Alava, M.; Moukarzel, C.

Transport on percolation clusters with power-law distributed bond strengths: when do blobs matter?

Published in:
Physical Review E

DOI:
10.1103/PhysRevE.67.056106

Published: 01/01/2003

Document Version
Publisher's PDF, also known as Version of record

Please cite the original version:
Alava, M., & Moukarzel, C. (2003). Transport on percolation clusters with power-law distributed bond strengths: when do blobs matter? Physical Review E, 67(5), 1-8. [056106]. https://doi.org/10.1103/PhysRevE.67.056106

Transport on percolation clusters with power-law distributed bond strengths

Mikko Alava
Helsinki University of Technology, Laboratory of Physics, P.O. Box 1100, FIN-02015 HUT, Finland and NORDITA, Blegdamsvej 17, DK-2100 Copenhagen, Denmark

Cristian F. Moukarzel
Física Aplicada, CINVESTAV Mérida, 97310 Mérida Yucatán, Mexico
(Received 4 December 2002; published 9 May 2003)

The simplest transport problem, namely finding the maximum flow of current, or maxflow, is investigated on critical percolation clusters in two and three dimensions, using a combination of extremal statistics arguments and exact numerical computations, for power-law distributed bond strengths of the type $P(\sigma)\sim\sigma^{-\alpha}$. Assuming that only cutting bonds determine the flow, the maxflow critical exponent v is found to be $v(\alpha)=(d-1)\nu+1/(1-\alpha)$. This prediction is confirmed with excellent accuracy using large-scale numerical simulation in two and three dimensions. However, in the region of anomalous bond capacity distributions ($0\leqslant\alpha\leqslant1$) we demonstrate that, due to cluster-structure fluctuations, it is not the cutting bonds but the blobs that set the transport properties of the backbone. This "blob dominance" avoids a crossover to a regime where structural details, the distribution of the number of red or cutting bonds, would set the scaling. The restored scaling exponents, however, still follow the simplistic red bond estimate. This is argued to be due to the existence of a hierarchy of so-called minimum cut configurations, for which cutting bonds form the lowest level, and whose transport properties scale all in the same way. We point out the relevance of our findings to other scalar transport problems (i.e., conductivity).

DOI: 10.1103/PhysRevE.67.056106 PACS number(s): 64.60.Ak, 74.25.Fy, 73.50.−h

I. INTRODUCTION

The transport properties of percolation clusters have been a subject of interest for many years [1,2]. A natural problem to study is, e.g., conductivity, and one often complicates it further by using random bond "strengths" σ with a power-law tail of the form $\sigma^{-\alpha}$ [3–11]. In the first place, because this allows one to represent continuum percolation [5,7] and thus get closer to some actual physical realizations of percolation. A second reason why these systems are interesting is the equivalence [12] between transport on strongly disordered systems and percolative transport [35].

Transport critical exponents on these systems are found to depend on α, which means that strict universality is lost. The original observation that transport exponents become nonuniversal is due to Kogut and Straley [3], who used mean-field-type arguments. Later Straley [4], with the help of the nodes-links-blobs [13–16] picture of the backbone, concluded that the conductivity exponent t, such that $\Sigma\sim(p-p_c)^t$, is the maximum of the universal exponent $t_0=(d-2)\nu+\zeta$ and the α-dependent exponent $\overline{t}(\alpha)=(d-2)\nu+1/(1-\alpha)$. Here ν is the correlation length exponent and ζ measures the contribution of blobs to the resistance between two points on the backbone, for the case of constant conductances. For the conductivity problem thus there is a crossover from the universal exponent t_0 for $\alpha<\alpha_c$ to $\overline{t}(\alpha)$ in the "anomalous regime" $\alpha>\alpha_c$. Although not without some controversy initially [6,8,9,17], this result is by now well established [8–11].

It is somehow surprising that $\overline{t}(\alpha)$ can be analytically calculated in the anomalous regime, given that the universal exponent t_0, which applies to the arguably simpler case of constant conductance, has not been analytically derived up to now. The difficulty in deriving t_0 resides in that ζ is determined by the blobs, and thus one would require detailed information [18] about the internal structure of the blobs.

On the contrary, it has been argued by several authors [6,11,18] that $\overline{t}(\alpha)$ in the anomalous regime is determined by the cutting bonds alone. Since these form linear chains of typically $L^{1/\nu}$ bonds at p_c [15], the resulting conductivity exponent is easily derived. The argument to support the belief that blobs are irrelevant in the anomalous regime seems to be roughly the following: an exceedingly small conductivity falling on a blob has little effect on the overall conductance, because there are many alternative parallel paths. On the other hand, if this small conductivity is located on a cutting bond it will certainly dominate the system conductance. While this argument is true in principle, this reasoning misses the fact that the number of cutting bonds is itself a fluctuating quantity. The issue of blob irrelevance has been considered by Machta *et al.* [6] using a hierarchical model for the backbone, to reach similar conclusions. However, as noted by the authors, their model does not include structural fluctuations. We will demonstrate in this work that it is in fact the blobs and not the cutting bonds that determine the critical transport properties, even in the anomalous regime. This, as we will see, is due to structural fluctuations. However, the resulting transport properties turn out to be the same as those given by the most simplistic red bond estimate.

A related critical transport problem, which is relevant for disordered superconductors, is that of determining the critical current density $J_c=I_cL^{-(d-1)}$ that a percolation network can sustain, and which above p_c behaves as $J_c\sim(p-p_c)^v$ [19–24]. This problem has a simple geometrical interpretation. Finding the maximum flow of current, or *maxflow*, is equivalent to finding the surface across the system, on which the sum of critical currents of the bonds is maximized. As we will draw advantage of this analogy later on, we note that this surface is called a *mincut* in computer science language [25–27].

