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# Propagating $q$-field and $q$-ball solution

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One possible solution of the cosmological constant problem involves a so-called $q$-field, which self-adjusts so as to give a vanishing gravitating vacuum energy density (cosmological constant) in equilibrium. We show that this $q$-field can manifest itself in other ways. Specifically, we establish a propagating mode ($q$-wave) in the nontrivial vacuum and find a particular soliton-type solution in flat spacetime, which we call a $q$-ball by analogy with the well-known $Q$-ball solution. Both $q$-waves and $q$-balls are expected to play a role for the equilibration of the $q$-field in the very early universe.

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## 1. Introduction

A novel approach to the cosmological constant problem, guided by thermodynamics and Lorentz invariance, is provided by the $q$-theory of the quantum vacuum.[1] The variable $q$ describes the phenomenology of the quantum vacuum. Various realizations of $q$ have been discussed in Refs. 1–7. Here, we only consider the four-form-field-strength realization based on a three-form gauge field.[8–15]

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In Refs. 2, 3 and 6, we have considered $q$-theory applied to the behavior of the quantum vacuum in an expanding universe. Now, we concentrate on the properties of the $q$-field itself and address the following two questions: does the $q$-field have a propagating degree of freedom in the nontrivial vacuum and are there soliton-type solutions for the $q$-field? This paper gives an affirmative answer to both questions. Throughout, we use natural units with $c = \hbar = 1$ and take the metric signature $(-+++)$.

## 2. Kinetic Term

For the $q$-theory approach to the cosmological constant problem, it suffices to consider only a potential term of the (pseudo-)scalar composite field $q$ in the low-energy effective action. But, nothing excludes having a kinetic term of the composite $q$ in the effective action. In fact, the $q$-field effective action with kinetic term can be chosen as follows:

$$S = -\int_{\mathbb{R}^4} d^4x \sqrt{-g} \left[ \frac{R}{16\pi G(q)} + \epsilon(q) + \frac{1}{8} K(q) g^{\alpha\beta} \nabla_\alpha(q^2) \nabla_\beta(q^2) \right], \quad (2.1a)$$

$$q^2 \equiv -\frac{1}{24} F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta}, \qquad F_{\alpha\beta\gamma\delta} \equiv \nabla_{[\alpha} A_{\beta\gamma\delta]}, \quad (2.1b)$$

$$F_{\alpha\beta\gamma\delta} = q\sqrt{-g}\,\epsilon_{\alpha\beta\gamma\delta}, \qquad F^{\alpha\beta\gamma\delta} = q\epsilon^{\alpha\beta\gamma\delta}/\sqrt{-g}, \quad (2.1c)$$

where the functions $G(q)$, $\epsilon(q)$, and $K(q)$ in (2.1a) involve only even powers of $q$, as $q$ is a pseudoscalar according to (2.1c) with the Levi-Civita symbol $\epsilon_{\alpha\beta\gamma\delta}$. In this realization, $q$ has mass dimension 2. A further term for the action (2.1a) will be presented in Sec. 5.1.

Variation of the action (2.1a) over the three-form gauge field $A$ gives a generalized Maxwell equation, which has been derived in Ref. 2 for the theory without direct $q$-kinetic term ($K = 0$). The presence of the $K$ term in the action (2.1a) does not change the general form of the field equation,

$$\nabla_\beta \left( \frac{\delta S}{\delta q} \right) = 0, \quad (2.2)$$

which gives for the case at hand

$$\nabla_\beta \left( \frac{d\epsilon(q)}{dq} + \frac{R}{16\pi} \frac{dG^{-1}(q)}{dq} + \frac{1}{8} \frac{dK(q)}{dq} \nabla_\alpha(q^2) \nabla^\alpha(q^2) - \frac{1}{2} q \nabla^\alpha [K(q)\nabla_\alpha(q^2)] \right)$$

$$= 0. \quad (2.3)$$

The $K$ term of (2.1a) also gives rise to an extra contribution in the generalized Einstein equation, which need not concern us for the moment.

The generalized Maxwell equation (2.3) has two constant solutions in flat Minkowski spacetime:

(1) the trivial solution $q = 0$, which corresponds to the "absolutely empty vacuum";

(2) the nontrivial solution $q = q_0 \neq 0$ (with an integration constant $\mu = \mu_0$), which corresponds to the "physical vacuum."

