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Andreev levels as a quantum dissipative environment

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We argue that at subgap energies quantum behavior of superconducting weak links can be exactly accounted for by an effective Hamiltonian for a Josephson particle in a quantum dissipative environment formed by Andreev levels. This environment can constitute an important source for intrinsicinelastic relaxation and dephasing in highly transparent weak links. We investigate the problem of macroscopic quantum tunneling in such weak links demonstrating that—depending on the barrier transmission—the supercurrent decay can be described by three different regimes: (i) weak intrinsic dissipation, (ii) strong intrinsic dissipation, and (iii) strong capacitance renormalization. Crossover between quantum and thermally assisted supercurrent decay regimes can also be strongly affected by the Andreev level environment.

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I. INTRODUCTION

Quantum dissipation is known to occur as a result of interaction with an effective environment. Quite generally, this environment can be modeled as a set of harmonic oscillators. Tracing out the oscillator degrees of freedom one naturally arrives at the Feynman-Vernon influence functional theory [1] and the Caldeira-Leggett description of quantum dissipation [2,3]. Electrons in metallic conductors can also play the role of a quantum dissipative environment, as it is illustrated, e.g., by the Ambegaokar-Eckern-Schön (AES) effective action treatment of metallic tunnel junctions [4,5]. Further extension of the influence functional technique also allows us to directly account for Fermi statistics [6] in the situation when electrons in a metal form an effective environment “for themselves.”

In superconducting tunnel junctions dissipation at low enough temperatures/energies can only occur extrinsically (e.g., by attaching an Ohmic shunt resistor [2,3,5]) as there exist no states with energies below the superconducting gap Δ in such junctions. The situation changes if one goes beyond the tunneling limit. In this case subgap Andreev bound states [7] are formed inside superconducting weak links for sufficiently short junctions the corresponding bound state energies are ±εn(ϕ), where

εn(ϕ) = Δ sin2(ϕ/2λ),

(1)

Tn denote normal transmissions of conducting channels and ϕ is the superconducting phase difference across the junction. While in the tunneling limit Tn ≪ 1 one has εn(ϕ) ≈ Δ for any value ϕ, at higher transmissions the energies of Andreev levels (1) can drop well below Δ and may even tend to zero for fully open channels and ϕ ≈ π.

Here we will demonstrate that Andreev bound states along with model oscillators [1–3] or electrons in a metal [4–6]—may act as intrinsic quantum dissipative environment for the Josephson phase ϕ strongly affecting quantum properties of superconducting weak links.

II. EFFECTIVE HAMILTONIAN FOR A WEAK LINK

In what follows we will consider a current biased superconducting junction characterized by a geometric capacitance C0 and an arbitrary distribution of normal transmissions Tn among N transport channels. Provided the phase ϕ does not fluctuate the junction may conduct the supercurrent [8]

I(ϕ) = \frac{eΔ^2 sin ϕ}{2} \sum_n \frac{T_n}{ε_n(ϕ)} tanh \frac{ε_n(ϕ)}{2T} \cdot \tag{2}

Here and below the sum runs over all channels from n = 1 to n = N. In order to describe quantum fluctuation effects it is necessary to treat the phase ϕ as a quantum variable [5]. The generalization of the AES type-of-approach can be worked out also beyond the tunneling limit employing both Matsubara [9] and Keldysh [10] techniques, however the resulting effective action becomes tractable only in certain physical situations.

One of such situations is realized if phase fluctuations remain sufficiently weak [11,12]. Splitting the phase variable into constant and fluctuating parts ϕ(t) = χ + ϕ(t) and assuming |ϕ(t)| ≪ 1 one can express the kernel of the Keldysh evolution operator J as a double path integral

J = \int Dϕ_F Dϕ_B \exp [iS_0[ϕ_F] - iS_0[ϕ_B] + iS_R - S_0],

(3)

where the phase variables ϕ_F, B are defined respectively on the forward and backward branches of the Keldysh contour and S_0[ϕ_F, B] define local in time contributions

S_0[ϕ] = \int_0^t dt' \left[ \frac{C \dot{ϕ}_0^2}{8ε^2} - U(\chi + ϕ(t')) \right].