In this paper our goal is to present a comprehensive study of the maxflow problem on percolation clusters. This is motivated by the following observations. First, this is the simplest transport problem that one can think about, and has not been as such discussed much in the literature. Second, we are able to use to our advantage recent developments [28] on combinatorial optimization algorithms, in the context of disordered systems. Here one can use a three-step approach, in which first a critical spanning cluster is set, its backbone is pruned out, and finally that is used for the maxflow-mincut problem. Each stage is solved with one of the powerful graph optimization algorithms for the particular problem, as discussed later.

In the simplest version of the maxflow problem, all present bonds have the same critical current, or *capacity* i_c and absent bonds have $i_c=0$. At criticality, a typical percolating cluster is a linear chain of cutting bonds and thus $I_c=i_c$. From this observation plus the usual scaling relation $J_c(L)\sim L^{-v/\nu}$, one concludes that [19] $v=(d-1)\nu$. This result is consistent with experiments [19,22] and numerical simulation [21,24]. In a more realistic model, each present bond has a random capacity i_c with power-law distribution $P(i_c)\sim i_c^{-\alpha}$. This is, for example, the case for continuum percolation models [20,21,23]. A simple extension of the ‘‘typical cutting-bond string’’ argument gives $v(\alpha)=(d-1)\nu+1/(1-\alpha)$ as we show later in Sec. II.

In the following we will find it useful to compare the conductivity and critical current problems to each other. This comparison is done by interpreting the random bond variables i_c alternatively as bond conductances σ or as bond capacities i_c. Consider, for example, two bonds with σ_1 and σ_2 connected in parallel. The resulting conductance $\sigma_{par}=\sigma_1+\sigma_2$ is then the same as the maximum current I_{max} that can flow if σ_j are capacities. If these bonds are instead connected in series, then $\sigma_{series}=(\sigma_1^{-1}+\sigma_2^{-1})^{-1}$ and $I_{max}=\min(\sigma_1,\sigma_2)$ are no longer equal. However, the series conductance can be written as [29] $\sigma_{series}=\min(\sigma_1,\sigma_2)(1+\beta)^{-1}$, with $\beta=\min(\sigma_1,\sigma_2)/\max(\sigma_1,\sigma_2)$. In the limit of strong disorder ($\alpha\rightarrow 1$), β is typically negligible. We conclude that, in this limit, also in the series case the conductance equals exactly the maximum current obtained by interpreting σ_j as capacities i_c. Therefore in the $\alpha\rightarrow 1$ limit, the resistive current problem and the superconducting current problem (maxflow) are equivalent, at least for all structures that can be solved by a combination of series and parallel bond reductions [36]. Moreover, as shown in Sec. III B, we find that the equivalence noticed above is valid not only in the $\alpha\rightarrow 1$ limit but for a range of α values, for strings of bonds in series.

In deriving α-dependent exponents, both for $v(\alpha)$ and $\overline{t}(\alpha)$, the assumption is made that the backbone always contains $L^{1/\nu}$ cutting bonds. While this is true typically, the number of cutting bonds is in fact a fluctuating variable whose distribution may extend down to zero in the form of a power law (see later). The existence of such fluctuations has been noted by some works previously [30,31], but their role in transport properties has not been considered. These number fluctuations, we will show in Sec. II, do modify the transport exponent that results from a string of cutting bonds. Then by analyzing the conceptually and numerically simple maxflow problem, we will be able to show that in fact blobs cannot be neglected. The net outcome, which we justify by a heuristic hierarchical picture, is that although the simplest cutting-bond scaling (without fluctuations) is restored, it is in fact the blobs that set this scaling behavior.

The structure of the rest of the paper is as follows. Section II presents the analytical discussion, based on a ‘‘fluctuating number of cutting-bonds’’ picture. In Sec. III we go through one by one the numerical methods employed, the findings about structural fluctuations, and some further numerical analysis of the extremal statistics aspects. Section IV contains the results concerning the maxflow problem, and some details of interest that can be determined from analyzing large statistics. Section V finishes the paper with a discussion.

II. CRITICAL CURRENT DENSITY

We consider diluted lattices where the maximum supercurrent i_c that a present bond can sustain is a random variable distributed between 0 and 1 according to

$$P(i_c)=(1-\alpha)c^{-\alpha}, \tag{1}$$

with $\alpha<1$.

Let I_c be the maximum supercurrent (or maxflow) that the whole system, given a set of values $\{i_c\}$, can sustain. The average current density J_c is then $J_c=\langle I_c\rangle/L^{(d-1)}$, and goes to zero at p_c as

$$J_c\sim(p-p_c)^{v}. \tag{2}$$

Right at p_c, and for a system of finite linear size L, usual finite-size scaling arguments [18,32] imply that

$$J_c(p_c,L)\sim L^{-v/\nu}, \tag{3}$$

where ν is the percolation correlation length exponent. The nodes-links-blobs picture of the percolation cluster [13–16] tells us that, right at p_c, there is typically a single connected path through the sample. This path is a sequence of multiply connected regions (blobs) connected by strings of singly connected bonds, also called *cutting bonds*. The average number of cutting bonds is of the order of $L^{1/\nu}$ at p_c [15].

We now start by considering the maximum flow f^* allowed by a string of n cutting bonds, and which obviously equals the least capacity among the n bonds. The typical least value f_n^* among a collection of $n\gg 1$ random numbers i_c with probability $P(i_c)$ satisfies

$$\int_0^{f_n^*}P(i)di=1/n. \tag{4}$$

Thus

$$f_n^* = n^{-1/(1-\alpha)}. \tag{5}$$

On a system of linear size L at p_c, the average number of cutting bonds is $L^{1/\nu}$ [15]. In replacing this one obtains $f_n^* \sim L^{-1/\nu(1-\alpha)} \sim L^{d-1} J_c \sim L^{(d-1)-v(\alpha)/\nu}$ and thus from Eq. (3),

$$v(\alpha) = (d-1)\nu + 1/(1-\alpha) \tag{6}$$

as advanced in the Introduction.