The last type of constant solution can be explained as follows. For the homogeneous vacuum, we have $d\epsilon(q)/dq = \mu$, which determines $q$ for a fixed integration constant $\mu$. Hence, $\mu$ plays the role of a chemical potential, which is thermodynamically conjugate to the density $q$ of the conserved total charge $Q$. This emergence of a conservation law makes the four-form field $F$ one of the possible realizations of the vacuum field $q$. In the Minkowski vacuum, the equilibrium value $\mu = \mu_0$ and the corresponding $q$-field value $q_0 \neq 0$ naturally give a zero value for the gravitating vacuum energy density (effective cosmological constant).[1,2]

The solution $q = 0$, on the other hand, may be considered as the "absolutely empty vacuum," that is, a vacuum which is devoid of any type of quantum field and of the corresponding quantum fluctuations. In the condensed matter analogy, where the role of the variable $q$ is played by the number density $n$ of atoms, this vacuum corresponds to the state with $n = 0$, i.e. to empty space devoid of atoms and, thus, of any type of emergent quantum field.

The aim of this paper is to look for nonconstant solutions of the $q$-field.

## 3. Propagating Mode

We now make the linear expansion

$$q(x) = q_0[1 - \varphi(x)] \tag{3.1}$$

and consider the integrated Maxwell equation (2.3) in the Minkowski spacetime background with integration constant $\mu$ fixed to the equilibrium value $\mu_0$. The effect of $\varphi$ on the metric $g_{\mu\nu}$ in the generalized Einstein equation is quadratic in $\varphi$ and can be neglected in linear approximation. As a result, we obtain the linear equation for the mode $\varphi$ of the $q$-field from the four-form field $F$ propagating in the Minkowski background. This equation corresponds to the Klein–Gordon equation of a massive scalar field.

The expression for the mass-square of the $\varphi$-mode can also be obtained from (2.1a) by expanding in $\varphi$ and considering the quadratic terms,

$$S = -\int_{\mathbb{R}^4} d^4x \left( \frac{1}{2\chi_0}\varphi^2 + \frac{1}{2}q_0^4 K(q_0)\nabla_\alpha\varphi\nabla^\alpha\varphi + \cdots \right), \tag{3.2}$$

where $\chi_0$ is the compressibility of the vacuum introduced in Ref. 1,

$$(\chi_0)^{-1} \equiv \left[ q^2 \frac{d^2\epsilon(q)}{dq^2} \right]_{q=q_0} > 0 \,. \tag{3.3}$$

From (3.2), the mass-square of the $\varphi$-mode is then given by

$$M^2 = \frac{1}{\chi_0 K(q_0)q_0^4} \,. \tag{3.4}$$

In the absence of the kinetic term, that is for $K = 0$, the mass $M$ becomes infinite and the $q$-wave does not propagate. Also, note that the equivalence between the $q$-field from the four-form field $F$ and the scalar field $\varphi$ exists only at the perturbative level. In general, this $q$-field is a composite pseudoscalar field with different properties compared to those of a fundamental pseudoscalar field (see e.g. Sec. 2 of Ref. 7).

If the $q$-field is of the Planck scale [$q_0 \sim (E_P)^2 \equiv 1/G_N$, with Newton's gravitational constant $G_N$], then the mass $M$ is also of the Planck scale, $M^2 \sim G_N/\chi_0 \sim 1/G_N = (E_P)^2$. In the absolutely empty vacuum with $q = 0$ replacing $q_0 \neq 0$, the vacuum $q$-field from the four-form field $F$ is not propagating according to (3.4), which corresponds to the standard result for the three-form gauge field $A$.[8,9]

It appears that there is an important difference between the ground state of condensed matter and the relativistic vacuum. In condensed matter physics, the compressibility of the ground state determines the speed of sound in the medium (cf. Chap. VIII in Ref. 16). In the relativistic vacuum, this is impossible due to the Lorentz invariance of the vacuum, which selects the Lorentz-invariant form for the spectrum of propagating modes, $E^2 = M^2 + c^2 p^2$. As $c$ is fixed by the Lorentz invariance, only the vacuum compressibility may enter the mass $M$ of the vacuum mode, just as we have obtained in (3.4) for the particular case of the four-form realization of the vacuum $q$-field. The same should be valid in the general case: for any realization of the vacuum field $q$, the mass of the propagating mode is determined by the vacuum compressibility.

