Aalto University

OPEN ACCESS

This is an electronic reprint of the original article.
This reprint may differ from the original in pagination and typographic detail.

Makhlin, Y.; Silaev, M.; Volovik, G.E.

Topology of the planar phase of superfluid He-3 and bulk-boundary correspondence for three-dimensional topological superconductors

Published in:
Physical Review B

DOI:
10.1103/PhysRevB.89.174502

Published: 01/01/2014

Document Version
Publisher's PDF, also known as Version of record

Please cite the original version:
Makhlin, Y., Silaev, M., & Volovik, G. E. (2014). Topology of the planar phase of superfluid He-3 and bulk-boundary correspondence for three-dimensional topological superconductors. Physical Review B, 89(17), 1-8. [174502]. https://doi.org/10.1103/PhysRevB.89.174502

This material is protected by copyright and other intellectual property rights, and duplication or sale of all or part of any of the repository collections is not permitted, except that material may be duplicated by you for your research use or educational purposes in electronic or print form. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone who is not an authorised user.

Topology of the planar phase of superfluid ^{3}He and bulk-boundary correspondence for three-dimensional topological superconductors

Yuriy Makhlin,[1,2,3] Mikhail Silaev,[1,4] and G. E. Volovik[1,2]
[1] *Low Temperature Laboratory, Aalto University, P.O. Box 15100, FI-00076 Aalto, Finland*
[2] *Landau Institute for Theoretical Physics, Akademika Semyonova Avenue, 1-A, 142432 Chernogolovka, Russia*
[3] *Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Russia*
[4] *Institute for Physics of Microstructures RAS, 603950 Nizhny Novgorod, Russia*
(Received 2 February 2014; revised manuscript received 9 April 2014; published 5 May 2014)

We provide topological classification of possible phases with the symmetry of the planar phase of superfluid ^{3}He. Compared to the B phase [class DIII in classification of A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997)], it has an additional symmetry, which modifies the topology. We analyze the topology in terms of explicit mappings from the momentum space and also discuss explicitly topological invariants for the B phase. We further show how the bulk-boundary correspondence for the three-dimensional (3D) B phase can be inferred from that for the 2D planar phase. A general condition is derived for the existence of topologically stable zero modes at the surfaces of 3D superconductors with class-DIII symmetries.

DOI: 10.1103/PhysRevB.89.174502 PACS number(s): 67.30.hp, 73.20.−r, 74.90.+n

I. INTRODUCTION

Recently, topological classification for generic symmetry classes of topological insulators and superconductors was given by Schnyder *et al.* [1] and Kitaev [2]. Depending on the presence and properties of generic symmetries, namely the time-reversal and particle-hole symmetry, ten symmetry classes can be identified, and for each of them, depending on the space dimensionality, topological classification gives an integer invariant ($\mathbb{Z}$), a binary invariant ($\mathbb{Z}_2$), or no invariant (only trivial, "nontopological" insulators in this class). One example of a topological insulator is provided by the time-reversal invariant B phase of superfluid helium-3 which belongs to the DIII symmetry class; a Hamiltonian in this class respects time-reversal and particle-hole symmetries [1–3] (see Sec. II A). In thin films of superfluid ^{3}He, the time-reversal invariant planar phase [4] can become stable. In this phase, the superfluid gap, isotropic in the B phase, is anisotropic and vanishes for the direction transverse to the film. Nevertheless, in two dimensions (2D), this system is gapful ("insulating"). While this phase has not been identified experimentally yet, in recent experiments [5–9] strong suppression of the transverse gap has been observed.

The planar phase has an extra discrete symmetry, a combination of spin and phase rotations, which may modify the topological classification, adding extra topological invariants. In this paper, we set out to provide the topological classification of insulators with this additional symmetry. We are partially motivated by the above-mentioned experiments [5–9]. However, it is also interesting to see how additional symmetries modify the results for one of the ten classes. Although, in principle, additional (to time and charge reversal) unitary symmetries can be dealt with in the Altland-Zirnbauer (AZ) approach (cf. Ref. [10]), we also want to understand the relevant topology and topological invariants directly in the explicit language of the homotopy theory, that is, by analyzing the homotopy equivalence of relevant mappings (cf. Ref. [11]). This allows one to identify a topological class from the analysis of the topology of the band structure (in the case of no disorder) in the bulk. Such direct view is also of interest for the basic ten classes. We begin with similar analysis for the B phase (that is, for the DIII symmetry class) and reproduce the known results. The topological classification of topological insulators in class DIII in $d = 1,2,3$ dimensions is rederived, again in the explicit language of homotopy topology. We then account for the additional symmetry of the planar phase and modify the classification accordingly. The classification for the planar-phase symmetry in 2D is discussed, for which Volovik and Yakovenko give an integer ($\mathbb{Z}$) topological invariant in Ref. [12].

An extra motivation to study this particular case of the planar phase is that we use it to construct a dimensional reduction for general class-DIII topological superconductors. We show that the topological properties of a 3D system and an embedded $(2+1)$D system, which exist in any time-reversal invariant cross section of the momentum space, are connected. As an application of such a reduction, we derive a generalized index theorem for 3D topological superconductors, which provides an example of the bulk-boundary correspondence in odd spatial dimensions.

We consider here a particular additional symmetry, which is realized in the planar phase. This is a combination of a π spin rotation around some axis, followed by a $\pi/2$ phase rotation. This symmetry is satisfied in superfluid ^{3}He, in which the spin-orbit interaction is very weak, but not necessarily in other materials. Nevertheless, the method of dimensional reduction, discussed below, can be applied to other classes of topological superconductors and insulators with various additional exact or approximate symmetries, such as the point symmetry groups in crystals (cf. Refs. [13,14]) and pseudospin rotations in graphene.

II. TOPOLOGICAL CLASSIFICATION FOR PLANAR-PHASE SYMMETRY

A. Parametrization and symmetries of the Hamiltonian

Our considerations of the planar-phase symmetry are based on those for the DIII symmetry class, defined below, and we begin with the latter case. We consider noninteracting translationally invariant systems. This allows us to characterize

 ©2014 American Physical Society

the system by a single-particle Bloch Hamiltonian $H(\mathbf{k})$. Here $H(\mathbf{k})$ is a mapping from the momentum space to the space of Hamiltonians with certain constraints, depending on symmetry. This is easily extended to interacting systems using the Green-function formalism, with the effective Hamiltonian being given by the Green function at zero frequency; see, e.g., Ref. [15]. With such constraints, topologically (homotopically) nonequivalent mappings are possible, and the goal of topological classification is to provide a complete list of the equivalence classes of such mappings (two mappings are considered equivalent—homotopic—if they can be continuously transformed to each other). In order to deal with the topology of the mappings, let us first discuss their properties in more detail.