This typical-n argument, however, neglects the fact that n is a fluctuating number. Since $\mathcal{P}(n)$ actually has a power-law tail extending down to $n=0$ [31], this neglect turns out to be not correct for quantities that depend on $1/n$ as Eq. (5).

We now present a more careful treatment, which takes into account the fluctuations in n. It is known [31] that $\mathcal{P}_L(n) = (n_L^*)^{-1} \hat{\mathcal{P}}(n/n_L^*)$, where $\hat{\mathcal{P}}(\hat{n})$ is a size-independent function, and $n_L^* \sim L^{1/\nu}$ [15]. Since for the purpose of our discussion all that matters is the behavior of $\mathcal{P}(n)$ as $\hat{n} \to 0$, we take for simplicity $\hat{\mathcal{P}}(\hat{n}) = (1+a)\hat{n}^a$, for $0 < \hat{n} \leqslant 1$. Thus,

$$\mathcal{P}(n) = (1+a)(n_L^*)^{-(1+a)} n^a, \tag{7}$$

for $1 \leqslant n \leqslant n_L^*$. We will for the moment assume that n cannot be zero.

Let now f be the minimum among n numbers x distributed with probability $P(x)$. The distribution $m_n(f)$ of f is determined as

$$\begin{aligned} m_n(f) &= nP(f)\left\{1 - \int_0^f P(x)dx\right\}^{n-1} \\ &\approx nP(f)\exp\left(-n\int_0^f P(x)dx\right). \end{aligned} \tag{8}$$

Because of the strong exponential suppression that occurs for f larger than f_n^* defined by Eq. (4), we can approximate $m_n(f)$ by

$$m_n(f) \approx \begin{cases} nP(f) & \text{if} \quad 0 < f \leqslant f_n^* \\ 0 & \text{if} \quad f > f_n^*. \end{cases} \tag{9}$$

Now allowing for the fact that n fluctuates, the probability distribution function (PDF) of the maxflow f through a string of cutting bonds is

$$\begin{aligned} m(f) &= \int_1^\infty dn \mathcal{P}(n) m_n(f) = (1+a) \\ &\quad \times (n_L^*)^{-(1+a)} \int_1^{n_L^*} dn\, n^a m_n(f), \end{aligned} \tag{10}$$

for $0 < f < 1$. From Eqs. (5) and (9) we conclude that, for a given value of f, the only nonzero contributions in Eq. (10) come from n values which are smaller than $\eta(f) = f^{-(1-\alpha)}$. Thus

$$m(f) = \frac{P(f)(1+a)}{(n_L^*)^{1+a}} \int_1^{\min[n_L^*, \eta(f)]} n^{a+1} dn. \tag{11}$$

Defining $f_{typ} = (n_L^*)^{-1/(1-\alpha)}$, $\kappa = (a+1)(1-\alpha)$, and $\lambda = \kappa/(a+2)$, this last expression can be written as

$$m(f) = \begin{cases} \lambda f_{typ}^{-1} \left(\dfrac{f}{f_{typ}}\right)^{-\alpha} & \text{if} \quad 0 < f < f_{typ} \\[2ex] \lambda f_{typ}^{-1} \left(\dfrac{f}{f_{typ}}\right)^{-(1+\kappa)} & \text{if} \quad f_{typ} < f < 1. \end{cases} \tag{12}$$

This gives the PDF for the maxflow f through a string of cutting bonds on a system of size L, allowing for fluctuations in the number n of bonds on the string. The strength of the fluctuations of $1/n$ is characterized by the exponent κ, which in turn depends on a. If $a \to \infty$ (nonfluctuating limit), $m(f)$ is nonzero only for $f < f_{typ}$. Thus $\langle f \rangle \sim f_{typ}$ and Eq. (6) is recovered in this case. However, it is known that $a \approx 0.22$ in two dimensions [31].

For general a and α, $m(f)$ has a power-law tail with exponent $(1+\kappa)$ for $f \gg f_{typ}$. The importance of this power-law tail is evidenced by considering the average flow

$$\langle f \rangle = \frac{\lambda}{2-\alpha} f_{typ} + \frac{\lambda}{1-\kappa} (f_{typ})^{\kappa}. \tag{13}$$

When $\kappa > 1$, $\langle f \rangle \sim f_{typ} \sim (n_L^*)^{-1/(1-\alpha)} \sim L^{-1/\nu(1-\alpha)}$ and Eq. (6) is recovered. However, if $\alpha > a/(a+1)$ $(\kappa < 1)$, the power-law tail dominates the average. In this case $\langle f \rangle f_{typ}^{\kappa} \gg f_{typ}$. Therefore $\langle f \rangle \sim L^{-(a+1)/\nu}$, and Eq. (3) implies that in this case,

$$v = (d-1)\nu + a + 1. \tag{14}$$

The meaning of this is clear. If α is large, typical cases with $O(n_L^*)$ cutting bonds will only allow an exceedingly small flux f. The average flow $\langle f \rangle$, however will be dominated by the very rare cases in which n is small and for which $f \sim O(1) \gg f_{typ}$. So finally we conclude that, if we idealize the backbone at p_c as a string of n cutting bonds, and if $\mathcal{P}(n)$ behaves for small n as n^a, one has that

$$v(\alpha) = \begin{cases} \dfrac{1}{1-\alpha} + \nu(d-1) & \text{if} \quad \alpha < \dfrac{a}{1+a} \\[2ex] a + 1 + \nu(d-1) & \text{if} \quad \alpha > \dfrac{a}{1+a}. \end{cases} \tag{15}$$

III. NUMERICAL RESULTS

A. Algorithms

In this section we test our analytical derivation of $v(\alpha)$ of Sec. II in two and three dimensions on large systems, with the help of powerful combinatorial algorithms [28]. Percolation backbones are first generated by means of a matching algorithm [28,33], for square and cubic lattices. We do this by randomly adding bonds one at a time until a percolation