At this moment, we have not much to say about the generation of these $q$-waves. For the action (2.1a) with $q$-independent Standard Model fields added,[6] the emission of $q$-waves requires strong gravitational fields, which may occur in the very early universe.

## 4. Heuristics of the $q$-Ball

The $q$-ball solution of $q$-theory is a ball with radius $R$ having $q \neq 0$ inside ($r < R$) and the empty vacuum value $q = 0$ outside ($r \geq R$).

Generally speaking, $Q$-balls are non-topological solitons which carry a conserved global quantum number $Q$ (see Refs. 17–22 and references therein). The dynamics of the $q$-ball of $q$-theory is, however, rather different from that of the standard $Q$-ball in, for example, the SO(2)-invariant theory of two real scalar fields $\phi_1$ and $\phi_2$ with non-derivative interactions.[20] For this reason, we employ a somewhat different notation, "$q$-ball" instead of "$Q$-ball".

Incidentally, we could also consider an extended $q$-theory with two three-form gauge fields ($A_1$ and $A_2$) and an $SO(2)$-invariant action in terms of the corresponding composite pseudoscalars $q_1$ and $q_2$ replacing the fundamental scalars $\phi_1$ and $\phi_2$. In such an extended $q$-theory, there would be a direct analog of the standard $Q$-ball solution.[20] But, here, we keep the simplest possible theory with a single three-form gauge field $A$ and a single corresponding composite pseudoscalar field $q$.

The dynamics of the $q$-ball solution is essentially different from Coleman's $Q$-ball,[20] as we do not need the time-dependence of the scalar fields. The $q$-ball is similar to a droplet of quantum liquid, because our $q$ (see, in particular, Refs. 1, 3 and 7) is the analog of the number density $n$ of the conserved quantity $N$ (particle number). Such a liquid droplet with fixed $N$ has a fixed radius $R$ and the $q$-ball is similar.

The $q$-ball is expected to be stable for at least three reasons, two of which will be discussed now and the third of which will be presented at the end of this section. First, there are no propagating modes at all outside the $q$-ball, since the mass of the mode is infinite for $q = 0$, as discussed in Sec. 3. Second, the splitting of a $q$-ball into two $q$-balls would lead to an increase of the surface energy (for a fixed value of the total charge $Q \equiv qV$) and the splitting is, therefore, excluded.

Expanding on the role of the surface, we note that the vacuum pressure inside the $q$-ball is determined by the surface tension $\sigma$ and the radius $R$ of the $q$-ball. In the absence of gravity, we have (cf. Sec. IV D of Ref. 1)

$$P_\sigma = -\frac{2\sigma}{R}\,, \tag{4.1}$$

which corresponds to the effective equation of state[1,23]

$$P_\sigma = w_\sigma \epsilon_\sigma\,, \qquad w_\sigma = -\frac{2}{3}\,. \tag{4.2}$$

The surface tension $\sigma$ has an order of magnitude $\epsilon(q_0)/M$. For a Planck scale $q$-field, we have the pressure $P_\sigma \sim -(E_P)^3/R$. The magnitude of $P_\sigma$ is of the order of present cosmological constant if $R \sim E_P/H^2$, which is far beyond the horizon of order $1/H$.

The pressure (4.1) from the surface tension is compensated by the bulk vacuum pressure: $q$ inside the $q$-ball is slightly different from the perfect-vacuum value $q_0$ (see Sec. 6 for an explicit numerical result). This is exactly the same as for a droplet of liquid, where $n$ slightly deviates from $n_0$ inside the droplet.

Let us, finally, present the third argument for the stability of the $q$-ball solution (details of the solution are to follow in Secs. 5 and 6), where the argument parallels Coleman's discussion of the standard $Q$-ball solution.[20] The $q$-ball can, for large radius $R$, be considered as a structure with $q \approx q_0$ inside ($r < R$) and $q = 0$ outside ($r \geq R$). In the terminology of $Q$-balls, this $q$-ball is similar to the thin-wall $Q$-ball.[20]

Take, as a simple example, the following *Ansatz* for $\epsilon(q)$:

$$\epsilon(q) = \frac{1}{6}q^4 - \frac{1}{2}q_0^2 q^2\,, \tag{4.3}$$

where $q_0$ is again the equilibrium value of the $q$ field, which is now normalized to have mass dimension 1. The energy of the $q$-ball with volume $V$ and total charge $Q \equiv qV$ is then given by

$$E(V, Q) = V\left(\frac{1}{6}\frac{Q^4}{V^4} - \frac{1}{2}\frac{q_0^2 Q^2}{V^2}\right). \tag{4.4}$$