(4)

Here C is the effective junction capacitance which may differ from C0 due to retardation effects [11] and

U(ϕ) = -2T \sum_n ln \left( \cosh \frac{ε_n(ϕ)}{2T} \right) - \frac{I ϕ}{2ε} \tag{5}

is the effective potential where the first term is recovered by integrating the supercurrent (2) over ϕ and the second term accounts for the bias current I.

The remaining—nonlocal in time—terms

S_R = \int_0^t dt' \int_0^t dt'' R(t' - t'') ϕ_-(t') ϕ_+(t''),

S_I = \int_0^t dt' \int_0^t dt'' I(t' - t'') ϕ_-(t') ϕ_-(t'')

(6)

(7)
describe the influence functional for the phase variable. Here
we denote \( \phi_+ = (\phi_F + \phi_B)/2 \) and \( \phi_- = \phi_F - \phi_B \). Both ker-
nels \( R(t) \) and \( I(t) \) in Eqs. (6) and (7) are real functions related
to each other via the fluctuation-dissipation theorem. General
expressions for each of these kernels remain rather involved
[11] and contain three different contributions originating from
(i) the subgap Andreev bound states, (ii) the quasiparticle states
above the gap, and (iii) the interference between (i) and (ii).

Significant simplifications occur provided both temperature
and typical phase fluctuation frequencies remain well in
above the gap, and (iii) the interference between (i) and (ii).

\( \frac{R(t)}{I(t)} = \sum_n \frac{\gamma_n}{8 \epsilon_n(\chi)} \sin \left[ \epsilon_n(\chi) \tau \right]. \tag{9} \)

What remains is to account for the Andreev terms (i) in both
kernels \( R \) and \( I \). Making use of the results [11,12], we obtain

\[ R(t) = \sum_n \frac{\gamma_n}{2 \epsilon_n(\chi)} \sin[2 \epsilon_n(\chi) \tau], \]

\[ I(t) = \sum_n \frac{\gamma_n}{8 \epsilon_n(\chi)} \text{coth} \left( \frac{\epsilon_n(\chi)}{T} \right) \cos[2 \epsilon_n(\chi) \tau], \tag{10} \]

where

\[ \gamma_n = T_n^2 (1 - T_n) \frac{\Delta^4}{\epsilon_n(\chi)} \sin^4 \frac{\chi}{2} \tanh \frac{\epsilon_n(\chi)}{2T}. \tag{11} \]

Note that in the limit \( \omega \ll \Delta \) and provided \( \epsilon_n(\chi) \sim \Delta \) the
Fourier component of the kernel \( R(t) \) in Eq. (9) can also be
expanded in powers of \( \omega/\epsilon_n \) up to terms \( \sim \omega^2 \) giving rise to extra renormalization of \( C \) [11] (see below). For \( \epsilon_n(\chi) \ll \Delta \), however, this expansion is not justified anymore. For this
reason in what follows we will keep the kernel \( R(t) \) in its exact
form (9).

It is interesting to observe that the influence functional
defined in Eqs. (6)–(10) is exactly equivalent to that produced
by a bath of harmonic oscillators [1–3] coupled linearly to the
fluctuating part of the phase \( \phi \). In other words, our weak link
can also be described by an effective low energy Hamiltonian

\[ \hat{H} = -\frac{2e^2}{C} \frac{\partial^2}{\partial \phi^2} + U(\chi + \phi) \]

\[ + \sum_n \left[ \frac{\hbar^2}{2M_n} + \frac{M_n \omega_n^2}{2} \right] \left( Q_n - \frac{c_n \phi}{M_n \omega_n^2} \right)^2. \tag{12} \]

The first and the second lines of Eq. (12) account respect-
ively for the “Josephson particle” and for its Andreev level
environment which consists of \( N \) harmonic oscillators with
frequencies \( \omega_n = 2 \epsilon_n(\chi) \) coupled to the “particle coordinate” \( \phi \). The coupling constants \( c_n \) are identified by the condition \( c_n^2/M_n = \gamma_n \). It is important to emphasize that the Hamiltonian
(12) follows directly from a fully microscopic effective action