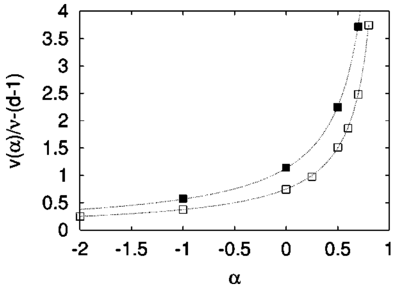

FIG. 1. Maxflow exponents measured for two-dimensional (2D) (empty squares) and 3D (filled squares) percolation clusters at p_c. Dotted lines indicate the theoretical result $v(\alpha)/\nu-(d-1)=1/\nu(1-\alpha)$, with $1/\nu=0.75$ (2D) and 1.13 (3D). The maximum linear size simulated was $L=4000$ in two dimensions and $L=120$ in three dimensions. Error bars are smaller than symbol sizes.

path is first present. At this point the matching algorithm identifies the conducting backbone, exactly at the percolation point for each sample. Alternatively, one could fix the density of present bonds to a value close to the infinite system critical density p_c, and then identify the percolating backbone with the same algorithm. However, our procedure has the advantage that no separate estimate is necessary for p_c.

For each percolating backbone, capacities are drawn from the given distribution, and the maxflow is calculated by means of a flow augmentation algorithm (see Ref. [28] for a review of the maxflow problem). The efficiency of the maxflow algorithm is highly increased when working on the backbone only, so we are able to analyze thousands of samples for each value of α. In this way we estimate numerically the average flow at p_c for several linear sizes L, and from its scaling properties $v(\alpha)$ is derived. The largest sample sizes studied were $L=4000$ in two dimensions and $L=120$ in three dimensions. These are mostly set by the CPU usage of the combination of the matching and flow algorithms, which in turn is dominated for L large by the scaling of the matching part. The maxflow code is actually sublinear in $n=L^2$ in CPU time, since the mass of the backbone scales with its fractal dimension. Notice that once the backbone of a sample has been established, it can be used for several consequent maxflow determinations for different α to save CPU time. In the Appendix, we present an idea for an optimal algorithm for this problem.

B. Results

Results are shown in Fig. 1. Our numerical simulation results confirm Eq. (6) nicely. However, the saturation of $v(\alpha)$ predicted by Eq. (15) for $\alpha>a/(a+1)$ does not occur. Notice that a is not a universal exponent but depends on the ensemble. For example, if the ensemble is determined by fixing $p=p_c$, numerical measurements and renormalization group calculations [31] give $a\approx0.22$. Additionally, roughly 20% of the connected samples have zero cutting bonds [30,31], that is, $\mathcal{P}(n)\sim0.20\delta(n)+cn^{0.22}$ for small n.

However, other ensembles can be considered. Consider, for example, the percolation cluster defined by the construction of Ambegaokar *et al.*, in which conductances are laid down on the lattice in order of increasing conductivity until a

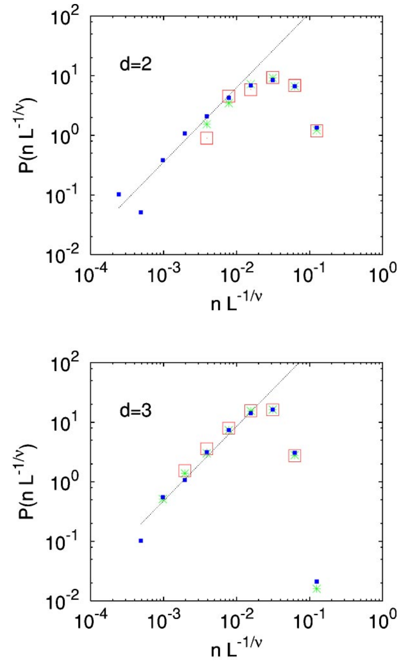

FIG. 2. Distribution of the number n of cutting bonds on percolation clusters in two [top, $L=32$ (empty squares), 128 (asterisks) and 1024 (full squares)] and three [bottom, $L=16$ (empty squares), 32 (asterisks) and 64 (full squares)] dimensions, in terms of the reduced variable $N=n/L^{1/\nu}$. We find that $P(n)$ is consistent with a power law n^a for small n (dashed lines). Within numerical accuracy, the exponent $a=1.25$ both in two and three dimensions.

percolating path is created [12]. At least the last conductance to be laid down is a cutting bond, so $\mathcal{P}(0)=0$. Experimentally this situation is realized when superconductive samples grow percolatively by deposition [23]. In this case the point at which the supercurrent is nonzero for the first time is defined by the first appearance of a connected path, not by a fixed density of occupied bonds.

As we add bonds one at a time, our numerical simulations correspond to this case rather than to fixing $p=p_c$. Our measurements of the distribution of the number of cutting bonds indicate (Fig. 2) that $\mathcal{P}(n)\sim n^a$ for small n, with $a\approx1.25$ in two and three dimensions.

Different ensembles give rise to different distributions of the number of cutting bonds, and specifically to different values of a so, if Eq. (15) were to hold for percolation clusters, the resulting transport exponent would be ensemble dependent. However, our maxflow measurements on percolation clusters are consistent with Eq. (6) for all α, without any sign of saturation.

