-spectral density matrix can be obtained from the blackbody cross-spectral density with an additional term. The term is the difference of a matrix with its own transpose and hence it is sufficient to determine that matrix, which is given by

$$\begin{aligned}
\mathbf{W}^{(\mathrm{sp})}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega) &= \frac{1}{2\pi}\int_0^{\pi/2}\int_0^{2\pi}\begin{bmatrix}\sin\theta\cos\theta\sin\phi\cos\phi & \sin\theta\cos\theta\sin^2\phi & -\sin^2\theta\sin\phi\\ -\sin\theta\cos\theta\cos^2\phi & -\sin\theta\cos\theta\sin\phi\cos\phi & \sin^2\theta\cos\phi\\ 0 & 0 & 0\end{bmatrix}\\
&\quad\times\exp[ik\sin\theta(\xi\cos\phi+\eta\sin\phi)]\,d\phi\,d\theta\\
&= \frac{1}{2\pi}\int_0^{\pi/2}\int_0^{2\pi}\begin{bmatrix}-(k^2\sin\theta)^{-1}\cos\theta\,\partial_\xi\partial_\eta & -(k^2\sin\theta)^{-1}\cos\theta\,\partial_\eta^2 & ik^{-1}\sin\theta\,\partial_\eta\\ (k^2\sin\theta)^{-1}\cos\theta\,\partial_\xi^2 & (k^2\sin\theta)^{-1}\cos\theta\,\partial_\xi\partial_\eta & -ik^{-1}\sin\theta\,\partial_\xi\\ 0 & 0 & 0\end{bmatrix}\\
&\quad\times\exp[ik\Delta\sin\theta\cos(\phi_\Delta-\phi)]\,d\phi\,d\theta\\
&= \int_0^{\pi/2}\begin{bmatrix}-(k^2\sin\theta)^{-1}\cos\theta\,\partial_\xi\partial_\eta & -(k^2\sin\theta)^{-1}\cos\theta\,\partial_\eta^2 & ik^{-1}\sin\theta\,\partial_\eta\\ (k^2\sin\theta)^{-1}\cos\theta\,\partial_\xi^2 & (k^2\sin\theta)^{-1}\cos\theta\,\partial_\xi\partial_\eta & -ik^{-1}\sin\theta\,\partial_\xi\\ 0 & 0 & 0\end{bmatrix}J_0(k\Delta\sin\theta)\,d\theta\\
&= \begin{bmatrix}\partial_\xi\{\eta(k\Delta)^{-2}[1-J_0(k\Delta)]\} & \partial_\eta\{\eta(k\Delta)^{-2}[1-J_0(k\Delta)]\} & ik^{-1}\partial_\eta j_0(k\Delta)\\ -\partial_\xi\{\xi(k\Delta)^{-2}[1-J_0(k\Delta)]\} & -\partial_\eta\{\xi(k\Delta)^{-2}[1-J_0(k\Delta)]\} & -ik^{-1}\partial_\xi j_0(k\Delta)\\ 0 & 0 & 0\end{bmatrix}\\
&= -\frac{J_1(k\Delta)}{k\Delta}(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}})\hat{\boldsymbol{\Delta}}+\frac{1-J_0(k\Delta)}{k\Delta}[\hat{\boldsymbol{\Delta}}(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}})+(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}})\hat{\mathbf{z}}]+ij_1(k\Delta)(\hat{\mathbf{z}}\times\hat{\boldsymbol{\Delta}})\hat{\mathbf{z}},
\end{aligned}\tag{B1}$$

where $\boldsymbol{\Delta}$ is as in Appendix A, and we have used the relations

$$\int_0^{\pi/2}\sin\theta\,J_0(z\sin\theta)\,d\theta = j_0(z),\tag{B2}$$

$$\int_0^{\pi/2}\sin\theta\cos^2\theta\,J_0(z\sin\theta)\,d\theta = \frac{j_1(z)}{z}.\tag{B3}$$

The source distribution for the circularly polarized far field can be written as

$$\mathbf{W}^{(\pm)}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega) = \mathbf{W}^{(\mathrm{bb})}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega)\pm\frac{i}{2}\big[\mathbf{W}^{(\mathrm{sp})}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega)-\mathbf{W}^{(\mathrm{sp})^{\mathrm{T}}}_{\mathrm{NE}}(\boldsymbol{\Delta},\omega)\big],\tag{B4}$$

where T denotes the matrix transpose. Substituting the two constituent matrices from Eqs. (B1) and (53) results in Eq. (69).

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