Before dwelling into further calculations it is instructive to
also construct the grand partition function for our weak link
\( Z = \text{Sp} e^{-\hat{H}/T} \). Making use of the effective Hamiltonian in

Eq. (12) we can express \( Z \) in terms of the path integral over
both \( \phi \) and the oscillator coordinates \( Q_n \). Integrating out all
\( Q_n \) variables we obtain

\[ Z = \int \text{D} \phi \exp \left\{ -S_{\text{eff}}[\phi(\tau)] \right\}, \tag{13} \]

where

\[ S_{\text{eff}} = \int_0^{1/T} d\tau \left[ C \frac{\phi^2}{8 e^2 T^2} + U[\chi + \phi(\tau)] \right] \]

\[ + \int_0^{1/T} d\tau_1 \int_0^{1/T} d\tau_2 Y(\tau_1 - \tau_2) \phi(\tau_1) \phi(\tau_2). \tag{14} \]

is the imaginary time effective action for our superconducting
contact and

\[ Y(\tau) = \sum_n \frac{\gamma_n}{8 \epsilon_n} \left( \frac{\delta(t) - \frac{\cosh[2 \epsilon_n(\chi)(\tau - \frac{1}{2T})]}{\sinh[\epsilon_n(\chi)/T]}}{\epsilon_n} \right). \tag{15} \]

Expanding the kernel (15) in the Fourier series \( Y(\tau) = T \sum_{\omega_n} Y_{\omega_n} e^{-i \omega_n \tau} \) with \( \omega_n = 2 \pi m T \), we get

\[ Y_{\omega_n}(\tau) = \sum_n \frac{\gamma_n}{8 \epsilon_n^2(\chi)} \frac{\omega_n^2}{\omega_n^2 + 4 \epsilon_n^2(\chi)}. \tag{16} \]

### III. ANDREEV LEVEL BATH SPECTRUM AND INELASTIC RELAXATION

To begin with, we observe that the coupling constants \( c_n^2 \propto \gamma_n \) (11) vanish in the limit \( T_n \to 1 \). Thus, environmental
modes corresponding to fully open transport channels are
totally decoupled from the phase variable \( \phi \) and, hence, cannot
influence its quantum dynamics.

On the other hand, channels with \( T_n < 1 \) do affect the
behavior of \( \phi \). As long as \( \omega_n \) remains much smaller than
\( \omega_n \) the \( n \)th environmental mode may only contribute to extra
capacitance renormalization. Provided the inequality \( \omega_n \ll \omega_n \)
is fulfilled for all \( n \) we again recover the corresponding
result for the renormalized capacitance [11]. In the opposite
limit \( \omega_n \gg \omega_n \) Andreev levels may already act as a quantum
dissipative environment for the fluctuating phase. This is
because the oscillators can get excited to higher energy states
as a result of their interaction with \( \phi(t) \). Note that since phase
fluctuations remain small the conjugate charge variable, in
contrast, fluctuates strongly implying that multiple electron
charge transfer is possible through each conducting channel.
Accordingly, many of such electrons can in general get excited
to the higher of the two Andreev levels while passing through the
\( n \)th channel. These processes translate into the excitation of
harmonic oscillators to higher levels illustrating the physical
reason why the effect of Andreev doublets is equivalent to that
of such oscillators.

The frequency spectrum of this quantum dissipative envi-
ronment depends on the particular distribution of normal
transmissions \( T_n \). Let us introduce

\[ J(\omega, \chi) = \frac{\pi}{2} \sum_n \frac{\gamma_n}{\omega_n} \delta(\omega - \omega_n), \quad \omega_n = 2 \epsilon_n(\chi). \tag{17} \]

For a junction with large number of channels \( N \gg 1 \) and with transmissions distributed in the interval \( 0 < T_n < 1 \) with the
probability \( P(T_o) \) after a simple algebra for \( \omega_{min}(\chi) < \omega < 2\Delta \) we get
\[
J(\omega, \chi) = \frac{\pi(\Delta^2 - \omega^2/4)}{2\omega} P(T_o) T_o(1 - T_o) \tanh \frac{\omega}{4T} \tag{18}\]
and \( J(\omega, \chi) = 0 \) otherwise. Here we define
\[
T_o = \frac{1 - \omega^2/(2\Delta)^2}{\sin^2(\chi/2)}, \quad \omega_{min}(\chi) = 2\Delta|\cos(\chi/2)|. \tag{19}\]

