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Non-Abelian geometric phases in ground-state Josephson devices

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I. INTRODUCTION

Accurate control and measurement of few-level quantum systems has recently attracted great experimental and theoretical interest with possible applications in quantum information processing (QIP). Geometric phases arising from adiabatic and cyclic quantum evolution can provide robustness against, e.g., timing errors. Recently, it was shown that such evolution in a nondegenerate ground state is immune to decoherence from a low-temperature environment suggesting that it may provide an important tool for controlling quantum systems.

In the nondegenerate case, the accumulated geometric phase, the Berry phase, is a shift of the complex phase of the eigenstate and hence cannot be used as such for QIP. Non-Abelian phases corresponding to unitary matrices operating in a degenerate subspace of the system Hamiltonian, thus providing means for universal QIP. Although geometric phases capable of entangling two quantum bits, qubits, have been observed in liquid-state nuclear magnetic resonance experiments, this kind of geometric quantum computing (GQC) is yet to be demonstrated. In fact, the geometric phases using nuclear magnetic resonance, and in more recent experiments demonstrating nonadiabatic Aharonov-Anyandi phases in fullerene spin qubits, accumulate in a rotating frame, and hence there is no true degeneracy in the system.

The initial proposals for the experimental realization of GQC (Refs. 10 and 11) rely on a fully degenerate subspace to build the logical operators and it has been extended to many quantum systems including Josephson junction devices. In similar systems, a way to observe the non-Abelian evolution by measuring the charge pumped through the device has been recently proposed.

However, all the schemes assume typically a so-called tripod Hamiltonian which has degeneracy only in its excited states [see Fig. 1(a)] rendering the system prone to decoherence even in the low-temperature limit. This is potentially a serious limitation in condensed-matter systems in which the coupling between the system and environment is strong and unavoidable.

In this paper, we present an experimentally realizable Josephson device and show that it can be used to observe adiabatic non-Abelian geometric phases. In contrast to the above pioneering works, we employ a conceptually different Hamiltonian allowing us to work on the ground state manifold of the system. This scheme provides a clear extension to the theoretical proposals and experiments on the Berry phase in superconducting circuits.

II. NON-ABELIAN ADIABATIC EVOLUTION

We denote the parameters of the system Hamiltonian in a general cyclic loop by a vector \( \vec{\lambda} \). The instantaneous eigenstates of the Hamiltonian \( H[\lambda(t)] \) along this loop for all \( t \in [0, T] \) are denoted by \( \{ |\phi_{e}(t)\rangle \} \), where \( T \) is the period of the cycle. Generally, any temporal evolution of the system state can be represented using the time-evolution operator, \( \hat{U}(T) \), such that \( |\Psi(T)\rangle = \hat{U}(T)|\Psi(0)\rangle \), where \( |\Psi(0)\rangle \) is the state of the system at time \( t = 0 \). The charge \( Q \) transferred through a superconducting system in a parameter cycle can be obtained by integration of the current operator \( \hat{I} = \frac{-2e}{\hbar} \partial \Phi H(t) \) as \( Q = \int_{0}^{T} dt (\langle \Psi(t) | \hat{I} | \Psi(t) \rangle) \), where \( \varphi \) is the superconducting phase difference across the system. Using the Schrödinger equation and the definition of the time-evolution operator, this can be written in the form

\[
Q = -2ie \langle \Psi(0) | U(T) \{ \partial_\varphi \hat{U}(T) \} | \Psi(0) \rangle.
\]

FIG. 1. (Color online) (a) Eigenenergy diagram of the so-called tripod Hamiltonian consisting of two bright states, \( B \), and two degenerate dark states, \( D \). (b) Eigenenergy diagram of the circuit Hamiltonian along the considered cycle. The energy difference between the ground, \( g \), and excited, \( e \), state is denoted by \( \Delta \) and the ground state degeneracy splitting by \( \delta \). In the ideal case, \( \delta = 0 \). (c) Circuit diagram of the non-Abelian superconducting pump. Green (left) and blue (right) denote the superconducting islands and red the Josephson junctions.
However, if the Hamiltonian parameters are changed adiabatically along the cycle, the evolution can be restricted to the initial eigenspace. In an \( n \)-fold degenerate eigenspace, the state of the system after a parameter change is \( \Psi(T) = U_{\text{def}}(T)|\Psi(0)\rangle + \mathcal{O}(1/T) \), where \( |\Psi(t)\rangle = \sum_{i=1}^{m} C_i(t) |\psi_i(t)\rangle \). If the instantaneous eigenvectors are defined globally and continuously, the operator \( U_{\text{def}}(t) \) is represented in this basis as

\[
U_{\text{def}}(t) = e^{-i(t\hbar)\int_0^t dt' E(t') T e^{-i(t\hbar)\int_0^t dt' E(t')}}, \tag{2}
\]

where \( E(t) \) is the energy of the degenerate eigenspace, \( T \) is the time-ordering operator, and the connection \( \Gamma(t) \) is given by \( [\Gamma(t)]_{ab} = \langle \psi_a(t) | \psi_b(t) \rangle \). The first exponential function in Eq. (2) yields the accumulated dynamic phase shift, \( U_{\text{dyn}}(t) \), and the second one provides the geometric transformation, \( U_{\text{geo}}(t) \), which is non-Abelian, in general.