In view of the failure of percolation clusters to show the predicted exponent saturation, we first confirmed the validity of Eq. (15) for strings of cutting bonds. We did so by numerically studying strings of bonds whose number n is distributed according to Eq. (7), and whose conductances (or capacities, for the maxflow problem) are distributed according to Eq. (1). The maximum flow is simply the least critical current i_c. Alternatively, bond capacities i_c may be interpreted as conductances, in which case the resulting conductance for the whole string is simply $\sigma=1/\Sigma_{j=1}^{n}1/i_c(j)$. We

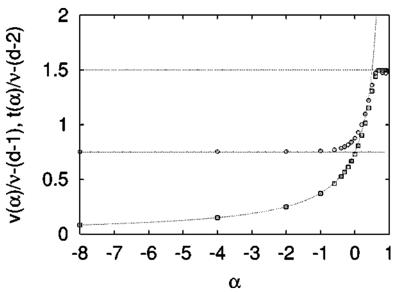

FIG. 3. Maxflow (squares) and ohmic current (circles) scaling exponents, as numerically estimated for strings of cutting bonds. The number of bonds n on the string is distributed according to Eq. (7), with $a=1.00$ and $n_L^*=L^{1/\nu}$ with $1/\nu=0.75$. Averages were taken over 10^7 samples, for $L=32,128,512,2048$, and 8192. Notice that (apart from a trivial shift) both critical exponents saturate to $(a+1)/\nu$ for large α. For the maxflow exponent this is the behavior predicted by Eq. (15). The fact that the (shifted) conductivity exponent has the same behavior for all $\alpha>0$ indicates that the sum of resistances along the string is dominated by the largest one, in this regime.

find for these strings of cutting bonds (Fig. 3) that Eq. (15) is satisfied very accurately. Figure 3 also shows that the conductivity and maxflow exponents are the same for $\alpha>0$, indicating that the conductance is dominated by the least i_c value in that regime. We conclude that Eq. (15) is exact for strings of cutting bonds. Thus the failure of Eq. (15) for percolation clusters simply means that these *do not* behave as strings of cutting bonds do. In other words, for α near 1, it is not correct to approximate a percolation cluster as a string of cutting bonds.

IV. THE ROLE OF BLOBS

A. Structural fluctuations

Our results (Fig. 1) show that the maxflow exponent $v(\alpha)$ follows Eq. (6), although fluctuations in the number n of cutting bonds, which exist and are relevant in real percolation clusters, were disregarded in its derivation. So we face a somehow paradoxical situation, since a naive calculation gives the correct result [Eq. (6)], while a seemingly more careful calculation that takes into account the fluctuations in n [Eq. (15)] does not. As mentioned in the preceding section, this means that our assumption that the maximum flow is determined by the cutting bonds alone needs to be revised. In order to test this assumption, we separately measure the maximum flow allowed by cutting bonds and by blobs, which we call m_c and m_b, respectively, for each percolation cluster. The overall maximum flow is the minimum of these. The procedure works such that one picks first the smallest of the cutting-bond capacities, and then assigns to it an infinite capacity. Then the maxflow is found, which is now given by the minimal blob mincut (configuration). Figures 4(a–c) show how the PDF of m_c (cutting-bond flow) and the total PDF vary with α. For nonanomalous values $\alpha=0$ [Fig. 4(a)], the distribution is centered around a well-defined mean value. With increasing α one enters the anomalous regime, and the PDF develops a power-law tail. This would be ex-

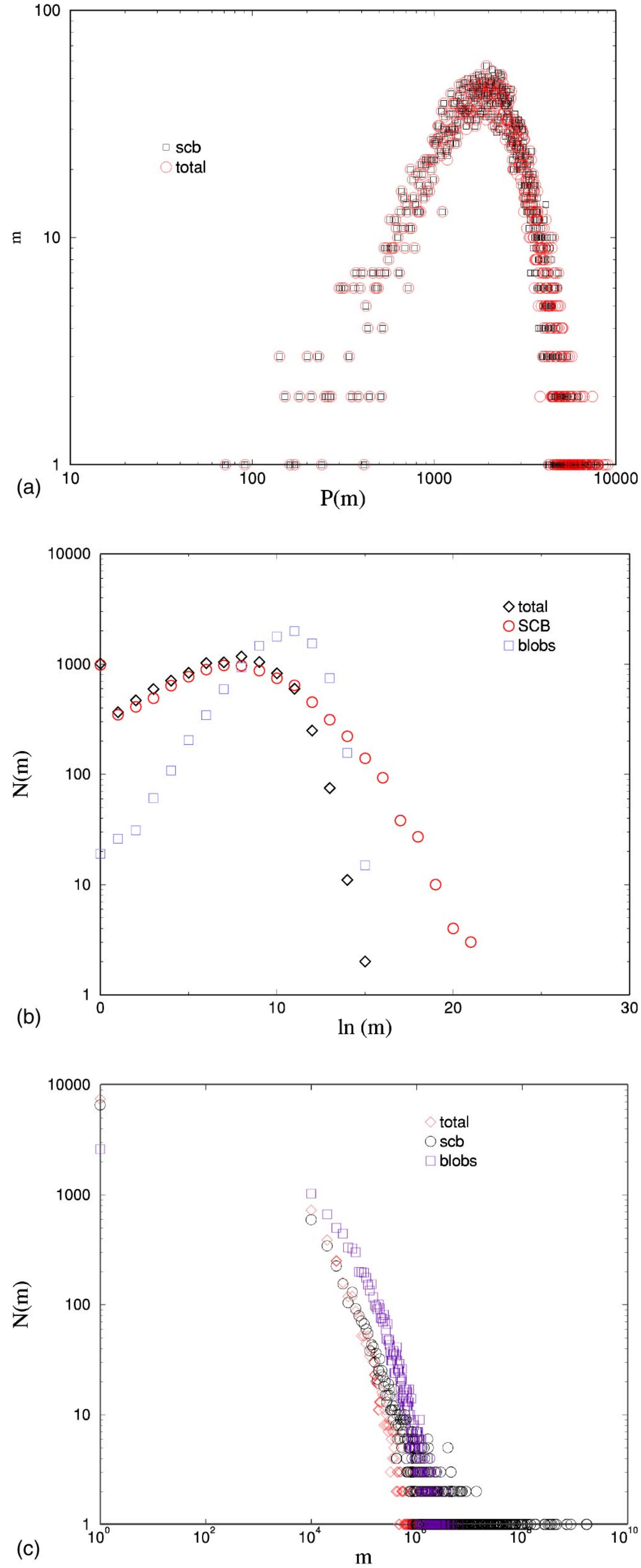

FIG. 4. Probability distribution for the maxflow allowed by cutting bonds (squares), blobs (circles), and resulting maxflow (crosses), which is the minimum of both. Results are shown for $L=256$ in two dimensions. From top to bottom, the disorder exponent is $\alpha=0.0$, 0.5, and 0.7.

