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Localization and diffusion in Ising-type quantum networks

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We investigate the effect of phase randomness in Ising-type quantum networks. These networks model a large class of physical systems. They describe micro- and nanostructures or arrays of optical elements such as beam splitters (interferometers) or parametric amplifiers. Most of these structures are promising candidates for quantum information processing networks. We demonstrate that such systems exhibit two very distinct types of behavior. For certain network configurations (parameters), they show quantum localization similar to Anderson localization whereas classical stochastic behavior is observed in other cases. We relate these findings to the standard theory of quantum localization.

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I. INTRODUCTION

Maturing of the techniques for controlling single-quantum systems such as atoms, photons, or quantum dots has made it possible to design more complicated structures of these basic elements: quantum networks. Interesting examples of such micro- and nanostructures are for instance thin wires used as atom guides [1], trapped chains of atoms and ions [2], or structures grown on a substrate such as quantum dots [3] or arrays of nanomagnets. Also optical elements such as beam splitters can be used to construct networks where interesting classical or quantum interference effects occur for classical beams and single photons, respectively [4]. Optical multiports are extensively used in quantum optics, especially in the manipulation, measurement, and detection of quantum states. Small networks are used for instance to measure the phase of light or to identify Bell states in quantum teleportation schemes [5]. Medium-size optical networks could be used to demonstrate elementary tasks of quantum computing. All these are among the first steps of laboratory demonstrations leading to quantum network engineering and finally to matured and scalable quantum technology. Due to these possible applications investigations of quantum networks have become topical.

In this paper, we consider a quantum network configuration which is perhaps the simplest nontrivial one: a network which realizes nearest-neighbor interactions by coupling the individual quantum systems *pairwise*. As a possible implementation one can think of light modes coupled together by an optical element, for example, a beam splitter or a mode coupler. The network is schematically presented in Fig. 1. Another common form of nearest-neighbor interaction is simultaneous interaction with many neighbors, like electrons interacting with Coulomb interaction. Here, however, we consider pairwise interaction which is typical for light modes, and for many quantum networks in general.

The network in Fig. 1 has an interesting and potentially useful connection to the Ising model [6,7]. Interference effects are typical in this kind of network and have been investigated by many authors [8–11]. The Hamiltonian struc-

ture of the network was studied in [12]. In this paper, we consider the effect of randomness in such a network. Randomness in certain quantum systems is known to lead to intriguing effects such as Anderson localization and dynamical localization [15–17]. Also classical wave behavior combined with randomness can lead to localization, c.f. localization of light [18,19].

We can view the considered network also as a interferometer. The effects demonstrated can be understood as the result of interferences between paths available for the excitations in the network. However, interferences are known to be very fragile towards phase perturbations. We show that the typical interference effects are modified or completely destroyed as randomness is introduced in the network.

The way in which the randomness enters the network has a profound effect on the behavior. Therefore we consider two basic variants of the network. First, a two-dimensional (2D) network of the type shown in Fig. 1, with randomness all over the network. This means that along each connecting line between two interaction nodes, the phase (free-time evolu-

FIG. 1. A schematic presentation of the considered quantum network. The nodes are connected to their nearest neighbors. The boxes A and B denote the transformations performed at the nodes; they could correspond for instance to beam splitters in case of an optical realization.

tion) is changed randomly within the possible range of $\langle 0, 2\pi \rangle$. A corresponding physical system could be, for example, a network of optical elements with phase fluctuations arising from different optical lengths between the elements. The fluctuations could be produced at will with phase shifters to make systematic studies of the effect of randomness. Alternatively, our calculations can model undesired phase fluctuations in optical networks and can be used to study the stability of interference effects in a complicated interferometer. We show that in the random 2D network classical stochastic diffusion is typical.