In a completely translationally invariant d-dimensional system, $\mathbf{k}$ runs over the infinite momentum space. In the latter case, for a spherically symmetric system (^{3}He-B) at $k\to\infty$, the Hamiltonian has a fixed matrix form (up to an inessential positive constant; cf. Sec. III) and this allows one to compactify the momentum space by identifying all the points at $k\to\infty$ to a single point, which reduces the d-dimensional momentum space $\mathbb{R}^d$ to a sphere S^d. Similarly, in a system with discrete translational symmetry (i.e., a periodic crystal, a topological band insulator), the (quasi)momentum space is the Brillouin zone (BZ), which is a d-dimensional torus, $\mathbb{T}^d$. However, if we disregard the so-called weak topological invariants and focus only on the strong topological invariants robust to disorder [2,16], one can again replace $\mathbb{T}^d$ by a sphere S^d by gluing together all points at the BZ boundary. Thus, below we consider mappings from the d-dimensional "spherical Brillouin zone" $S^d_{\rm BZ}$. Below, we sometimes use the "spherical" language, assuming that $\mathbf{k}=0$ is the north pole, and $k=\infty$ (or the boundary of BZ) corresponds to the south pole. We further assume that opposite points, $\pm k$, correspond to the points on the opposite ends of a same-latitude line (that is, differ by a π rotation about the z axis).

Further, the properties of the mapping are fixed by the conditions that the Hamiltonian is Hermitian and nondegenerate (i.e., has no zero-energy eigenvalues, a gapful spectrum) and by the symmetry constraints. In class DIII of the tenfold Altland-Zirnbauer classification [3] (see Refs. [1] for a summary and further references), the system—the Bogolyubov-de-Gennes (BdG) Hamiltonian in the case of our current interest—possesses two symmetries, which are the time reversal and charge conjugation. We consider a $4=2\times 2$-dimensional space of states (we show below that increasing this dimensionality does not modify the result). This corresponds to the spin and two Bogolyubov indices, each taking one of two values; below we use the Pauli matrices σ_i and τ_i for these indices, respectively. In our notations below, the BdG Hamiltonian has the form $\left(\begin{smallmatrix}\varepsilon_{\mathbf{k}} & \hat\Delta_{\mathbf{k}}\\ \hat\Delta^\dagger_{\mathbf{k}} & -\varepsilon^T_{-\mathbf{k}}\end{smallmatrix}\right)$, where $\hat\Delta$ is a spin-symmetric and momentum-odd matrix (triplet pairing) of the superfluid order parameter, while $\varepsilon_{\mathbf{k}}$ is the spectrum of excitations in the normal state, e.g., $\varepsilon_{\mathbf{k}}=(\mathbf{k}^2/2m)-\mu$. The simplest form of $\hat\Delta$ for ^{3}He-B and the planar phase is $\hat\Delta_B=(\Delta_B/k_F)(\sigma_x p_x+\sigma_y p_y+\sigma_z p_z)g_\sigma$ and $\hat\Delta_P=(\Delta_P/k_F)(\sigma_x p_x+\sigma_y p_y)g_\sigma$, respectively. Here, $g_\sigma=i\sigma_y$. The time-reversal symmetry (TRS) implies that

$$H(-\mathbf{k})=g_\sigma H^T(\mathbf{k})g_\sigma^{-1}.\qquad \text{(TRS)}\tag{1}$$

(We follow the notations of Schnyder *et al.* [1].) The charge-conjugation symmetry [or particle-hole symmetry (PHS)] also relates $H(\mathbf{k})$ and $H(-\mathbf{k})$: specifically, $H(-\mathbf{k})=-\tau_1 H^T(\mathbf{k})\tau_1$. Since TRS and PHS are antiunitary symmetries, for convenience we use their combination, which basically constrains the structure of $H(\mathbf{k})$ at each $\mathbf{k}$ (a chiral symmetry):

$$H(\mathbf{k})=-PH(\mathbf{k})P,\quad P=\tau_1\sigma_y,\quad (\text{PHS}*\text{TRS}),\tag{2}$$

that is, $H(\mathbf{k})$ anticommutes with P. This implies that $H(\mathbf{k})$ is block-off-diagonal in the eigenbasis of P, and thus it is completely defined by its block M above the diagonal (the block below the diagonal being $M^\dagger$; cf. Ref. [1]). Equivalently and more specifically, one easily finds from Eq. (2) that H is a real linear combination of eight (instead of the initial 16) terms: $H=a\sigma_x+b\sigma_z+c\tau_1\sigma_x+d\tau_1\sigma_z+e\tau_2+f\tau_3+g\tau_2\sigma_y+h\tau_3\sigma_y$. Then, the TRS relates the values of these coefficients at $\mathbf{k}$ and $-\mathbf{k}$: the coefficients f and g in front of τ_3 and $\tau_2\sigma_y$ are the same, while the other six change sign. It is convenient to combine these eight real numbers, a, b, c, d, e, f, g, h, into the following matrix:

$$M=\begin{pmatrix}A+iB & C+iD\\ C-iD & -A+iB\end{pmatrix},\tag{3}$$

where $A=a-id$, $B=b+ic$, $C=h-ie$, and $D=-g-if$. Then the condition that H is gapped ($\det H\neq 0$ or, equivalently, $A^2+B^2+C^2+D^2\neq 0$) reads

$$\det M\neq 0,\tag{4}$$

and the PHS maps $M\to -M^T$.

Thus, our problem reduces to finding the classes of topologically equivalent mappings $M:\text{BZ}\to\text{GL}(2,\mathbb{C})$ with the property

$$M_{-\mathbf{k}}=-M^T_{\mathbf{k}}.\tag{5}$$

We refer to this property by saying that the mapping M is *odd*.

It is useful to further simplify the problem by reducing (continuously tightening) GL(2,$\mathbb{C}$) to U(2) in a standard manner. Namely, each $M\in\text{GL}(2,\mathbb{C})$ can be uniquely presented as a product $M=\check{P}U$ of a unitary U and a positive hermitian $\check{P}$ (polar decomposition). Then $\check{P}$ can be (e.g., linearly) retracted to the identity $\hat{1}$. Formally speaking, this means that U(2) is a deformation retract [17,18] of GL(2,$\mathbb{C}$). Thus, we have to classify mappings $U:\text{BZ}\to\text{U}(2)$ with $U_{-\mathbf{k}}=-U^T_{\mathbf{k}}$ (one can easily check all the details of this reduction).

B. ^{3}He-B: A $\mathbb{Z}_2$ invariant in 2D

Each Hamiltonian is described by a unitary U, and we have to classify mappings BZ$\to$U(2) with the proper symmetries. To find the classification, we recall that $\text{U}(2)=(S^1\times S^3)/\mathbb{Z}_2$; more specifically, each unitary U can be presented as a product of a phase factor and a special unitary matrix, $U=e^{i\phi}S$, $S\in\text{SU}(2)$ (i.e., $\det S=1$). This presentation is not unique, since one can change simultaneously the sign of both terms in the product. However, if we choose some presentation at one point, say, $\mathbf{k}=0$, we can follow how the phase factor and the matrix S vary continuously over the BZ [33].