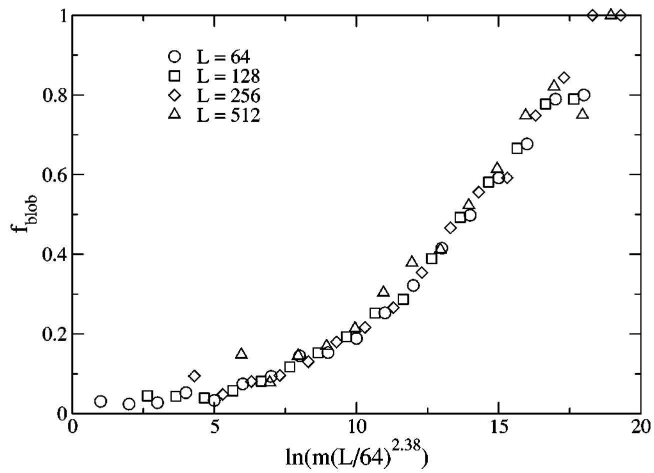

FIG. 5. Probability distributions for the blob-dominated fraction. The data are collapsed by scaling with the average maxflow. $m = -0.7$.

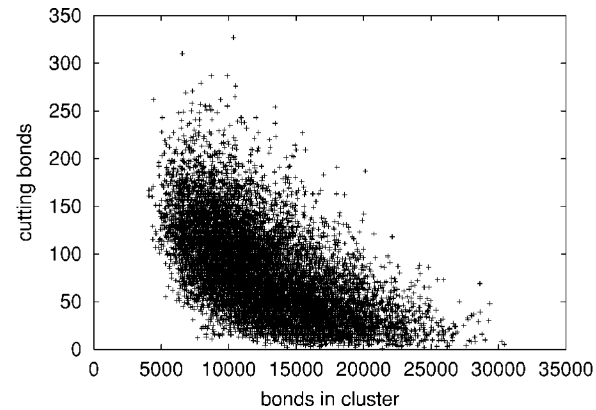

FIG. 6. The number of singly connected bonds for $L=256$ vs the backbone mass, sample to sample. 10 000 samples.

pected to result from the cutting bonds, while the blob flows m_b have a much narrower distribution, decaying roughly exponentially for large flows. This means that, when m_c is large, most probably m_b will be much smaller and thus the overall flow will be determined by m_b. Thus, although our derivation of Eq. (15) is correct for strings of cutting bonds, it is the blobs that determine the flow in those rare cases in which m_c is large. Therefore, the power-law tail in $P(m_c)$, which is responsible for the saturation of $v(\alpha)$ at large values of α in Eq. (15), is suppressed by blobs on percolation clusters. Figure 5 illustrates this by showing that the fraction of cases—for a given maxflow *m*—that are dominated by the blob contribution follow a separate PDF. The collapse is not completely perfect, since there may be a very slight trend in the total fraction of blob-dominated cases with increasing L. On the other hand, the variances of the maxflow distributions scale as expected (as the mean). It is worth mentioning that the distribution of k cuts (number of bonds in the mincut) is roughly exponential, so that $\langle k \rangle$ is of the order of 1.4–1.5 for $\alpha = 0.7$.

It is also worth pointing out that there are cross correlations between the structural quantities on one hand, and between the structure and the maxflow on the other hand. These are illustrated in Figs. 6 and 7. In a system with a given L it is after a moment's deliberation rather clear that there may be an inverse correlation between the *number* of cutting bonds and the sample-to-sample weight of the backbone. We have not tried to measure this relationship quantitatively, but given such a relation it is no surprise (Fig. 7) that the mass of the backbone correlates strongly with the maxflow value.

B. Blob dominance

In order to prove that our hypothesis, namely, that the blobs set the maxflow scale, is correct, we still have to show that the blob flow m_b has the right scaling properties, i.e., $\overline{m_b} \sim L^{-1/\nu(1-\alpha)}$. A complete calculation of the maximum flow allowed by blobs would require detailed information about the blob's internal structure. However, an estimate can be obtained from the following arguments. It is known that the backbone at p_c has a hierarchical, or self-similar, structure [15]. At the top level of this hierarchy, the backbone itself can be thought of as a string of singly connected (cutting) bonds interspersed with blobs. Blobs in turn are loops made of doubly connected bonds interspersed with smaller blobs and so on, as depicted in Fig. 8. This hierarchical structure has its counterpart in a similar classification of surfaces that separate the backbone into two pieces (cuts). At the top level of this hierarchy are the surfaces $\{S_1\}$ that cut the backbone at just one bond, next come those surfaces $\{S_2\}$ that cut the backbone at exactly two bonds, etc. The capacity $C(S)$ of a cut S is defined as the sum of the capacities i_c of the bonds crossed by it. Because of the maxflow-mincut theorem, the maximum flow equals the minimum of the cuts' capacities. Our assumption that cutting bonds alone determine the maximum flow is equivalent to minimizing the capacities among the S_1 alone. We now describe how the next level S_2 in this hierarchy can be analyzed. Coniglio [15] has shown that the

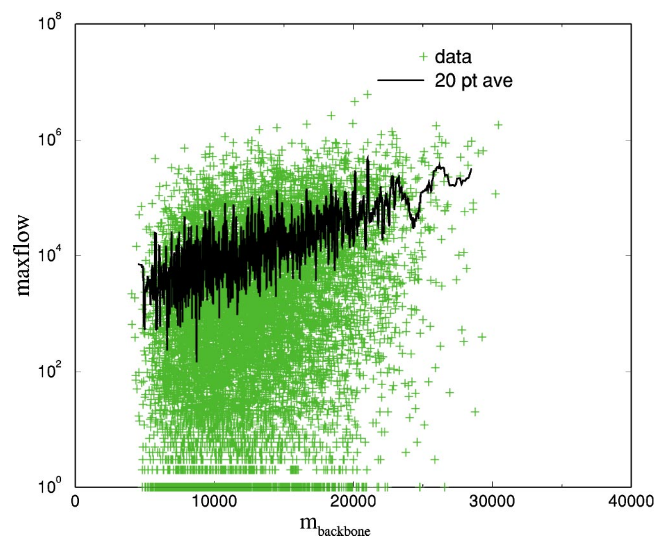

FIG. 7. The maxflow, for $L=256$ and $m=-0.7$, vs the actual backbone mass. The average is a running average over 20 samples, with consecutive masses from 10^4 samples.