Minimization of this energy with respect to the volume $V$ at fixed total charge $Q$ gives

$$\frac{dE}{dV}\bigg|_Q = \epsilon(q) - q\frac{d\epsilon}{dq} = \frac{1}{2}(q_0^2 q^2 - q^4) = 0\,. \tag{4.5}$$

For a nontrivial $q$-ball, we thus find the equilibrium value $q = \pm q_0$ for the interior region. The corresponding equilibrium volume of the $q$-ball equals

$$V = Q/|q_0|\,. \tag{4.6}$$

With the charge $Q$ being conserved[2,9,12] (see also Sec. 5.1) and the quantity $q_0$ being a "constant of nature," the volume $V$ of the $q$-ball is fixed, according to (4.6). With constant volume $V$ of the spherically symmetric $q$-ball, the edge of the $q$-ball cannot move inwards or outwards, so that the spherically symmetric $q$-ball solution is stable.

The pressure $P_V$ of the vacuum is zero in the thin-wall approximation, according to the Gibbs–Duhem relation $\epsilon - q\,d\epsilon/dq = -P_V$ and the result (4.5). A nonzero value for the pressure comes from the surface term (4.1). To summarize, the $q$-ball is stable due to the conservation of the total charge $Q$, while the surface term is only responsible for the pressure inside the $q$-ball.

## 5. Analytical Results

The profile of the thin wall separating the true vacuum and the absolutely empty vacuum can be found analytically in the limit of large $R$, where the wall can be considered as a flat plane and the profile depends on the spatial coordinate $x$ across the wall with $q = 0$ at $-\infty < x \le 0$ and $q^2 > 0$ at $0 < x < \infty$. The thickness of the transition region is $d \sim M^{-1}$, with $M^2$ given by (3.4).

### 5.1. *Surface term*

In order to find this $q(x)$ profile, we need to establish the relevant equation. Consider the function $q(x)$ at $x > 0$ with the boundary conditions $q(0) = 0$ and $q(\infty) = q_0$. For $x > 0$ in flat spacetime, the generalized Maxwell equation (2.3) reads

$$\partial_x\left(\frac{d\epsilon(q)}{dq} + \frac{1}{8}\frac{dK(q)}{dq}[\partial_x(q^2)]^2 - \frac{1}{2}q\partial_x[K(q)\partial_x(q^2)]\right) = 0 \tag{5.1}$$

and has the solution

$$\frac{d\epsilon(q)}{dq} + \frac{1}{8}\frac{dK(q)}{dq}[\partial_x(q^2)]^2 - \frac{1}{2}q\partial_x[K(q)\partial_x(q^2)] = \mu \tag{5.2}$$

with integration constant (chemical potential) $\mu$. In the limit of infinite radius $R$, the chemical potential $\mu$ is the same as for the infinite-volume equilibrium vacuum and we have $\mu = \mu_0$ in (5.2) for $x > 0$. For $x < 0$, we have $q = 0$ and thus $\mu = 0$

in (5.2). In short, the $q$-ball dynamics follows from a single ordinary differential equation (ODE), given by (5.2) with

$$\mu = \begin{cases} 0\,, & \text{for } x \leq 0\,, \\ \mu_0\,, & \text{for } x > 0\,. \end{cases} \tag{5.3}$$

This jump in the chemical potential at $x = 0$ (from $\mu = 0$ for negative $x$ values to $\mu = \mu_0$ for positive $x$ values) corresponds to a delta-function source term $\mu_0\,\delta(x)$ on the right-hand side of (5.1). Such a source term may be obtained by the introduction of a surface contribution in the action (2.1a),

$$S_{\text{surface}-\mathbb{S}} = -\mu_{\mathbb{S}} \int_{\mathbb{S}} dS^{\alpha\beta\gamma} A_{\alpha\beta\gamma}\,, \tag{5.4}$$

for a boundaryless three-dimensional surface $\mathbb{S}$ and a constant $\mu_{\mathbb{S}}$. This surface term describes the conservation of the flux of the four-form $F$-field. Since

$$\int_{\mathbb{S}=\partial\mathbb{V}} dS^{\alpha\beta\gamma} A_{\alpha\beta\gamma} = \int_{\mathbb{V}} dS^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta}\,, \tag{5.5}$$

we obtain, for the time-independent $q$-field of the infinite-radius $q$-ball and with $\mu_{\mathbb{S}} = \mu_0$, a term corresponding to the conservation of the total charge $Q$ of the $q$-ball:

$$S_{\text{surface}-\mathbb{S}}^{(q\text{-ball},R=\infty)} = -\mu_0 \int dt \int d^3x\, q = -\mu_0 \int dt\, Q\,. \tag{5.6}$$