If we parametrize S with a 4D unit vector $\mathbf{m}$ as $S=m_0\hat{1}+i(m_x\sigma_x+m_y\sigma_y+m_z\sigma_z)$, the symmetry properties of $U(\mathbf{k})$ (the odd parity) translate in the following: considering

$\det U(\mathbf{k})$, we find that $e^{i\varphi_{\mathbf{k}}}$ is either equal or opposite to $e^{i\varphi_{-\mathbf{k}}}$. Since they are equal at $\mathbf{k}=0$, we find that $e^{i\varphi_{\mathbf{k}}}$ is even: $\varphi_{-\mathbf{k}}=\varphi_{\mathbf{k}}$. This implies that $S_{\mathbf{k}}$ is odd, which means that m_0, m_x, m_z are odd and m_y is even:

$$m_0(-\mathbf{k}) = -m_0(\mathbf{k}),\ \ m_x(-\mathbf{k}) = -m_x(\mathbf{k}),$$
$$m_z(-\mathbf{k}) = -m_z(\mathbf{k}),\ \ m_y(-\mathbf{k}) = m_y(\mathbf{k}). \tag{6}$$

Hence we have to classify mappings $e^{i\phi}:\mathrm{BZ}\to S^1$ and $S:\mathrm{BZ}\to\mathrm{SU}(2)$ with these symmetry properties (6).

The first mapping is obviously always topologically trivial because it is even. The mapping $S: S^2_{\mathrm{BZ}}\to S^3_{\mathrm{SU}(2)}$, however, may be nontrivial: Because of the symmetry relation (6), the points $\mathbf{k}=0$ and $k=\infty$ (the poles of S^2_{BZ}) are mapped to $\pm i\sigma_y$. Thus there is a topological $\mathbb{Z}_2$ invariant that shows *whether they are mapped to the same or to different points*.

Clearly, both values of this invariant can be realized and this gives a complete classification, that is, any odd mappings $\mathrm{BZ}\to S^3$ with the same value of this $\mathbb{Z}_2$ invariant are homotopic to each other within the class of odd mappings. To see this, let us first consider the 1D case of the mapping $S: S^1_{\mathrm{BZ}}\to S^3_{\mathrm{SU}(2)}$ from a 1D sphere (a circle). In this case, one can consider the mapping of one-half of S^1_{BZ} between the north and south poles: (i) In the even case (with the zero $\mathbb{Z}_2$ invariant), this can be contracted to its initial point, and the other half gets contracted as well by the odd symmetry; hence the mapping is topologically trivial. (ii) In the odd case ($\mathbb{Z}_2$-invariant 1), this half starts at one pole of $S^3_{\mathrm{SU}(2)}$ and ends at the other, and it can be deformed to any standard path between the poles (e.g., a meridian line), and the other half gets deformed by symmetry to the other half of the meridian; hence all odd mappings can be deformed to each other continuously. A similar consideration can be found in the next section in more detail.

Going to the 2D case, one can imagine cutting S^2_{BZ} by a meridian circle in two hemispheres, thinking of this circle as the 1D BZ. By first deforming this 1D BZ as above, and then gluing to it the two 2D halves of the 2-sphere, we arrive at the conclusion above. Notice, however, that by going to $d=3$ and using the same procedure, one finds out that the 3D hemispheres of S^3_{BZ} can be attached to the 2D frame in many topologically different ways, described by the degree of the mapping. Thus we can see that in this case, the (integer) degree of the mapping $S: S^3_{\mathrm{BZ}}\to S^3_{\mathrm{SU}(2)}$ is the only topological invariant. In summary, in agreement with Refs. [1,2], we find for class DIII a $\mathbb{Z}_2$ invariant in 1D and 2D and a $\mathbb{Z}$ invariant in 3D.

Let us also remark that if, in addition, we assert that far away from the Fermi surface (i.e., "far above and deep inside the Fermi sea"), the Hamiltonian is dominated by the normal-state part $\propto\tau_3$, so that we fix also the images of $k=0,\infty$ in S^1_φ (to ±1), then we have an additional $\mathbb{Z}$ invariant that tells us how many "half times" the image of S^d_{BZ} encircles S^1_φ. In other words, this invariant is given by the winding number of $\det M$ along an arbitrary path from $k=0$ to $k=\infty$, which is $\int \mathrm{tr}(PH^{-1}dH)/(4\pi)$.

C. Planar phase

As we have discussed in Sec. I, the planar phase of superfluid ^{3}He has an extra symmetry as compared to ^{3}He-B. While in the bulk helium-3, only A and B phases are known to be stable (in zero magnetic field), the planar phase may be stabilized in thin films. In recent experiments [5–9], indications of the strongly distorted B phase were found, and the planar phase may become observable too. As estimated in Ref. [19], the superfluid gap, which is isotropic in the B phase in the bulk, becomes anisotropic in the films with the gap in the transverse direction suppressed by a factor of about 0.4.

This motivates us to analyze the topological classification for symmetry classes with "extra" symmetries on top of the basic TRS and/or PHS symmetries. On one hand, this can be analyzed within the general frame of the AZ approach [10]. However, explicit results for symmetries of interest and especially explicit expressions for respective topological invariants are of great interest (cf. the discussion for crystalline solids [20–24]).

We analyze the symmetry class of the planar phase, that is, the DIII class with an extra symmetry described below, and provide a complete classification. We show below that the $\mathbb{Z}_2$ invariant, found in the previous section for a 2D class-DIII system, survives. Moreover, the complete classification within this symmetry class gives rise to an integer topological invariant, with the $\mathbb{Z}_2$ invariant being its parity [34]. We explain these results below in this section.

The additional symmetry of the BdG Hamiltonian in the planar phase in our notation is $C=\sigma_z$ (that is, $C=\tau_0\sigma_z$),

$$H(\mathbf{k}) = CH(\mathbf{k})C,\ \ C=\sigma_z, \tag{7}$$

and we consider the symmetry class with this additional symmetry constraint. Here, C is a combined Z_2 symmetry—a combination of the spin π rotation about the z axis and the phase rotation by $\pi/2$. Note that the single-particle Hamiltonian (and Green function) commutes also with the transformations generated by C, $\exp(i\alpha C)$, which form a continuous U(1) symmetry group in similarity, e.g., to spin rotations about z generated by S_z. This, however, does not impose additional constraints on H and hence does not change the topology of these single-particle quantities. Furthermore, the many-body Hamiltonian and multiparticle quantities (e.g., the two-particle Green function) obey only the discrete, but not the continuous, symmetry.

The Hamiltonian satisfying (7) can be transformed to the off-diagonal form with

$$M = -M^\dagger, \tag{8}$$

and thus $U=-U^\dagger$. Hence U is either $\pm i$ (the case of little interest) or $U=i\mathbf{n}\boldsymbol{\sigma}$, a spin rotation by π around an arbitrary axis $\mathbf{n}$. This second choice provides the nontrivial topology. We have to classify mappings of the sphere S^2_{BZ} to the sphere $S^2_{\mathbf{n}}$ that are odd: opposite points are mapped to opposite points and, for both spheres, *opposite* refers to points related by a π rotation around a specific axis (the z axis for S^2_{BZ} and the y axis [35] for $S^2_{\mathbf{n}}$).