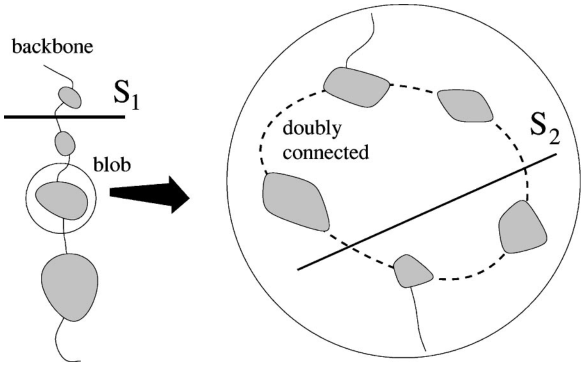

FIG. 8. Backbone structure.

derivative of the spanning probability p' with respect to p is proportional to the average number of cutting bonds $\langle n\rangle$. An extension of his reasoning, due to Kantor [30], allows one to write the second derivative of $p'(p)$ with respect to p at p_c as $\partial^2 p'/\partial p^2|_{p_c}\sim\langle n(n-1)-2N_2\rangle|_{p_c}$, where n is the number of cutting bonds and N_2 is the number of pairs of doubly connected bonds. Because by definition $\partial^2 p'/\partial p^2=0$ at p_c [37], one finds that $2\langle N_2\rangle=\langle n^2\rangle-\langle n\rangle$ at p_c. Since $\langle n^k\rangle\sim L^{k/\nu}$ [38], we conclude that the typical number of pairs of doubly connected bonds at p_c is $N_2\sim L^{2/\nu}$. However, this alone is not enough to estimate the typical maximum flow allowed by doubly connected bonds, for they might be grouped into blobs in different ways. Fortunately the total number n_2 of doubly connected bonds at p_c can also be calculated [15], and it turns out to be $\langle n_2\rangle\sim L^{1/\nu}$. This means that the blob statistics is dominated by one large ring of roughly $L^{1/\nu}$ bonds and therefore contains a number of pairs of doubly connected bonds, which is of the order of $L^{2/\nu}$. Using this information we can now estimate the maximum flow allowed by blobs at the level of doubly connected bonds. This large blob dominates the maximum flow since lesser blobs, located somewhere else along the backbone, will allow a larger flow. Thus one has to find the maximum flow for two parallel strings, each containing $L^{1/\nu}$ cutting bonds. The typical flow allowed by each string is of the order of $L^{-1/\nu(1-\alpha)}$ and therefore the typical maximum flow allowed by doubly connected blobs, which is twice this, is of the right order.

Our reasoning for doubly connected bonds only considers typical cases, i.e., fluctuations in the number of doubly connected bonds are disregarded. If a particular cluster, in addition to having a small number of cutting bonds, also has a small number of doubly connected bonds, then the next levels in this hierarchy would be relevant. The same sort of reasoning can be used at all levels in the hierarchy of cuts, but because the algebra becomes too complicated for triply connected bonds already, we did not test this in detail. However, it seems safe to assume that $\min\{C(S_k)\}\sim L^{-1/\nu(1-\alpha)}$ for all $k>2$ as well. Additionally notice that, in order for the mincut to be located at triply connected bonds, it is necessary that the numbers of singly and doubly connected bonds be simultaneously small, an occurrence which arguably has a small probability.

We then see that, in those rare cases in which the number of singly connected bonds is small (they allow a large flow), blobs take their role thus limiting the flow to a value that is typically of the order of $L^{-1/\nu(1-\alpha)}$. This then shows that the correct value of the exponent $v(\alpha)$ is given by Eq. (6). Similar behavior is of course expected for other transport properties, e.g., conductivity, in the limit of anomalous distributions of bond strengths. This is so because in this limit the resistance of the whole cluster is dominated by that of the mincut, where conductivities are interpreted as critical currents.

V. CONCLUSIONS

In this paper we have demonstrated that the transport problem on percolation clusters still holds surprises. Our findings deny the widespread notion that, in the limit of anomalous strength distributions, it is the cutting bonds alone that determine the transport properties. We show analytically, and confirm numerically that, if blobs could be neglected (because of their allowing a larger maxflow than cutting bonds) then the overall system's behavior would be strongly dependent on the ensemble (the cutting-bond PDF tail exponent). This ensemble dependence would come about because the number of cutting bonds has a "broad" distribution extending down to zero. However the predicted ensemble dependence is not there, as we show numerically on large two- and three-dimensional systems. Using scaling arguments we then demonstrate that it is in fact the blobs that finally determine the average maxflow. However, we are forced to finish with the paradoxical conclusion that though the expected mechanism for the maxflow, namely, cutting-bond dominance, does not work in the anomalous regime (large α), the original cutting-bond estimate for the transport exponent is nevertheless restored by the limiting effect of the blobs.

ACKNOWLEDGMENTS

This collaboration was partially supported by the Väisälän Foundation, Finland. C.F.M. acknowledges financial support by CONACYT, Mexico, through a research project under Grant No. 36256-E. C.F.M. also wishes to acknowledge the kind hospitality of the Laboratory of Physics, HUT, where parts of this work were done. M.A. was supported by the Academy of Finland Center of Excellence program.