For the flat $x = 0$ surface between empty and equilibrium vacua, the variation of (5.4) with $\mu_{\mathbb{S}} = \mu_0$ over the three-form gauge field $A$ gives the a term $\mu_0\,\delta(x)$ on the right-hand side of (5.1), so that we get for the infinite-radius $q$-ball

$$\partial_x \left( \frac{d\epsilon(q)}{dq} + \frac{1}{8}\frac{dK(q)}{dq}[\partial_x(q^2)]^2 - \frac{1}{2}q\partial_x[K(q)\partial_x(q^2)] \right) = \mu_0\,\delta(x) \tag{5.7}$$

with the empty-vacuum boundary condition $\lim_{x\to-\infty} q(x) = 0$.

The previous discussion was specialized to the case of the infinite-radius $q$-ball, even though the surface term (5.4) was given in its most general form. The same discussion holds for the finite-radius $q$-ball with the coordinate $x$ replaced by the radial coordinate $r$ and the constant $\mu_{\mathbb{S}}$ taking the appropriate value $\mu_c$ in order to cancel the surface pressure (4.1); see Sec. 6 for details.

To summarize, the surface term (5.4) gives precisely the delta-function source term in the generalized Maxwell equation. It properly reflects the conservation law for the charge $Q \equiv \int_V d^3x\, q(x)$ of the $q$-ball solution surrounded by the absolutely empty vacuum as discussed in Sec. 2.

### 5.2. *Profile of the infinite-radius q-ball*

Let us now determine the detailed behavior of the $q$-field on the true-vacuum side at $x > 0$. For the theory (2.1), the function $q(x)$ is the extremum of the following

thermodynamic potential which can be interpreted as the surface tension of the
$q$-ball:

$$\sigma^{(R\to\infty)} = \int_0^\infty dx\left(\epsilon(q) - \mu_0 q + \frac{1}{8}K(q)\partial_x(q^2)\partial_x(q^2)\right).$$ (5.8)

In the limit of infinite-radius $R$, the chemical potential on the positive $x$ side is the
same as for the infinite-volume equilibrium vacuum, which is why we use $\mu = \mu_0$
in (5.8). In Sec. 6, we will see that the finite-radius $q$-ball has $\mu \neq \mu_0$.

The solution of the second-order equation for $q$ (found by variation of (5.8)) can
be obtained from the first integral, which gives

$$\epsilon(q) - \mu_0 q - \frac{1}{8}K(q)\partial_x(q^2)\partial_x(q^2) = \text{const.} = 0\,.$$ (5.9)

From the above equation, we get

$$\int_0^x dx' = \int_0^{q^2 < q_0^2} d(q'^2)\sqrt{\frac{K(q')}{8[\epsilon(q') - \mu_0 q']}}\,.$$ (5.10)

For $x \to \infty$, $q^2$ approaches $q_0^2$ from below. For given functions of $\epsilon(q)$ and $K(q)$,
we can always find the solution of (5.10) numerically. Still, keeping $\epsilon(q)$ and $K(q)$
generic, the asymptotic behavior for $x \to \infty$ and $x \to 0$ can be found analytically.

The integral over $q^2$ is concentrated near $q_0^2$, where it diverges logarithmically.
In terms of the variable $\varphi$ from (3.1), we obtain the following asymptotic behavior
as $x \to \infty$:

$$x \to \int_\varphi^1 \frac{d\varphi'}{M\varphi'} + \text{const.} = -\frac{1}{M}\ln\varphi + \text{const.}$$ (5.11)

or

$$\varphi(x) \propto e^{-Mx} \quad \text{for} \quad x \to \infty\,.$$ (5.12)

The asymptote at $x \to 0$ is obtained from expansion near $q = 0$. If $K(0)$ is
positive and finite, we have

$$x = \int_0^q q'\,dq'\sqrt{\frac{K(0)}{2q'|\mu_0|}} = \frac{2}{3}\sqrt{\frac{K(0)}{2|\mu_0|}}q^{3/2}\,,$$ (5.13)

which gives

$$q(x) \propto x^{2/3} \quad \text{for} \quad x \to 0\,.$$ (5.14)

If $K(q) \propto q^\beta$ as $q \to 0$ with $\beta > 0$, we have $q(x) \propto x^{2/(3+\beta)}$ as $x \to 0$.