Intrinsic dissipation due to subgap Andreev levels can be identified if we consider, e.g., the two lowest energy levels in the Josephson potential well \( U(\phi) \) which can also be treated as a qubit. The corresponding inelastic relaxation rate for such a qubit \( \gamma_{in}(\chi) \) reads
\[
\gamma_{in} = \frac{E_C}{\omega_0} J(\omega_0, \chi) \coth \frac{\omega_0}{2T}. \tag{20}\]

Here \( E_C = \epsilon^2/2C \) is the junction charging energy, \( \omega_0 = \sqrt{8E_C U''(\chi)} \) is the plasma oscillation frequency, and the phase \( \chi \) is fixed by the condition \( I_S(\chi) = I \).

In many cases \( \omega_{min} \) remains of order \( \Delta \). For example, in diffusive junctions at \( T \to 0 \) and \( I \) close to \( I_c \) one finds \( \omega_{min} \approx 1.1\Delta \). Only provided \( \chi \) is close to \( \pi \) and, in addition, there exist transmitting channels with \( T_n \approx 1 \) the frequency \( \omega_{min} \) can go well below \( \Delta \). This situation occurs, e.g., in the problem of macroscopic quantum tunneling (MQT) to be addressed below.

### IV. MQT IN HIGHLY TRANSPARENT WEAK LINKS

Provided a superconducting weak link with \( N \gg 1 \) is biased by the current \( I \) close to \( I_c \) the zero resistance state becomes unstable and can decay into a resistive state due to quantum tunneling of the phase across the potential barrier from the potential \( U(\phi) \). In the case of superconducting tunnel barriers with \( T_s < 1 \) this MQT problem with extrinsic dissipation was thoroughly studied in the past [2,3]. Here we consider the opposite limit of highly transparent weak links with \( T_n \approx 1 \) in which case the critical current \( I_c = I_S(\chi_c) \) is reached at the phase value \( \chi = \chi_c \), close to \( \pi \). Such weak links can now be fabricated in a controlled manner employing a variety of different materials including, e.g., atomic point contacts [13], graphene-based weak links [14,15], high transparency Al/\( \text{BiTe/Al} \) double barrier heterostructures [16], or InAs nanowire Josephson junctions [17]. In some of these experiments effective channel transmissions with values very close to unity were demonstrated.

In what follows for simplicity we will assume that all the junction transmissions have the same value \( T_n = T \) and, hence, \( \epsilon_\sigma(\chi) = \epsilon(\chi) \), cf. Eq. (1). We also define the reflection coefficient \( r = 1 - T \ll 1 \) and the parameter \( q(T) = 1 - I/I_c(T) \ll 1 \).

Let us evaluate the supercurrent decay rate with the exponential accuracy \( \Gamma \propto \exp(-A) \). At \( T \to 0 \) the effective potential (5) reduces to a simple form
\[
U(\phi) = -I \phi/2e - N\Delta \sqrt{1 - T} \sin^2(\phi/2), \tag{21}\]
For \( q \ll \sqrt{T} \) Eq. (21) can be simplified further by expanding it in powers of \( \phi \) around \( \chi = \chi_c = \pi - \arccos[(1 - \sqrt{T})^2/T] \).
where \( \alpha \ll 1 \) accounts for dissipation [18]. As long as \( \sqrt{r} \lesssim q \) the value \( A \) grows with \( r \) merely due to the potential profile change [as accounted for by the parameter \( \kappa \) in Eq. (23) and also illustrated in Fig. 1] rather than due to dissipation. Hence, in the limit \( \sqrt{r} \ll q \) the function \( \alpha \) can be safely neglected.