In complete analogy with the 3D case for class DIII above, we find that in 2D such mappings are completely classified by the *degree of the mapping*, which could assume any integer value. Moreover, whether this value is even or odd is related to the $\mathbb{Z}_2$ invariant, defined above. Indeed, both $k=0$ and $k=\infty$ in BZ are mapped to $\pm i\sigma_y$, and the $\mathbb{Z}_2$ invariant above determines whether they are mapped to the same or to the opposite points of the $\mathbf{n}$ sphere. We show below that in the former case the degree is even and in the latter case the degree is odd.

To prove this statement, in analogy to the previous section, let us cut the BZ sphere in two halves with a line through $k=0$ and $k=\infty$, for instance, with $k_x=0$ [equivalently, in the language of spherical BZ S^2_{BZ}, with a full meridian circle on the BZ sphere, for instance, the zero and 180° meridians, which are the blue solid and red dashed lines in Fig. 1(b)]. Because of the odd parity, the mappings of one hemisphere completely define the full mapping. However, the mapping of the hemisphere is constrained at the boundary (the 0 and 180° meridians)—the opposite points of the boundary, $\pm\mathbf{k}$, should be mapped to the opposite points of $S^2_{\mathbf{n}}$. Such mappings can always be thought of in the following way: (i) We have a mapping of one half meridian (say, the 0 meridian) between the poles to $S^2_{\mathbf{n}}$ (with the ends mapped to the same or opposite poles, depending on the $\mathbb{Z}_2$ invariant). (ii) The mapping of the other half meridian (180° meridian) is determined by the symmetry (odd parity). (iii) And the mapping from the interior of the hemisphere somehow (arbitrarily) extends the mapping of the boundary.

First, each mapping of the hemisphere can be continuously deformed under the constraint of odd parity to a simpler mapping. Specifically, we can modify continuously the mapping of the half meridian (with the mapping fixed at its ends), with the other half meridian being modified in accordance with the odd parity. One can easily see that because of the odd parity, this modification does not change the total area on $S^2_{\mathbf{n}}$ covered by the image of the hemisphere in BZ (and, again by odd parity, the other hemisphere covers the same area). It is more convenient to describe this modification separately for two possible values of the $\mathbb{Z}_2$ invariant.

In the case of the zero value of the $\mathbb{Z}_2$ invariant, when two ends of the zero meridian are mapped to the same pole [Fig. 1(a)], the image of this meridian (a loop starting and ending at the same pole) can be continuously deformed to the pole itself. Then, the area spanned by the hemisphere is an integer, and the total area covered by the mapping $S^2_{\mathrm{BZ}}\to S^2_{\mathbf{n}}$ is even. (Note that any integer degree can be realized: to see that, one could just map the whole meridian to the pole.) In the other case of odd $\mathbb{Z}_2$ invariant [Fig. 1(c)], when two ends of the zero meridian on S^2_{BZ} are mapped to the opposite poles, the image of this meridian can be continuously deformed to "just a straight line," e.g., to the zero meridian on $S^2_{\mathbf{n}}$, as indicated in Fig. 1(c). Then, the area spanned by the hemisphere is half integer, and thus the full sphere S^2_{BZ} covers $S^2_{\mathbf{n}}$ an odd integer number of times.

Thus, *the degree of the mapping is the only invariant*. It can take any integer value. This value is even, when $k=0$ and $k=\infty$ are mapped to the same point, and odd, if they are mapped to different (then opposite) points. These two cases correspond to two values of the $\mathbb{Z}_2$ invariant from the previous section.

A comment is in order on higher-dimensional matrices. So far we considered the $4=2\times 2$-dimensional Hilbert space of possible states. In general, in these symmetry classes, we can consider higher $(2n\times 2)$-dimensional spaces, with the 2D Bogolyubov-Nambu and $2n$-dimensional "internal" space (spin and other degrees of freedom); large values

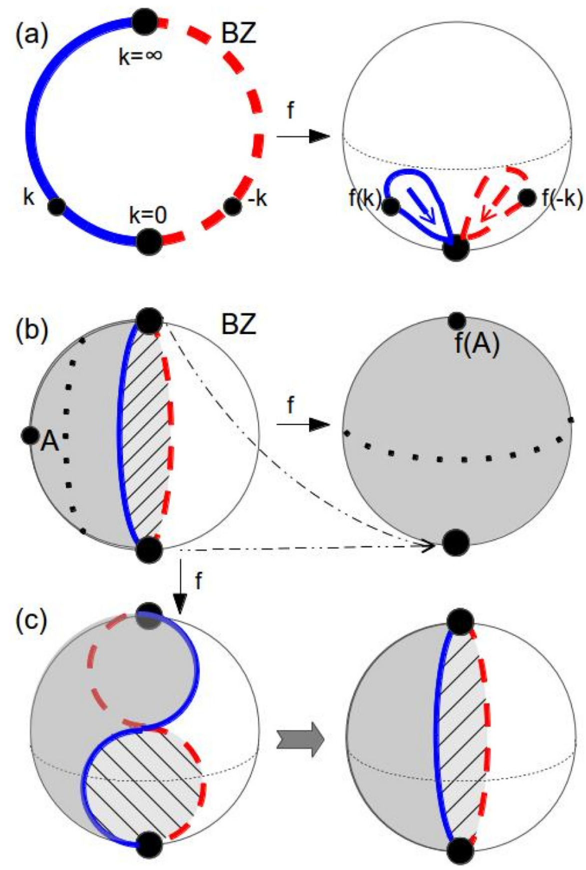

FIG. 1. (Color online) The mappings f, analyzed for classification of the planar-phase symmetry class, are from the Brillouin zone (BZ), compactified to a sphere S^d_{BZ} with the poles at $k=0$ and $k=\infty$, to the sphere $S^2_{\mathbf{n}}$ (see text). The mappings are odd: the images $f(\mathbf{k})$ and $f(-\mathbf{k})$ on $S^2_{\mathbf{n}}$ are related by a π rotation around the (vertical) axis. (a) The 1D BZ—a circle. For the zero value of the $\mathbb{Z}_2$ invariant, both poles of BZ are mapped to the same pole of $S^2_{\mathbf{n}}$. In this case, f can be continuously deformed to the trivial mapping, which maps the whole circle to one point. Indeed, the image of one semicircle (blue solid line) can be continuously contracted to the pole as indicated by the arrow; the other semicircle (red dashed line) is contracted at the same time because the mapping is odd. (b) The 2D BZ, a sphere, presented as a 1D frame (0 and 180° meridians) and two hemispheres. For the zero $\mathbb{Z}_2$ invariant, the 1D frame is contracted to a point (dashed arrows). Then the image of one hemisphere covers $S^2_{\mathbf{n}}$ an integer number of times (once in the figure), and, for an odd mapping, the other hemisphere—the same number of times, giving an even degree in total. (c) For the nonzero $\mathbb{Z}_2$ invariant, the two poles of BZ are mapped to two different poles of $S^2_{\mathbf{n}}$. Then the image of one semicircle from the 1D frame (blue solid line) can be deformed to a standard path (zero meridian). The image of each hemisphere in BZ covers the same half-integer part of $S^2_{\mathbf{n}}$, giving an odd degree in total (degree one in the figure).