It is also possible to obtain the complete $q$-ball profile analytically for certain
special *Ansätze* of the functions $K(q)$ and $\epsilon(q)$. Let us give an example. We change
to dimensionless variables ($x$ giving $s$, $q$ giving $f$, and $\mu_0$ giving $u_0$) and make the
following *Ansätze* for the dimensionless versions of the coupling $K(q)$ of the $q$ kinetic
term, the gravitational coupling $G(q)$, and the energy density $\epsilon(q)$ [cf. Eq. (4.3)]:

$$k(f) = 1\,,$$ (5.15a)

$$g(f) = 0\,, \tag{5.15b}$$

$$\epsilon(f) = \frac{1}{2}f^2\left(\frac{1}{3}f^2 - 1\right), \tag{5.15c}$$

which give the following dimensionless constants:

$$f_0 = 1\,, \tag{5.15d}$$

$$u_0 = -\frac{1}{3}\,, \tag{5.15e}$$

$$m^2 = 1 \tag{5.15f}$$

with $m^2$ corresponding to the dimensionless version of (3.4). Note that precisely this *Ansätz* for $\epsilon(f)$ has been used in Ref. 2 and that we set the dimensionless gravitational coupling $g$ to zero, so that Minkowski spacetime can be used consistently. For later use, we also give the dimensionless version of the integrand of (5.8) for the *Ansätze* (5.15),

$$r_\sigma[f] = \epsilon(f) - u_0 f + \frac{1}{2}f^2(\partial_s f)^2 \tag{5.16}$$

with $\epsilon(f)$ from (5.15c) and $u_0$ from (5.15e).

The dimensionless version of (5.10) for the *Ansätze* (5.15) then has the following analytic solution:

$$s = \frac{2}{\sqrt{3}}(1-f)\sqrt{\frac{2+f}{2-3f+f^3}}$$
$$\times\left(\sqrt{3}\,\mathrm{arctanh}\left(\sqrt{\frac{3f}{2+f}}\right) - 3\,\mathrm{arcsinh}\left(\sqrt{\frac{f}{2}}\right)\right). \tag{5.17}$$

The asymptotic behavior of the solution (5.17) agrees with the previous general results (5.12) and (5.14). Equation (5.17) defines the function $s(f)$ and a plot of the inverse function $f(s)$ is shown in Fig. 1 with reflected $s$ values for later comparison. Figure 1 also shows the energy density (5.16).
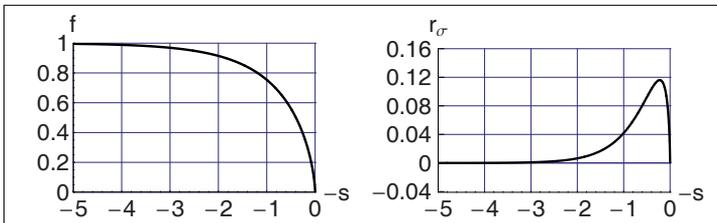


Fig. 1.    The left panel gives the analytic infinite-radius q-ball profile $f(s)$ from (5.17) with $f(0) = 0$ and $f \uparrow f_0 = 1$ for $-s \to -\infty$. Here, $f$ refers to the dimensionless q-field and $s$ to the dimensionless spatial coordinate orthogonal to the edge of the infinite-radius q-ball. The right panel gives the corresponding energy density (5.16).

## 6. Numerical Results

For the finite-radius $q$-ball in Minkowski spacetime, the generalized Maxwell equation (2.3) for $K(q) = K = \text{const.}$ reads in spherical coordinates

$$\partial_r \left( \frac{d\epsilon(q)}{dq} - \frac{1}{2} K q \frac{1}{r} \partial_r \partial_r [r q^2] \right) = 0 \tag{6.1}$$

with the following boundary conditions on $q(r)$:

$$q(0) = q_c > 0 \,, \tag{6.2a}$$

$$q'(0) = 0 \,, \tag{6.2b}$$

$$q(R) = 0 \,. \tag{6.2c}$$

Note that the two boundary conditions (6.2a) and (6.2c) exclude having a constant solution.