In the adiabatic limit, \( U(T) \) can be replaced with \( U_{\text{def}}(T) \) and substituting Eq. (2) into Eq. (1) yields the relation between the different transformations and the transferred charge

\[
Q = -2i e\{ \langle \Psi(0) | U_{\text{geo}}^\dagger(T) (\partial_t U_{\text{geo}}(T)) | \Psi(0) \rangle + \langle \Psi(0) | U_{\text{dyn}}^\dagger(T) \times [\partial_t U_{\text{dyn}}(T)] | \Psi(0) \rangle \}, \tag{3}
\]

where the first term is the geometric pumped charge and the second one is the dynamic charge due to the usual supercurrent. In the case of a nondegenerate eigenspace, \( n = 1 \), this reduces to the well-known relation \( Q = 2e\partial_t \phi_B(\Theta_g - \Theta_d) \), where the accumulated Berry phase, \( \Theta_g \), is related to \( U_{\text{geo}} \) by \( U_{\text{geo}} = e^{i\Theta_g} \) and the dynamic phase, \( \Theta_d \), to \( U_{\text{dyn}} \) by \( U_{\text{dyn}} = e^{-i\Theta_d} \). See Brosco et al. for an alternative way to obtain the pumped charge. Although the Berry phase induces just a phase shift to the state vector, it does not commute, in general, with the current operator \( \hat{I} \) which originates from a higher dimensional system.

**III. MODEL CIRCUIT**

The Cooper pair pump shown in Fig. 1(c) is considered here as the physical realization for observing non-Abelian geometric phases. It consists of three superconducting quantum interference devices (SQUIDs) in series with two superconducting islands between them. The SQUIDs are operated as tunable Josephson junctions which can be closed (Josephson energy \( E_j \) is zero) and opened \( (E_j \neq 0) \) by controlling the magnetic flux through them. The phase difference of the order parameter across the whole device, \( \varphi = \phi_R - \phi_L \), is kept constant by the magnetic flux \( \Phi \) through the outermost loop. The Hamiltonian has five external parameters which are controlled during a pumping cycle, i.e., three magnetic fluxes and two gate voltages.

The charging energy part of the Hamiltonian, \( H_{\text{ch}} \), is given by

\[
H_{\text{ch}} = E_{C_1} (\hat{n}_1 - n_{g1})^2 + E_{C_2} (\hat{n}_2 - n_{g2})^2 + E_m (\hat{n}_1 - n_{g1}) (\hat{n}_2 - n_{g2}), \tag{4}
\]

where \( \hat{n}_i \) is the operator for the excess number of Cooper pairs on the \( i \)th island and \( n_{g} \) is the corresponding gate charge given by \( n_{g} = C_g V_g / (2e) \). The charging energies are

\[
E_{C_1} = 2e^2 C_{s1} / C^2, \quad E_{C_2} = 2e^2 C_{s2} / C^2, \quad \text{and} \quad E_m = 4e^2 C_m / C^2.
\]

Here, \( C_{s} \) is the total capacitance of the \( i \)th island and \( C^2 = C_{s1} C_{s2} - C_m \).

The Josephson part of the Hamiltonian, \( H_J \), reads

\[
H_J = \sum_{n_1, n_2 = -\infty}^{\infty} (J_{eff,1} |n_1 + 1, n_2\rangle \langle n_1, n_2| + J_{eff,m} |n_1 + 1, n_2\rangle \langle n_1, n_2 + 1| + J_{eff,2} |n_1, n_2\rangle \langle n_1, n_2 + 1| + J_{eff,m} |n_1 + 1, n_2\rangle \langle n_1, n_2 + 1| + H.c.), \tag{5}
\]

where \( |n_1, n_2\rangle \) denotes the state with \( n_i \) excess Cooper pairs on the \( i \)th island, \( J_{eff,1} = -E_j(\Phi_0) e^{i\phi(T)}/2 \), \( J_{eff,2} = -E_j(\Phi_0) e^{-i\phi(T)}/2 \), and \( J_{eff,m} = -E_m(\Phi_m)/2 \). Here, \( E_1, E_2, \) and \( E_m \) are the tunable Josephson energies. The full Hamiltonian is given by

\[
H = H_{\text{ch}}(V_{g1}, V_{g2}) + H_J(\Phi_1, \Phi_2, \Phi_m, \Phi).
\]

**IV. NON-ABELIAN CYCLE**

If all the SQUIDs are closed, the conventional stability diagram with a hexagonal structure is recovered, see Fig. 2. In the vicinity of the triple degeneracy point of states \( |1,0\rangle \), \( |0,1\rangle \), and \( |1,1\rangle \), the adiabatic evolution is approximately restricted to these three states. The parameter cycle is composed of three symmetric paths in each of which a SQUID is opened, the gate voltages are shifted along a ground state degeneracy, and finally the SQUID is closed.