APPENDIX: AN OPTIMAL ALGORITHM

We note that the augmenting path method is better here than in the general maxflow problem (so-called push-relabel preflow algorithms [34] enjoy the most popularity). This is since the structure of the backbone is essentially one dimensional, the number of augmentations remains small, of the order of 1. To remind the reader, such an algorithm consists of *flow augmentations*, which are repeated until the mincut is formed (by a surface of blocked bonds) and maxflow is reached. For each augmentation one needs to establish a path from the "source" to the "sink," which can be done, e.g., by using shortest-distance path methods [28].

The one-dimensional nature means that the backbone can be decomposed into strings of subsequent cutting bonds C_i

and and blobs separating such strings B_i. Thus the structure is equivalent to the one-dimensional series $\ldots C_i B_i C_{i+1} \ldots$. In principle, one may thus write a more efficient algorithm by abandoning the lattice structure, and describing the internal geometry of each B_i separately. Thus an optimal version of the algorithm would entail the following steps.

(i) Establish the structure (B_i, C_i).

(ii) Find an augmenting path along the chain, across all B_i.

(iii) Augment flow, that is, find the smallest capacity in the C_i, and the smallest capacity in all the B_i. This is f_1.

(iv) If f_1 equals the minimal cutting-bond capacity stop, otherwise augment (subtract f_1 from C_i, and the paths inside B_i).

(v) Update the paths inside those B_j, only, where a bond was saturated by f_i ($i=1$ to begin with). Go to (iii).

We have used, instead, an Euclidean background for the maxflow part, since the scaling of the matching program is indeed the bottleneck.

[1] M. Sahimi, *Applications of Percolation Theory*, 1st ed. (Taylor & Francis, Bristol, 1994).
[2] M. Sahimi, Phys. Rep. **306**, 213 (1998).
[3] P. Kogut and J. Straley, J. Phys. C **12**, 2151 (1979).
[4] J.P. Straley, J. Phys. C **15**, 2343 (1982).
[5] B. Halperin, S. Feng, and P. Sen, Phys. Rev. Lett. **54**, 2391 (1985).
[6] J. Machta, R. Guyer, and S. Moore, Phys. Rev. B **33**, 4818 (1986).
[7] S. Feng, B. Halperin, and P. Sen, Phys. Rev. B **35**, 197 (1987).
[8] J. Machta, Phys. Rev. B **37**, 7892 (1988).
[9] T. Lubensky and A. Tremblay, Phys. Rev. B **37**, 7894 (1988).
[10] M. Octavio and C. Lobb, Phys. Rev. B **43**, 8233 (1991).
[11] O. Stenull and H. Janssen, Phys. Rev. E **64**, 056105 (2001).
[12] V. Ambegaokar, B. Halperin, and J. Langer, Phys. Rev. B **4**, 2612 (1971).
[13] A. Skal and B. Shklovskii, Sov. Phys. Semicond. **8**, 1029 (1975).
[14] H.E. Stanley, J. Phys. A **10**, L211 (1977).
[15] A. Coniglio, J. Phys. A **15**, 3829 (1982).
[16] P.G. Degennes, J. Phys. (France) Lett. **37**, L1 (1976).
[17] A. Benmizrahi and D. Bergman, J. Phys. C **14**, 909 (1981).
[18] *Fractals and Disordered Systems*, edited by A. Bunde and S. Havlin, 2nd ed. (Springer-Verlag, Heidelberg, 1996).
[19] G. Deutscher and M. Rappaport, J. Phys. (France) Lett. **40**, L219 (1979).
[20] C. Lobb, P. Hui, and D. Stroud, Phys. Rev. B **36**, 1956 (1987).
[21] M. Octavio *et al.*, Phys. Rev. B **37**, 9292 (1988).
[22] J. Gordon, A. Goldman, and B. Whitehead, Phys. Rev. B **38**, 12 019 (1988).
[23] M. Hernandez, M. Octavio, J. Aponte, and C. Rojas, Phys. Rev. B **49**, 674 (1994).
[24] R. Mulet, O. Diaz, and H. Herrmann, Physica A **268**, 1 (1999).
[25] C.H. Papadimitriou and K. Steiglitz, *Combinatorial Optimization: Algorithms and Complexity* (Dover, Mineola, NY, 1998).
[26] T.H. Cormen, C.E. Leiserson, and R.L. Rivest, in *Introduction to Algorithms*, 9th ed. (McGraw-Hill, New York, 1993), Chap. 6.
[27] J. Rhyner and G. Blatter, Phys. Rev. B **40**, 829 (1989).
[28] M. Alava, P.M. Duxbury, C. Moukarzel, and H. Rieger, in *Exact Combinatorial Algorithms: Ground States of Disordered Systems*, edited by C. Domb and J.L. Lebowitz, Phase Transitions and Critical Phenomena Vol. 18 (Academic Press, San Diego, 2001).
[29] P. Ledoussal, Phys. Rev. B **39**, 4816 (1989).
[30] Y. Kantor, J. Phys. A **19**, L497 (1986).
[31] J. Hovi and A. Aharony, Phys. Rev. E **56**, 172 (1997).
[32] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, 2nd ed. (Taylor & Francis, Bristol, 1994).
[33] C. Moukarzel, Int. J. Mod. Phys. C **9**, 887 (1998).
[34] A. Goldberg and R. Tarjan, J. Assoc. Comput. Mach. **35**, 921 (1988).
[35] Although this equivalence is strict only in the $\alpha \to 1$ limit, it holds with corrections for $1 > \alpha$ as well.
[36] Similar ideas were put forward in Ref. [6], for hierarchical structures which are reducible by series and parallel transformations.
[37] Here we depart slightly from Kantor's reasoning, who justifies this last point as due to duality. Duality is not necessary and therefore $2\langle N_2 \rangle = \langle n^2 \rangle - \langle n \rangle$ for any lattice and dimension, at p_c.
[38] This results from the fact that the distribution of n decays exponentially fast for large n. See Fig. 2 and Ref. [31].