of n pertain to realistic condensed-matter systems [2]. The symmetry conditions for TRS, PHS, and the extra planar-phase symmetry then look the same as above (cf. Ref. [1]). The anticommutation with P again implies the block-off-diagonal form of the Hamiltonian, and Eqs. (4), (5), and (8) are valid. Hence, as above, we have to classify odd mappings (5) $U: S^d_{\mathrm{BZ}}\to \mathrm{U}(2n)$. The analysis follows the same route as above:

for class DIII, first, presenting $U = e^{i\varphi} S$ with $\det S = 1$, we find that the map $e^{i\varphi}$ is always trivial; as for $S(\mathbf{k})$, each of the poles $\mathbf{k} = 0,\infty$ is mapped to one of the two connected components of antisymmetric matrices from SU($2n$) with the pfaffian Pf $S = \pm 1$, and this defines a $\mathbb{Z}_2$ invariant. As above, from direct homotopic-theoretical considerations, we see that for $d = 1,2$, there are no other invariants, while for $d = 3$, the topological class is again fully characterized by an integer, with the $\mathbb{Z}_2$ invariant above being its parity. Thus we find the same results for topological classification for class DIII ($\mathbb{Z}_2$ in $d = 1,2$ and $\mathbb{Z}$ in $d = 3$ dimensions).

Similar considerations apply for the planar-phase symmetry class; in this case, we again find a $\mathbb{Z}_2$ classification in 1D and $\mathbb{Z}$ in 2D, as opposed to a $\mathbb{Z}_2$ classification suggested earlier [25]. Note that this class contains $2n + 1$ disconnected components: iM (iU) is a Hermitian operator with l positive and $2n - l$ negative eigenvalues, where l may vary from 0 to $2n$ and is related to the signature $(2l,4n - 2l)$ of the Hermitian operator CH. Each component [the set of unitary anti-Hermitian $2n \times 2n$ matrices U with l eigenvalues $-i$ and $2n - l$ times i, i.e., the Grassmanian $\mathrm{U}(2n)/\mathrm{U}(l) \times \mathrm{U}(2n - l)$], except $l = 0,2n$, has the same second homotopy group $\pi_2 = \mathbb{Z}$, and hence a $\mathbb{Z}$ invariant in 2D arises. The cases $l = 0,2n$ correspond to $U = \pm i$ as above for $n = 1$ [see the discussion below Eq. (8)].

III. INDEX THEOREMS IN ODD SPATIAL DIMENSIONS

A. The problem

We have found a complete set of topological invariants for the planar-phase symmetry and for the B-phase symmetry, and further related questions need to be analyzed. In particular, it would be useful to have an explicit (integral) expression for the invariants. A further question of great current interest concerns the bulk-boundary correspondence between the topological invariants in the bulk and the properties of gapless boundary modes.

Since the discovery of a topological invariant for the integer quantum Hall state [26], there has been great interest in deriving index theorems that connect the topology of the fully gapped spectrum in the bulk with the number of gapless modes at the boundary of the system or inside topological defects (strings, domain walls, monopoles, etc.).

A full classification is still absent, though there is certain progress in understanding for even spatial dimensions, especially when the bulk system is characterized by an integer-valued topological invariant of group $\mathbb{Z}$. In 2D, one can mention three representatives of such systems: the integer quantum Hall effect state (class A according to the general tenfold classification), the ^{3}He-A phase and chiral $k_x + ik_y$ superconductivity (class D), and the planar phase of ^{3}He.

Among systems in odd spatial dimensions, of particular interest is the topological superfluid ^{3}He-B, which belongs to class DIII according to the general classification scheme. The ^{3}He-B Hamiltonian and the related Hamiltonians have the following form:

$$\mathcal{H}_{3\mathrm{D}} = \tau_3 \epsilon(\mathbf{k}) + \tau_1[\sigma_x f_x(\mathbf{k}) + \sigma_y f_y(\mathbf{k}) + \sigma_z f_z(\mathbf{k})]. \quad (9)$$

This Hamiltonian is gapful, when $\mathbf{f}(\mathbf{k})$ does not vanish at the Fermi surface, where $\epsilon(\mathbf{k}) = 0$. For convenience, here and below, we use the form of the BdG Hamiltonian, corresponding to an alternative definition of the Nambu spinor (see p. 77 in Ref. [27] for definition); it differs from the standard form by a unitary transformation,

$$\mathcal{H} = U^\dagger H U,\;\; U = \frac{1 + \tau_3}{2} + i\sigma_y \frac{1 - \tau_3}{2}. \quad (10)$$

For clarity, we use calligraphic letters for operators in this form. For the p-wave ^{3}He-B, the functions of the three-momentum $\mathbf{k}$ can be chosen as

$$\begin{aligned} \epsilon(\mathbf{k}) &= \mathbf{k}^2/2m - k_F^2/2m, \\ \mathbf{f}(\mathbf{k}) &= f\mathbf{k}, \end{aligned} \quad (11)$$

where $f > 0$. The Hamiltonian (11) has a symmetry-protected topological invariant (17) with $N_{\mathrm{B}} = 2$. The more general Hamiltonian (9) may have any even invariant N_{B} under the conditions that $f_i(\mathbf{k})/\epsilon(\mathbf{k}) \to 0$ at $k \to \infty$, where $i = x,y,z$, and $\epsilon(|\mathbf{k}| \to \infty) > 0$. These conditions allow compactification of the momentum space to S^3, but they are not needed in crystals, since in that case the Brillouin zone is a compact space. An example of a nontrivial mapping with a higher topological charge $N_{\mathrm{B}} = 2n$ is

$$\begin{aligned} f_z(\mathbf{k}) &= f k_z, \\ f_x(\mathbf{k}) &= f\,\mathrm{Re}(k_x \pm ik_y)^{|n|}, \\ f_y(\mathbf{k}) &= f\,\mathrm{Im}(k_x \pm ik_y)^{|n|}, \\ \epsilon(\mathbf{k}) &= \mu[(\mathbf{k}/k_F)^{2|n|} - 1], \end{aligned} \quad (12)$$

where $n \in \mathbb{Z}$ and the upper (lower) sign corresponds to $n > (<) 0$. The form of $\epsilon(\mathbf{k})$ dispersion in Eq. (12) is chosen in such a way to allow compactification of momentum space. Alternatively, higher values of the topological invariant can be obtained in a system consisting of several layers of the planar phase.

In order to derive the index theorem for ^{3}He-B (11) and related Hamiltonians (12), we assume that the boundary plane is $y = 0$, so that the conserved momentum projections are $k_{x,z}$. To find the complete spectrum of bound states $\varepsilon_b = \varepsilon_b(k_x,k_z)$, it is enough to consider a set of 2D spectral problems for the cross sections of momentum space,

$$k_z \cos\theta + k_x \sin\theta = 0, \quad (13)$$

where $2\pi > \theta \geqslant 0$. Indeed, the bound states at the interface between the nontopological insulator and ^{3}He-B are formed due to the subsequent Andreev and normal reflections of particles and holes, as shown schematically in Fig. 2. The momenta of both the incident particle and the one reflected from the boundary belong to the same cross section (13). Further, we will use the fact that the planes determined by Eq. (13) are time-reversal invariant in the sense that they contain states with opposite momenta $\mathbf{k}$ and $-\mathbf{k}$.