In the second configuration considered, the network models a 1D system. Now the basic unit of the network contains fixed randomness. The basic unit is formed by the two columns of A and B (“the motif” element for building the network) in Fig. 1, and along the connecting lines between the columns the phase is adjusted arbitrarily, i.e., within the range of $\langle 0, 2\pi \rangle$. In such a way we define a random motif matrix. Then this very same basic unit is precisely repeated M times. The horizontal dimension of the network can now be understood as time and the network actually models time evolution in one dimension (the vertical dimension is the real space or a mode space). Note that the configuration also corresponds to a thin layer of material inside a cavity, with a light pulse crossing it several times: the 1D propagation is now the spreading of the light along the material layer after many cavity crossings. We show that randomness in the 1D type configuration leads to quantum localization.

We explain the different behavior of the two configurations by relating our observations to the standard theory of localization. The theory of Anderson localization and dynamical localization is often formulated considering the properties of the Hamiltonian for the system of interest. In the quantum engineering context, the individual components of a more complicated network are described by a unitary transform. The transfer matrix for the total network is easily obtained by simple multiplications of the transfer matrices corresponding to the individual elements, but to find out the Hamiltonian corresponding to the complex system is non-trivial if not impossible [12]. Therefore we find it of interest to consider here the quantum localization problem by starting the theoretical description from the transfer matrices of the system. Transfer matrices [13], as well as Green’s functions [14], have been extensively applied to study Anderson localization. Transfer matrices in many of these studies originate from writing the Schrödinger equation as a difference equation and are used as a mathematical tool which reveals results via the theory of products of random matrices. In our case the product of transfer matrices reflects the fact that the physical system is made of a combination of blocks which individually are described by some (lower-dimensional) transfer matrices. The connection to the standard transfer-matrix approach depends on the configuration we consider, and will be explained in detail in the following section.

Note that in the mathematical level, the unitary transforms describing the network can also be understood as operations performed in the computational space of a quantum computer. Thus, our investigations also relate to decoherence affecting certain types of computational operations. Nearest-

neighbor coupling configurations have been considered in the quantum information context recently [20,21].

In Sec. II we introduce the transfer matrix which is the common starting point for both network configurations of interest. The first configuration, leading to classical diffusion, is considered in Sec. III, the second, showing localization, in Sec. IV. Differences in the quantum behavior in these two cases are explained with the theory of Anderson/dynamical localization in Sec. V. The conclusions are presented in Sec. VI.

II. THE TRANSFER MATRIX

The network we consider realizes the nearest-neighbor structure given in Fig. 1. Each element (box in the figure) represents a 2×2 coupler, e.g., a beam splitter in case of optical networks. The whole network is build up by repeating the motif structure consisting of a row of components A and B , i.e., we use the outputs of the motif as the input for the next layer. We assume periodic boundary conditions [6,7].

The transfer matrix describing the motif unit has the form

$$U = \begin{bmatrix} & & & & \\ & A & 0 & 0 & \cdots \\ & & 0 & 0 & \cdots \\ 0 & 0 & & A & \\ 0 & 0 & & & \\ \vdots & \vdots & & & \ddots \\ & & & & A \end{bmatrix} \times \begin{bmatrix} B_{22} & 0 & 0 & \cdots & B_{21} \\ 0 & & B & & \\ 0 & & & B & \\ \vdots & & & & \ddots \\ B_{12} & & & & B_{11} \end{bmatrix}, \quad (1)$$

where

$$A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix}. \quad (2)$$

When the parameters θ and ϕ are real, Eq. (2) describes a $SU(2)$ network. Purely imaginary θ and ϕ correspond to a $SU(1,1)$ network. Above, $\sin^2 x$ (x is θ or ϕ) gives the coupling strengths of the elementary units used and N is the number of the units in one layer. We will consider only the case $\cos^2 x = 0.5$. In optics, this corresponds to so-called balanced beam splitters (or 3-dB couplers).