As discussed in Sec. 5.1, the $q$-ball requires that the right-hand side of (6.1) be replaced by a delta-function. In order to obtain the numerical solution $q(r)$, we proceed as follows. We take the *Ansätze* (5.15) and use the same dimensionless variables as in Sec. 5.2 together with the dimensionless radial coordinate $v \in [0, \infty)$. Next, we replace the third-order ODE (6.1) by two second-order ODEs, one for the inside region and one for the outside region,

$$\frac{d\epsilon(f)}{df} - \frac{1}{2} f \frac{1}{v} \partial_v \partial_v [v f^2] = \begin{cases} u_c \,, & \text{for} \quad 0 \leq v < v_R \,, \\ 0 \,, & \text{for} \quad v \geq v_R \end{cases} \tag{6.3}$$

with $v_R > 0$ and $\epsilon(f)$ from (5.15c). The following boundary conditions on $f(v)$ hold for the inside region:

$$f(0) = f_c > 0 \,, \qquad \lim_{v \uparrow v_R} f(v) = 0 \,, \tag{6.4a}$$

and for the outside region:

$$f(v_R) = 0 \,, \qquad f'(v_R) = 0 \,. \tag{6.4b}$$

For later reference, we also give the dimensionless interior energy density functional,

$$r_c[f] \equiv \epsilon(f) - u_c f + \frac{1}{2} f^2 (\partial_v f)^2 \tag{6.5}$$

with $\epsilon(f)$ from (5.15c) and $u_c$ from (6.3).

The ODE (6.3) with boundary conditions (6.4) gives the $q$-ball solution with a sharp drop of the integration constant $u$ at the $q$-ball edge and the corresponding non-smoothness of the function $f(v)$. In line with the discussion of Sec. 5.1, we observe that the solution of (6.3) solves (6.1) for $r \neq R$ (or $v \neq v_R$), but *not* at $r = R$ (corresponding to the sphere $S_R$). The proper solution is determined by the surface term (5.4), which describes the interface between the physical vacuum and the absolutely empty vacuum.
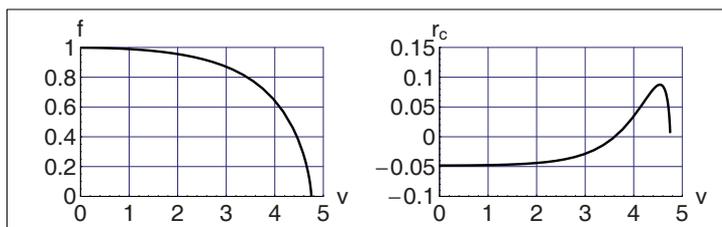
Fig. 2. The left panel gives the numeric finite-radius q-ball profile $f(v)$ from the second-order ODE (6.3) with parameters and boundary conditions (6.7). Here, $f$ refers to the dimensionless q-field and $v$ to the dimensionless radial coordinate of the q-ball. The right panel gives the corresponding energy density (6.5).

The ODE (6.3) for the outside region with boundary conditions (6.4b) gives immediately the solution

$$f(v) = 0, \quad \text{for} \quad v \geq v_R. \tag{6.6}$$

But, the ODE for the inside region is singular and requires some care. We solve the ODE (6.3) over the $v$ interval $\{v_{\text{start}}, v_{\text{end}}\}$ with $0 < v_{\text{start}} < v_{\text{end}}$ and take boundary conditions $\{f(v_{\text{start}}), f'(v_{\text{start}})\} = \{f_c, 0\}$ with $f_c > 0$. By choosing appropriate values for $f_c$ and $u_c$, we obtain a solution with $f(v_{\text{end}}) = 0$, so that we can identify $v_{\text{end}}$ as the dimensionless parameter $v_R$ corresponding to the radius $R$ of the q-ball.