An example of such a dimensional reduction to the plane $k_z = 0$ is shown in Fig. 2(a). The 2D Hamiltonian in this cross section reduced from the 3D phase (12) is given by

$$\mathcal{H}_{2\mathrm{D}} = \tau_3 \epsilon(k_x,k_y) + \tau_1[\sigma_x f_x(k_x,k_y) + \sigma_y f_y(k_x,k_y)]. \quad (14)$$

The 2D Hamiltonian of the form (14) has the symmetry of the generalized planar state (7). In our current representation

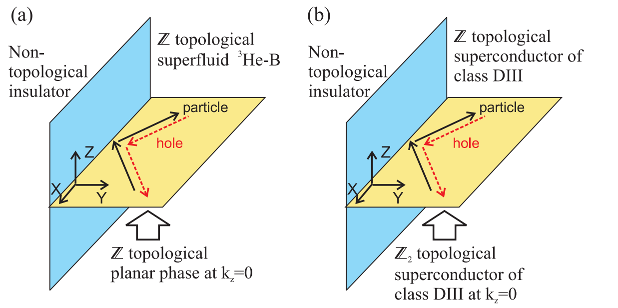

FIG. 2. (Color online) Dimensional reduction of the surface-state spectral problem in 3D to that in a time-reversal invariant cross section of momentum space $k_z = 0$. (a) Reduction from the $\mathbb{Z}$ topological superfluid ^{3}He-B results in the $\mathbb{Z}$ topological planar phase at $k_z = 0$. (b) Reduction from a general $\mathbb{Z}$ topological superconductor of class DIII produces a $\mathbb{Z}_2$ topological superconductor at $k_z = 0$.

of the BdG Hamiltonians [see below Eq. (9)], the operator of this additional symmetry—the matrix commuting with the Hamiltonian (14)—is $\mathcal{C} = U^\dagger C U = \tau_3\sigma_z$, with the unitary operator from Eq. (10). The Hamiltonians satisfying this additional symmetry are classified by an integer-valued topological invariant. The explicit form given by Eq. (19) expressed via the Green function gives an even-valued topological invariant $N_{\rm P}$. For the particular set of parameters (12), this invariant is $N_{\rm P} = 2n$. In general, it gives the Chern number $N_{\rm P}/2$ for each spin projection and therefore yields an index theorem for the number of edge states. Hence we conclude that $N_{\rm P}$ defines the number of zero edge modes at $k_z = 0$ in the parent 3D ^{3}He-B phase (12) as well.

Below we show that the topological invariants for the Hamiltonians (9) and (14) coincide, $N_{\rm B} = N_{\rm P}$, for quite a general form of the order parameter with arbitrary functions $f_{x,y}(\mathbf{k})$ and $f_z(\mathbf{k}) \propto k_z$. Then the index theorem for ^{3}He-B states that the number of zero modes at the $k_z = 0$ cross section of the momentum space is given by the 3D invariant $N_{\rm B}$. This particular choice of a cross section is determined by the specific form of the ^{3}He-B Hamiltonian given by Eqs. (9) and (12). Using this result, we argue that for a general Hamiltonian of class DIII, the time-reversal invariant cross sections (13) have a $\mathbb{Z}_2$ invariant, a nonzero value of which protects at least one stable zero of the bound-state spectrum $\varepsilon_b = \varepsilon_b(k_x,k_z)$ along each line from the one-parameter family (13).

We construct a map of the 3D slice in the 4D momentum-frequency space to the space of Green-function matrices $(k_x,k_y,t) \to {\rm GL}(4,\mathbb{C})$,

$$G = G(k_x,k_y,t,\alpha), \tag{15}$$

where $0 < \alpha < 2\pi$ is a parameter, as shown in Fig. 3. The map is designed to coincide with that for the planar state at $\alpha = 0$, when $t = \omega$, and for ^{3}He-B at $\alpha = \pi/2$, when $t = k_z$. They transform to each other by a continuous change of the orientation of the 3D slice in the 4D momentum-frequency $(\omega,\mathbf{k})$ space. We show below that the homotopy class is independent of α, and this proves that the generalized 3D ^{3}He-B is topologically equivalent to the generalized $2+1$ planar state in the $k_z = 0$ cross section.

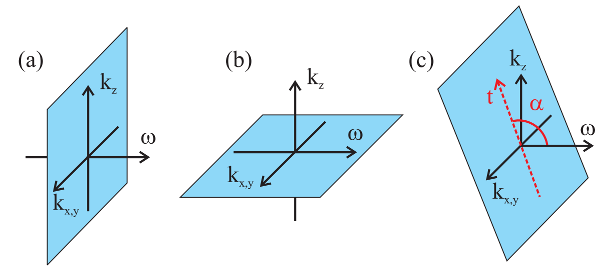

FIG. 3. (Color online) Slices of the $3+1$-dimensional space (ω,k_x,k_y,k_z) that are used to define the topological invariants for the (a) 3D ^{3}He-B phase, (b) $2+1$-dimensional planar phase, and (c) smooth transformation between 3D and $2+1$-dimensional invariants. The axillary axis is $t = \omega\cos\alpha + k_z \sin\alpha$.

B. Dimensional reduction from ^{3}He-B to the planar state

We start with a particular case of the simplified Green function describing the $(3+1)$-dimensional topological superfluid:

$$G^{-1}(\omega,\mathbf{k}) = i\omega - \mathcal{H}_{\rm 3D}(\mathbf{k}), \tag{16}$$

where $\mathcal{H}_{\rm 3D}(\mathbf{k})$ is given by Eq. (9) with $f_z \propto k_z$. There are two special cases: (i) $\omega = 0$ and (ii) $k_z = 0$.

In the case $\omega = 0$, the Green function $G^{-1}(0,k_x,k_y,k_z)$ represents a ^{3}He-B-like Hamiltonian, which anticommutes with the matrix $\mathcal{P} = \tau_2$, $\mathcal{P}^2 = 1$ (chiral symmetry). The Hamiltonians with such symmetry have the following symmetry-protected topological invariant:

$$N_{\rm B} = \frac{e_{ijl}}{24\pi^2}\,\mathbf{tr}\int_{\omega=0} d^3k\mathcal{P}G\partial_{k_i}G^{-1}G\partial_{k_j}G^{-1}G\partial_{k_l}G^{-1}, \tag{17}$$

where $\mathcal{P} = \tau_2$. Here the integral is taken over the momentum space, i.e., over the $\omega = 0$ slice in the 4D (ω,k_x,k_y,k_z) space, as shown in Fig. 3(a). The choice of slices in the frequency-momentum space of the Standard Model of particle physics is given in Ref. [15]. One can check by a direct calculation that the invariant (17) is *twice* the $\mathbb{Z}$ invariant from Sec. II B in 3D, that is, the degree of the mapping from BZ to the 3-sphere of SU(2) (it accumulates equal contributions from M and $M^\dagger$). For ^{3}He-B parameters (11) in Eq. (16), one finds that $N_{\rm B} = 2$ and all the higher values of the invariant $N_{\rm B} = 2n$, $n \in \mathbb{Z}$ are realized by the set (12).