The transfer matrix W of the whole network, in the case all motif structures are the same, is given as a power of U ,

$$W(M) = U^M. \quad (3)$$

This can be understood as a 2D network where all the matrices U are the same, or as a 1D system where the horizontal direction of the configuration represents discretized time

FIG. 2. A typical interference pattern in the case of no randomness in the network. The left picture shows the repetition of the motif structure of the network, the right the logarithm of the probability P to find the photon at a particular output n . We used 40 inputs and outputs. The horizontal axis shows the labels of the outputs, the vertical one the logarithm of the probability.

evolution. In the latter case, when U contains randomness, there is a certain connection to the transfer-matrix description of Anderson localization. E.g., in [13], the 1D Schrödinger equation is discretized and a difference equation is formed. Eigenfunctions of the system can then be obtained from $[\psi_{n+1} \psi_n]^T = S_n [\psi_0 \psi_{-1}]^T$, where ψ_n are wave functions corresponding to different (1D) lattice sites and S_n is a product of 2×2 random matrices $S_n = Y_n Y_{n-1} \cdots Y_0$. Properties of S_n are used for calculating the eigenfunctions: in a one-dimensional random potential they are localized. In our case, we have instead of a 2×2 product of random matrices a $2N \times M$ one. Results corresponding to 1D propagation can be obtained by choosing only one of the inputs of the network to be nonzero in the beginning. The matrices Y_n and our matrices do not have any simple correspondence. One may guess, however, that the 1D propagation in our case, with randomness added, could also show localization since the eigenfunctions in general are localized in a 1D random potential, as shown, e.g., in [13]. As a summary, there is a certain connection between the transfer-matrix approach presented in [13] and our network configuration, but no straightforward mapping. We choose to apply a numerical approach instead of one based on the theory of products of random matrices.

Apart from Eq. (3), the other possibility is that all the motif structures U_i , $i \in [1, M]$ are different due to overall randomness, then

$$W(M) = U_1 D_1 U_2 D_2 \cdots U_M. \quad (4)$$

The matrices D_i represent diagonal matrices giving the phase uncertainties between the random motifs. Here, M can be understood as the size ($2N \times M$) of the network in the case of a 2D network, or as time when the network models 1D propagation.

An example of the typical interference behavior in complete absence of randomness is presented in Fig. 2. In the following we will show how this interference pattern is modified by different forms of randomness.

III. RANDOM 2D NETWORK

We consider first a two-dimensional quantum network with randomness. This corresponds to the total transfer matrix

$$W(M) = U_1 D_1 U_2 D_2 \cdots U_M, \quad (5)$$

where U_i, D_i are all different. The randomness enters the motif structures either via the individual quantum components A and B , or by having an extra random-phase shift matrix between the matrices containing the N components A and B . For optical systems, the first case would correspond to having fluctuations in the transmittivities of the beam splitters, the latter to random-phase shifts occurring in the optical paths. Here we have analyzed phase randomness. The phases between all the connecting points change randomly within the $\langle 0, 2\pi \rangle$ interval.

We study the behavior of the network by following the propagation of an input state initially located at the input port which is at the center of the network. To illustrate our results we fixed the number of network input ports to 40 which corresponds to 20 elements in one layer. As the initial state we consider a single photon guided into the network from the 20th input port

$$|\psi(0)\rangle = |0\rangle_1 \cdots |0\rangle_{19} |1\rangle_{20} \cdots |0\rangle_{40}. \quad (6)$$

The spread of a photon given by this initial state in the network is shown in Fig. 3.

The plot shows the logarithm of the probability for the photon to appear at the given output (given by the dots). Three choices for M are shown to illustrate how the distribution changes when the photon propagates in the network. The largest value of M equals ten; after ten layers the photon hits the border of the network, then traversing just another motif layer the photon would be recycled back into the system due to periodic boundary conditions. In calculating the probability we have averaged over 1000 runs. The solid line gives the least-squares fit to the calculated values to show that the distribution follows a quadratic dependence.