Specifically, we take the values

$$\{v_{\text{start}}, v_{\text{end}}\} = \{10^{-2}, 4.74\}, \tag{6.7a}$$

$$\{f(v_{\text{start}}), f'(v_{\text{start}}), u_c\} = \{0.998, 0, -0.285\}. \tag{6.7b}$$

Figure 2 shows the resulting numerical solution $f(v)$ for $v < v_R \approx 4.74$ (similar results have been obtained with $v_{\text{start}} = 10^{-3}$, the other parameters and boundary conditions being kept the same). As mentioned above, we have $f(v) = 0$ for $v > v_R$. The energy density $r_c$ in Fig. 2 is seen to remain finite for $v < v_R$ and to approach (within numerical errors) a zero value for $v \uparrow v_R$ [the exact solution (6.6) has $r[f] \equiv \epsilon(f) + \frac{1}{2}f^2(\partial_v f)^2 = 0$ for $v \geq v_R$]. Note that, especially near the edge at $f = 0$, the $f$-profile of the finite-radius q-ball in Fig. 2 is close to that of the infinite-radius q-ball in Fig. 1.

## 7. Outlook

We have used, in this paper, a q-field realized by a four-form field strength $F$ as a phenomenological description of the deep vacuum. But, there are also other types of q-fields, provided they obey the condition necessary to represent a stable quantum vacuum. This necessary condition is the conservation law for the vacuum variable $q$, which is represented by (2.3) here. Note that a fundamental scalar field does not obey this condition. The structure of (2.3) holds also for the other types of vacuum q-fields. This demonstrates the universality of the phenomenology of the quantum

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vacuum: the phenomenology does not really depend on the details at the microscopic scale (be it Planckian or trans-Planckian), but is determined by phenomenological functions of the vacuum variable $q$, such as the energy density $\epsilon(q)$, the gravitational coupling $G(q)$, and the rigidity $K(q)$. These functions determine, in particular, the compressibility $\chi(q)$ of the quantum vacuum, the vacuum pressure $P(q)$, and the mass $M(q)$ of the propagating mode.

This universal behavior of $q$-theory is similar to the universal description of stable quantum many-body systems, such as quantum liquids formulated in terms of the number density $n$ of atoms (the nonrelativistic counterpart of our vacuum variable $q$). The vacuum $q$-ball considered in this paper is, in fact, the analog of a droplet of quantum liquid, which has a stable ground state at nonzero $n_c$ inside the droplet and $n = 0$ outside with a radius $R$ determined by the total number $N$ of atoms in the droplet. The chemical potential $\mu$ (being conjugate to $n$ in thermodynamics and acting as a Lagrange multiplier in the Hamiltonian) adjusts in order to match the external pressure in equilibrium. Without external forces on the droplet, the internal pressure is determined by the surface tension $\sigma$. In the limit of a large droplet radius, $R \to \infty$, the thermodynamic potential $\epsilon(n) - \mu n$ is nullified: $\epsilon(n) - \mu n = -P = P_\sigma \to 0$ and $\mu \to \mu_0$. This thermodynamic potential in terms of the atom number density $n$ is the analog of the gravitating vacuum energy density (cosmological constant) in terms of the vacuum variable $q$.

The space outside a droplet of quantum liquid, being free of atoms, is analogous to the absolutely empty vacuum of our discussion. The latter is a new concept and corresponds to a vacuum without any field and corresponding quantum fluctuations. The $q$-ball considered in this paper is a droplet of physical vacuum inside this absolutely empty vacuum. The extended $q$-theory with gradient terms allows us to describe the structure of the boundary of the $q$-ball. The $q$-ball is stable due to the conservation of the total charge $Q$, while the surface term is only responsible for the pressure inside the $q$-ball.

In the above discussion, it has been assumed that the physical vacuum is in full equilibrium and that its chemical potential $\mu$ has the equilibrium value $\mu_0$. The next step is to study the dynamics of the $q$-ball, which describes the relaxation to equilibrium from an arbitrary initial state of the $q$-ball. We expect a similar behavior as for the droplet of quantum liquid: non-equilibrium processes take place, accompanied by oscillations of the droplet shape and by radiation of the surplus droplet energy, and the equilibrium state of the liquid droplet is reached in the end. In this equilibration process, the chemical potential is no longer a Lagrange multiplier, but becomes a time-dependent variable, which relaxes to its equilibrium value.

It will be interesting to study the process of equilibration of the $q$-field, both in the absence and in the presence of gravity. The case with gravity may give a hint of how our Universe has relaxed to its present state, which is extremely close to equilibrium.

## Note Added

After completion of this paper, a further potential manifestation of the $q$-field with effective action (2.1a) has been found, namely, as a possible description of cold dark matter.[24] In addition, it was shown that the higher-derivative $q$-theory (2.1) does not suffer from the Ostrogradsky instability at the classical level.[25]

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