In the case $k_z = 0$, Eq. (16) represents the Green function of the generalized planar phase,

$$G^{-1}(\omega,k_x,k_y) = i\omega - \mathcal{H}_{\rm 2D}(k_x,k_y), \tag{18}$$

where the Hamiltonian $\mathcal{H}_{\rm 2D} = \mathcal{H}_{\rm 2D}(k_x,k_y)$ is given by Eq. (14). The Green function of a $2+1$ system with symmetry $\mathcal{C}$ has a symmetry-protected topological invariant, which determines transport properties of the $2+1$ system. Indeed, it defines the quantized spin Hall conductivity in the absence of external magnetic field [12]. The invariant involves the symmetry operator $\mathcal{C} = \tau_3\sigma_z$, which commutes with the Green function:

$$N_{\rm P} = \frac{e_{ijl}}{24\pi^2}\,\mathbf{tr}\int d^2k d\omega \mathcal{C}G\partial_{k_i}G^{-1}G\partial_{k_j}G^{-1}G\partial_{k_l}G^{-1}. \tag{19}$$

Here the integral is taken over the $k_z = 0$ slice, $k_i = (\omega,k_x,k_y)$, in the 4D (ω,k_x,k_y,k_z) space, as shown in Fig. 3(b). Again, one checks by a direct calculation that this invariant is twice the $\mathbb{Z}$ invariant from Sec. II C in 2D, that is, the degree of the mapping from BZ to the 2-sphere $S^2_{\mathbf{n}}$.

For particular cases, e.g., for the set of Hamiltonians with the order parameter (12), one finds that the invariant (19) in the 2D cross section $k_z = 0$ coincides with the value of the invariant in the parent 3D phase (17), $N_{\rm P} = N_{\rm B} = 2n$. This is not a coincidence. Let us show that in a more general case of arbitrary functions $f_{x,y}(\mathbf{k})$ and $f_z(\mathbf{k}) \propto k_z$, the even-valued integrals (17) and (19) can be continuously transformed to each other by the rotation of the 3D slices in the 4D space, shown schematically in Fig. 3. Such transformation allows us to connect topological properties of the Hamiltonians in 3D and in 2D at $k_z = 0$.

To construct the connection between topological invariants (17) and (19), let us use the Green function in the form $\tilde{G}^{-1} = -iG^{-1}\tau_2$:

$$\tilde{G}^{-1}(\omega,\mathbf{k}) = \omega\tau_2 - \tau_3\sigma_z f k_z - \tilde{H}, \tag{20}$$

$$\tilde{H} = -\tau_1\epsilon(\mathbf{k}) + \tau_3[\sigma_x f_x(\mathbf{k}) + \sigma_y f_y(\mathbf{k})]. \tag{21}$$

The "Hamiltonian" $\tilde{H}$ in Eq. (21) anticommutes both with $\mathcal{P} = \tau_2$ $(P^2 = 1)$ and $\mathcal{C} = \tau_3\sigma_z$ $(C^2 = 1)$, while the whole Green function (20) anticommutes with $f k_z \tau_2 + \omega\tau_3\sigma_z$. As a result there exist 3D slices $(\omega = t\sin\alpha, f k_z = t\cos\alpha)$ with fixed α [shown in Fig. 3(c)], where $\tilde{H}$ anticommutes with the constant matrix

$$\mathcal{Q}_\alpha = \tau_2\cos\alpha + \tau_3\sigma_z\sin\alpha,\ \mathcal{Q}_\alpha^2 = 1. \tag{22}$$

The topological charge for a given parameter α is

$$N_\alpha = \frac{e_{ijl}}{24\pi^2}\,\mathbf{tr}\int d^2k\,dt\,\mathcal{Q}_\alpha\tilde{G}\partial_{k_i}\tilde{G}^{-1}\tilde{G}\partial_{k_j}\tilde{G}^{-1}\tilde{G}\partial_{k_l}\tilde{G}^{-1}, \tag{23}$$

where $k_i = (t,k_x,k_y)$ and

$$\tilde{G}^{-1}(t,k_x,k_y|\alpha) = t(\tau_2\sin\alpha - \tau_3\sigma_z\cos\alpha) + \tau_1\epsilon(\mathbf{k}) - \tau_3[\sigma_x f_x(\mathbf{k}) + \sigma_y f_y(\mathbf{k})], \tag{24}$$

where $\mathbf{k} = (k_x,k_y,f^{-1}t\cos\alpha)$. When the parameter α changes from 0 to $\pi/2$, the topological charge (23) transforms from Eq. (17) for the $(3+1)$-dimensional ^{3}He-B to Eq. (19) for the $(2+1)$-dimensional planar phase. Naturally, along this path, the topological invariant is $N_\alpha = 2n$, including the cases $\alpha = 0,\pi/2$, so that $N_{\rm B} = N_{\rm P}$.

The constructed connection between topological properties of ^{3}He-B and the planar phase can be generalized to include all Hamiltonians within the DIII symmetry class, as we show in the next section.

C. Bulk-boundary correspondence for general DIII topological superconductors

In Sec. III A, we have discussed the index theorem for a subclass of DIII topological superconductors in 3D described by Hamiltonians, similar to that of ^{3}He-B. In particular, we assumed that the Hamiltonian at the $k_z = 0$ cross section of momentum space is equivalent to that of the planar phase, which has a topological invariant protected by an additional symmetry. However, this is not the case for the general Hamiltonian of class DIII. The reduction to $k_z = 0$ shown in Fig. 2(b) and, in general, to any time-reversal invariant plane (13) produces, in this case, a DIII topological superconductor in 2D. As discussed in Sec. II B, the only symmetries that exist in general for the 2D case are the TRS and PHS, which allow only the $\mathbb{Z}_2$ classification given by $\nu = (N_{\rm B}/2) \mod 2$.

The proof can be constructed as follows. First, we note that the set of Hamiltonians given by Eqs. (9) and (12) with $n \in \mathbb{Z}$ contains representatives from all topological classes of DIII symmetric Hamiltonians. Thus we can continuously transform any given DIII Hamiltonian to one of this set, preserving the value of $N_{\rm B}$. This generates a deformation of the 2D Hamiltonian in the $k_z = 0$ cross section to that of the generalized planar phase, given by Eqs. (14) and (12). This deformation does not change the value of the $\mathbb{Z}_2$ invariant, which coincides with that of the generalized planar state: $\nu = (N_{\rm B}/2) \mod 2 = (N_{\rm P}/2) \mod 2$ in accordance with the dimensional reduction arguments of Sec. III B.