For \( \sqrt{r} > q \) effective coupling of \( \phi \) to the Andreev level bath becomes strong and the last term in Eq. (14) should be treated nonperturbatively. It is well known that quantum tunneling can be described in terms of classical dynamics of the particle \( \phi \) propagating in the inverted potential \( -U \) along the so-called bounce trajectory. Identifying the bounce frequency with that of small oscillations \( \tilde{\omega}_0 \) near the bottom of this potential at \( \phi = \sqrt{2q} \) and setting again \( C \simeq e^2 \sqrt{N/(4\Delta)} \), after trivial algebra we get
\[
\tilde{\omega}_0 = \Omega(q,r^{1/4}), \tag{25}
\]
with \( Z = \sqrt{8q} - y^2 - r/y^3 \). For the potential (22) we find
\[
A \simeq 36U_0/(5\tilde{\omega}_0). \tag{27}
\]
This formula together with Eqs. (25) and (26) accounts for a trade-off between two different tunneling regimes. Let us define the value \( q_x \) from the equation \( \Omega(q_x,r^{1/4}) = 2\epsilon(\chi_x) = 2\Delta r^{1/4} \) which yields \( q_x = \sqrt{r}/32 + (r^{3/4} + r)/8 \). In the adiabatic limit \( \phi_x \ll 2\Delta r^{1/4} \) (or \( q < q_x \)) Andreev oscillators are “fast” and provide strong capacitance renormalization [19]
\[
C^* = C + e^2 N/(4\Delta r^{1/4}). \tag{28}
\]
The particle \( \phi \) then becomes heavier albeit its energy remains conserved during tunneling and Eq. (27) yields
\[
A \simeq (6/5)2^{1/4} N q^{5/4} r^{-1/8}. \tag{29}
\]
In the opposite antiadiabatic limit \( \phi_x \gg 2\Delta r^{1/4} \) (or \( q > q_x \)) Andreev oscillators become “slow” generating effective potential renormalization \( U \rightarrow U + N/(4\Delta r^{1/4}) \phi^2/8 \). Such oscillators can get excited to higher levels taking energy from the tunneling particle \( \phi \). Hence, for \( q < q \lesssim \sqrt{r} \) Eq. (27) describes a strong dissipation regime and matches smoothly with Eq. (24) at \( q \sim \sqrt{r} \).

V. QUANTUM-TO-CLASSICAL CROSSOVER

Quantum decay can only occur at low enough temperatures \( T \ll T_0 \), whereas at \( T > T_0 \) thermal activation takes over with \( \Gamma \propto \exp(-U_0/T) \). In order to analyze the crossover between these two regimes we will employ an approximate form (22) for the potential energy \( U \). At lower \( T \ll \epsilon(\chi_x) = \Delta r^{1/4} \) Eq. (22) holds for \( q \lesssim \sqrt{r} \), as we already indicated above. At higher temperatures \( \Delta r^{1/4} \ll T \lesssim \Delta \) the approximation (22) applies without any further restrictions. Indeed, at such \( T \) and for \( r \ll 1 \) Eq. (2) reduces to
\[
I_S(\psi) = e^2 N \Delta \sin \frac{\psi}{2} \tan \left( \frac{\Delta}{2T} \cos \frac{\psi}{2} \right) \tag{30}
\]
reaching its maximum \( I_S(\psi) = I_C(T) \) at \( \psi = \chi_x \), where now \( \chi_x = \pi - (2T/\Delta)W(2\Delta^2/T^2) \) and \( W(z) \) is the Lambert \( W \) function defined by the equation \( \exp(W) = z \). For \( q \ll \sqrt{T}/\Delta^2 \) we can again expand the potential \( U(\phi) \) in \( \phi \) around \( q \simeq q_x \) and reproduce Eq. (22) with \( v = 1 \).