As shown in Fig. 3, the quantum state in this network behaves like a classical distribution, i.e., it spreads into a broad Gaussian at the outputs. Classical and quantum behavior in this kind of network, in the absence of randomness, is studied in [8–11]. Gaussian outputs were found in the classical case [10] whereas complicated interference and recurrence phenomena are typical in the quantum case. Our results show that introducing randomness acts as decoherence which reduces the quantum behavior to the classical one. In Sec. V we explain the results further in the light of the theory of localization considering random Hamiltonian matrices.

IV. PROPAGATION IN 1D: A THIN LAYER OF MATERIAL

Next we consider the situation where the transfer matrix of one motif structure of the network has the form UD , where D is a diagonal matrix with *fixed* randomness, and the total transfer matrix is simply

$$W(M) = (UD)^M. \quad (7)$$

FIG. 3. The left picture shows the considered network structure. All U_i are random because of random phases introduced inside each motif structure; in the figure these phases are represented by the dotted arrows connecting the two columns of the elements A. Additional random phases are introduced in the layers D_i between each two motifs. The output distribution after ten layers is shown on the right. The dots represent the actual output probabilities, the solid line shows the least-squares fit. The bottom plot shows the evolution of the distribution after four, seven, and ten layers traversed. Note the change in the logarithmic (vertical) scale.

FIG. 4. The evolution of the output probability distribution for a network configuration formed by repeating a fixed random motif. The top right plot shows the distribution after ten traversed motifs (dots), compared to the corresponding least-squares fit. The evolution for different numbers of motifs is given by the three lower pairs of plots. They are for $M=5,10$, $M=15,20$, and $M=40,80$.

FIG. 5. The emergence of localization when the amount of randomness in the network is increased. The number of motifs is $M=10$. The interval of random phases changes from top to bottom and left to right as $\langle 0, \pi/16 \rangle$, $\langle 0, \pi/8 \rangle$, $\langle 0, \pi/4 \rangle$, $\langle 0, \pi/2 \rangle$, $\langle 0, \pi \rangle$, and $\langle 0, 2\pi \rangle$. The upper-left picture shows the almost unperturbed interference pattern appearing for very weak randomness, the lower-right picture the localized distribution in the case of strong randomness.

Physically this corresponds to quantum propagation in one dimension, or for example, a thin layer of material inside a cavity through which light makes several passages. The situation is illustrated in Fig. 4.

We consider the behavior of the same input as given in the previous section. Figure 4 shows that the distribution becomes exponentially peaked at the input position 20. To see the emergence of this exponential localization we present the evolution of the initial photon distribution in dependence on the number of motifs traversed (M).

We have found that the complicated interference pattern typical for the ideal network turns gradually into quantum localization when the amount of randomness is increased. This is demonstrated in Fig. 5.

We compare the distributions at $M=10$ for weak and strong randomness. The strength of randomness is controlled by the extend to which the phases in the random diagonal matrices can change, for instance, for weak randomness only within the interval $\langle 0, \pi/16 \rangle$ and for moderate randomness the interval would be larger $\langle 0, \pi/4 \rangle$. The main result is that we observe exponential localization for strong enough randomness. This remarkable quantum feature will be explained in Sec. V by making a connection to the standard theory of Anderson and dynamical localization.

Intermediate case: Extra noise in 1D propagation or semirandom 2D network

As an intermediate case we consider a network with the transfer matrix

$$W(M) = U D D_1 U D D_2 U D D_3 \dots, \quad (8)$$

where D_i are random and different from each other. The matrix D is also random. Note that with all $D_i=1$, $W(M)$ would be the same as in the previous section, that is, expected to lead to localization. This intermediate case models a network with phase fluctuations in between exactly similar motif structures. Examples of this are an optical network where the nearest-neighbor coupling could be essentially perfect but the path lengths in between two nearest-neighbor coupling layers fluctuate, or fluctuating cavity length in the 1D case of a layer of material in a cavity. In modeling 1D propagation, this is also a test of numerical stability of the localization phenomenon when very small extra randomness is introduced.