Finally, we note that the choice of the cross section $k_z = 0$ is arbitrary. Instead, we can choose any time-reversal invariant plane of the form (13), which allows one to investigate the properties of bound states at the $y = 0$ interface between a topological superconductor of class DIII and a nontopological insulator. In this case, all of the 2D Hamiltonians describing quasiparticles in each cross section (13) have the same $\mathbb{Z}_2$ invariant, $\nu = (N_{\rm B}/2) \mod 2$. According to the bulk-boundary correspondence in 2D time-reversal invariant topological insulators, a nonzero value of the $\mathbb{Z}_2$ invariant protects topologically stable Kramers pairs of zero-energy surface states [28]. Therefore, we conclude that the bulk-boundary correspondence in DIII topological superconductors can be formulated as follows: Provided the value of the bulk 3D invariant $N_{\rm B}/2$ is odd, the spectrum $\varepsilon_b = \varepsilon_b(k_x,k_z)$ of surface states at the boundary plane $y = 0$ with a nontopological insulator has at least one Kramers pair of topologically stable zero modes along each line (13).

IV. CONCLUSION

It is known that there exists a dimensional reduction, which connects the classification of fully gapped topological materials and the classification of nodal systems with topologically protected zeros in the energy spectrum (Fermi surfaces) described by Hořava using the K-theory [29]. An example of this connection is provided by the relation between the topological invariant (19), which describes the fully gapped $2+1$ planar phase, and the topological invariant, that protects the point nodes in the gapless $3+1$ planar phase [27]. The invariants are given by the same integral, but instead of integration over the whole (ω,k_x,k_y) space in Eq. (19), the integral is taken over the sphere S^3 around the node in the (ω,k_x,k_y,k_y) space.

Here we explicitly demonstrated a dimensional reduction, which connects the fully gapped time-reversal invariant topological materials of different classes: the $2+1$ planar phase of superfluid ^{3}He and the $3+1$ superfluid ^{3}He-B. As a result, two $3+1$ topological systems become connected: the $3+1$ gapless planar phase and the gapful ^{3}He-B. It is noteworthy that the planar phase of ^{3}He is topologically equivalent to the vacuum of the Standard Model of particle physics in its massless phase of topological semimetal [27,30,31], while the superfluid ^{3}He-B is topologically equivalent to the Standard Model vacuum in its massive phase of topological insulator [15,32]. That is why the discussed connection between the $3+1$ topological states can be useful for investigation of the topology of the Standard Model, which is also supported by symmetry. The phenomenon discussed here, when the discrete symmetry of the system leads to the continuous symmetry of the single-particle Green function and correspondingly to an integer topological invariant, is applicable to the vacuum of the Standard Model.

ACKNOWLEDGMENTS

Y.M. is grateful to J. E. Moore and A. Kitaev for valuable discussions. M.S. and G.E.V. acknowledge financial support by the Academy of Finland through its LTQ CoE grant (Project No. 250280). M.S. was supported by Russian Foundation for Basic Research and Russian President scholarship (SP-6811.2013.5).

[1] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008); AIP Conf. Proc. **1134**, 10 (2009).
[2] A. Kitaev, AIP Conf. Proc. **1134**, 22 (2009).
[3] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[4] D. Vollhardt and P. Woelfe, *The Superfluid Phases of Helium 3* (Taylor & Francis, London, 1990).
[5] L. Levitin, R. Bennett, A. Casey, B. Cowan, J. Parpia, and L. Saunders, J. Low Temp. Phys. **158**, 163 (2010).
[6] L. V. Levitin, R. G. Bennett, A. Casey, B. Cowan, J. Parpia, and L. Saunders, J. Low Temp. Phys. **158**, 159 (2010).
[7] L. Levitin, R. Bennett, A. Casey, B. Cowan, L. Saunders, D. Drung, T. Schurig, and J. Parpia, Science **340**, 841 (2013).
[8] L. V. Levitin, R. G. Bennett, E. V. Surovtsev, J. M. Parpia, B. Cowan, A. J. Casey, and J. Saunders, Phys. Rev. Lett. **111**, 235304 (2013)
[9] L. V. Levitin, R. G. Bennett, A. Casey, B. Cowan, J. Saunders, D. Drung, Th. Schurig, J. M. Parpia, B. Ilic, and N. Zhelev, J. Low Temp. Phys. (2014), doi:10.1007/s10909-014-1145-1.
[10] P. Heinzner, A. Huckleberry, and M. Zirnbauer, Comm. Math. Phys. **257**, 725 (2005).
[11] J. E. Moore and L. Balents, Phys. Rev. B **75**, 121306(R) (2007).
[12] G. Volovik and V. Yakovenko, J. Phys.: Condens. Matter **1**, 5263 (1989).
[13] J. C. Y. Teo and T. L. Hughes, Phys. Rev. Lett. **111**, 047006 (2013).
[14] Y. Ueno, A. Yamakage, Y. Tanaka, and M. Sato, Phys. Rev. Lett. **111**, 087002 (2013).
[15] G. E. Volovik, Pis'ma ZhETF **91**, 61 (2010) [JETP Lett. **91**, 55 (2010)].
[16] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007).
[17] R. M. Switzer, *Algebraic Topology—Homotopy and Homology* (Springer, Berlin, 1975).
[18] A. T. Fomenko, D. B. Fuchs, and V. L. Gutenmacher, *Homotopic Topology* (Ak. Kiado, Budapest, 1986).
[19] G. E. Volovik, Pis'ma ZhETF **91**, 215 (2010) [JETP Lett. **91**, 201 (2010)].
[20] R. S. K. Mong, A. M. Essin, and J. E. Moore, Phys. Rev. B **81**, 245209 (2010).
[21] L. Fu, Phys. Rev. Lett. **106**, 106802 (2011).
[22] T. H. Hsieh and L. Fu, Phys. Rev. Lett. **108**, 107005 (2012).
[23] R.-J. Slager, A. Mesaros, V. Juricic, and J. Zaanen, Nat. Phys. **9**, 98 (2013).
[24] T. Morimoto and A. Furusaki, Phys. Rev. B **88**, 125129 (2013).
[25] R. Roy, arXiv:0803.2868.
[26] Q. Niu, D. J. Thouless, and Yong-Shi Wu, Phys. Rev. B **31**, 3372 (1985); D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[27] G. E. Volovik, *The Universe in a Helium Droplet* (Clarendon, Oxford, 2003).
[28] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005); M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[29] P. Hořava, Phys. Rev. Lett. **95**, 016405 (2005).
[30] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B **185**, 20 (1981); **193**, 173 (1981); Phys. Lett. B **130**, 389 (1983).
[31] M. Creutz, J. High Energy Phys. 04 (2008) 017.
[32] D. B. Kaplan and Sichun Sun, Phys. Rev. Lett. **108**, 181807 (2012).
[33] More formally, each mapping U : BZ$\rightarrow$U(2) can be lifted to the covering $S^1 \times$ SU(2) of U(2) (*lifting theorem*). Of course, we have to choose the image of one point, say, $k=0$, between two possibilities, but after that the covering mapping is uniquely defined.
[34] In other words, in going from the planar phase to the general case, we lift the symmetry constraint, and all even mappings become mutually equivalent, as do the odd mappings.
[35] Of course, it does not matter for topological classification whether the axis is y or z. Here we have in mind the z axis for both spheres, and refer to its ends as the north and south poles.