At temperatures in the vicinity of the crossover to thermal activation quantum tunneling is described by the bounce trajectory \( \phi(\tau) \) which remains close to the local maximum of the potential \( U(\phi) \) at \( \phi = \sqrt{2q} \). The value \( T_0 \) is formally identified [3,20,21] as a temperature at which the bounce first reduces to \( \phi(\tau) = \sqrt{2q} \) meaning that \( A = U_0/T_0 \) at this point. This is achieved provided the corresponding eigenvalue of the operator \( \delta_S^2 S_{\text{eff}}/\delta \phi^2 \) vanishes, which yields
\[
\pi^2 T^2 C/e^2 - 2Y_{2\pi T}(\chi_x) = N\Delta\sqrt{q/2}. \tag{31}
\]
Resolving Eq. (31) one determines the crossover temperature \( T_0 \). If the geometric capacitance \( C_0 \) is large, it suffices to set \( C = C_0 \) and \( Y_{2\pi T} \rightarrow 0 \) in Eq. (31). Then we immediately recover the standard result [22] \( T_0 = \omega_0/(2\pi) \), where \( \omega_0 \) is defined below Eq. (23). This result holds for \( q \lesssim \sqrt{r} \). In the limit \( \sqrt{r} \ll q \) Eq. (31) cannot be applied anymore. In this case \( U_0/T \) reaches the value \( A \) in Eq. (23) at \( T \approx 2T_0 \).

Perhaps the most interesting situation occurs if geometric capacitance \( C_0 \) is small and \( C \) is dominated by the last term in Eq. (8). Substituting \( C = e^2 N/(4\Delta) \) together with Eq. (16) into Eq. (31) and resolving the latter with respect to \( T_0 \) we obtain
\[
T_0 = \Omega(q,y)/(2\pi), \quad y = \epsilon(\chi_x(T_0))/\Delta. \tag{32}
\]
In the limit \( T_0 \ll \Delta r^{1/4} / \pi \) we have \( y \simeq r^{1/4} \) and Eq. (32) reduces to \( T_0 = \omega_0/(2\pi) \). In particular, at small enough \( q \) we get \( T_0 = \Delta(8q)^{1/4}/r^{1/8} \). For \( T_0 \gtrsim \Delta r^{1/4} / \pi \) Eq. (32) approaches the \( r \)-independent result \( T_0 = \Delta(8q)^{1/4} / \pi \). The function \( T_0(q) \) (32) is displayed in Figs. 2 and 3 (inset) for different values of \( r \).

Note that for very small \( r \ll 10^{-4} \) the function \( T_0(q) \) becomes multivalued for some values of \( q \). This behavior, however, does not imply the presence of several crossover temperatures for a given bias \( I \) because the critical current \( I_C \) also depends on temperature and, hence, \( q = q(T) \). For each
quantum-to-classical crossover occurs at a given temperature. The function $I_0(T)$ is plotted in Fig. 3 for $r = 0.1$ together with $I_c(T)$. With decreasing $r$ classical activation region shrinks and $I_0(T)$ rapidly approaches $I_c(T)$.

In summary, we demonstrated that subgap Andreev bound states in superconducting weak links form an intrinsic quantum dissipative environment for the Josephson phase $\varphi$ and derived a microscopic low energy Hamiltonian for such weak links. Effective coupling between $\varphi$ and Andreev oscillators depends on transport channel transmissions $T_n$ and vanishes for fully open channels with $T_n \rightarrow 1$. In the case of highly transparent weak links we analyzed both quantum and thermally assisted decay of the supercurrent and identified the MQT regimes of weak intrinsic dissipation, strong intrinsic dissipation, and strong capacitance renormalization. Our predictions can be directly tested by routinely measuring the statistics of switching currents (see, e.g., recent experiment [23]) in highly transparent superconducting weak links in combination with Andreev level spectroscopy [24].

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[18] In order to estimate $\alpha$ in Eq. (24) we employ Eq. (16) with $\chi$ corresponding to the local minimum of the potential (21). Then we obtain $\alpha(z) \sim z^2 \ll 1$. Choosing $\chi = \chi_c$ would give $\alpha(z) \sim \sqrt{z} \ll 1$, however this choice is inaccurate for $\sqrt{\varphi} \ll q$ and $T \rightarrow 0$ since in this limit typical phase fluctuations $\delta \varphi \sim \sqrt{\varphi}$ exceed $\pi - \chi_c \sim r^{1/4}$.
[19] The combination of Eqs. (8) and (28) is equivalent to the analogous result [11] in the corresponding limit.