We now fix the amount of randomness in the matrix D that appears repeatedly in $W(M)$ (phases chosen between $\langle 0, 2\pi \rangle$), and vary the amount of randomness in D_i . We observe that for weak randomness in D_i (phases chosen between $\langle 0, 0.1\pi \rangle$), the localization is preserved [note that in this case we probably have approximately the situation $W \sim (UD)^M$]. For strong randomness (phases chosen between $\langle 0, 2\pi \rangle$) Gaussian diffusion is observed. This transition from localization into diffusion is illustrated in Fig. 6.

V. CONNECTION TO RANDOM HAMILTONIANS

The theory of Anderson and dynamical localization tells that exponential localization of the system eigenvectors, and consequently, of quantum wave packets, occurs when the system Hamiltonian is a random band-diagonal matrix [16]. Band diagonality means that only a few neighboring sites (thinking in terms of a lattice) are coupled — in the Lloyd model [16], which is the simplest model showing localization, only the nearest neighbors are connected. Randomness

FIG. 6. The transition from localization into diffusion. The number of traversed motifs is $M = 10$ (top) and $M = 20$ (bottom). The left column represents weak extra randomization (interval of the D_i phases only $\langle 0, 0.1\pi \rangle$). The middle is an intermediate regime with phases varying $\langle 0, 0.1\pi \rangle$. The right column was calculated for phases randomly chosen from the interval $\langle 0, 2\pi \rangle$. In this case no localization is observed.

usually means that the energies of the sites, that is, the diagonal terms of the Hamiltonian, are randomly chosen according to a certain distribution. But randomness can equally well appear in the off-diagonal couplings.

We describe the network of interest by a transfer matrix rather than a Hamiltonian. Connection between these two is nontrivial to obtain for large networks, cf. [12]. But some overall characteristics are easy to sketch. One motif structure of our network clearly couples nearest neighbors, which corresponds to a Hamiltonian H_{motif} with only nearest-neighbor coupling. In the case of 1D propagation, or a thin layer of material in a cavity, the total transfer matrix is

$$W(M) = U^M = e^{iH_{\text{motif}}M}. \quad (9)$$

The network size M can be interpreted as time, and the Hamiltonian of the system is the one describing one motif structure: random and band diagonal. Therefore it is in accordance with the theory of localization to observe an exponentially localized distribution at the output. The case with overall randomness in a 2D network is completely different. The total transfer matrix

$$W(M) = U_1 D_1 U_2 D_2 \cdots U_M \quad (10)$$

can no longer be expressed as a M th power of a certain band-diagonal Hamiltonian. By diagonalizing $W(M)$ one can obtain the corresponding Hamiltonian, but it is clear that it will not any more be band diagonal, but instead couples M neighbors. For large M , the Hamiltonian approaches a full random matrix. It is known that for full random matrices there is no localization [16]. Instead, we observe classical-type diffusion.

VI. CONCLUSIONS

We have considered randomness and localization in a specific quantum network which is characterized by nearest-neighbor structure and has a relation to models of statistical mechanics with nearest-neighbor interaction. The physical systems that can be modeled by such a network are for instance beam-splitter networks [4], micro- and nanostructures, and networks of intersecting energy levels [11,22]. We showed that in a 2D network with overall randomness, the quantum behavior was reduced to the classical one. In contrast, in 1D propagation, or a system like a layer of material in a cavity, we observe exponential localization. An intermediate case was showing a smooth transition between these two completely different behaviors.

We relate our findings to the standard theory of localization described by random Hamiltonians. Our discussion emphasises the features which arise when the systems of interest are described by a combination of the transfer matrices of the individual components rather than Hamiltonians. This would be the natural approach in case of engineered quantum networks.

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