Along each path, the effective three-level Hamiltonian has a 2×2 block and can be written as \( H_{eff} = \hat{\Theta}_{ij} \hat{B}(t) + \hat{\epsilon}(t) |i\rangle \langle j| \), where \( \hat{\Theta}_{ij} = \{ \hat{\Theta}_{i1}, \hat{\Theta}_{i2}, \hat{\Theta}_{i3} \} \) is a vector composed of the Pauli matrices for the states \( i, j \) (for example, \( \hat{\Theta}_{12} = |j\rangle \langle j| + |i\rangle \langle i| \) ). \( \hat{B}(t) \) is an effective magnetic field, \( \hat{\epsilon}(t) \) is the eigenvalue of the third charge state, and \( \{ |i\rangle \langle j| \} = \{ |1,0\rangle, |0,1\rangle, |1,1\rangle \} \).

The condition of the ground state double degeneracy is satisfied by tuning the smaller eigenvalue of the 2×2 block of \( H_{eff} \) to be equal to \( \hat{\epsilon}(t) \) along the evolution. In the three-level approximation this implies that the degenerate gate-voltage channels are hyperbolas in the gate-voltage plane with only a single SQUID kept open. Along the opening and closing of the SQUIDs, we choose to change voltages linearly with the SQUID energies. In this way, a nontrivial loop encircling the triple degeneracy point can be traversed along a path with a doubly degenerate ground state.

Using the eigenstates along the three paths, we can construct a continuous global basis (defined in the whole parameter space) and calculate the connection \( \Gamma^{\text{eff}}(t) \). If the SQUIDs can be perfectly closed, the supercurrent due to the dynamic phase in Eq. (3) vanishes since the energies of the eigenstates do not depend on \( \varphi \). In this case, the transferred charge has only a geometric contribution which can be calculated analytically from the \( \varphi \) dependence of the \( U_{\text{geo}}(T) \) operator. For a cycle starting from the degeneracy line between the states \( |1,0\rangle \) and \( |0,1\rangle \), this yields for the geometric transformation

\[
U_{\text{geo}}(T) = \begin{bmatrix} 0 & e^{i\varphi} \\ 1 & 0 \end{bmatrix}, \tag{6}
\]

represented in the basis \( \{ |1,0\rangle, |0,1\rangle \} \). This result was confirmed by solving numerically the Schrödinger equation us-
into the gate-voltage plane. In the ideal cycle. The inset shows the parameter cycle projected on the $E_i$ axis. Maximum value $E_{i_{\text{max}}}$.

The evolution can be made Abelian by increasing the maximum Josephson energies and the cycle can be traversed slower such that no transitions occur.

The system can be initialized to the state $\{1,0\}$ by the following procedure. First, all the SQUIDs are closed to $E_{i_{\text{min}}}$ and gate voltages tuned to have $\{1,0\}$ as a nondegenerate ground state. After the system has relaxed to the ground state, the gate voltages are suddenly shifted, $T_{\text{shift}}<\hbar/\delta$, to the degeneracy line between the states $\{1,0\}$ and $\{0,1\}$. The sudden shift keeps the system in the state $\{1,0\}$ and the non-Abelian cycle can be traversed starting from a well-known initial state. The system can be initialized to the state $\{0,1\}$ with a similar procedure.

To describe the adiabaticity of the evolution, we introduce the adiabaticity parameter $\alpha$ defined as the population of the initial state after a back and forth cycle. In the perfectly non-Abelian regime, the geometric transformations induced by the forward and backward cycles exactly cancel each other. Thus, the total transformation is proportional to the identity implying that $\alpha=1$. For the perfectly Abelian limit, no transitions occur between the eigenstates and again $\alpha=1$ if the initial state is an eigenstate. Between these two regimes, no easy theoretical prediction can be made since the states are only partially mixed during the evolution.

In all numerical simulations, we fix the phase across the device $\varphi$ to zero and $E_{j_{\text{min}}}=0.2$ meV. Figure 5(a) suggests that for non-Abelian cycle with period $5 \leq T < 10$ ns the evolution is adiabatic and $\alpha$ is close to unity even if the SQUIDs cannot be perfectly closed with $k=E_{i_{\text{min}}}^{\text{max}}/E_{i_{\text{min}}}=1000$. In this regime, the pumped charge shown in Fig. 3 reaches the value $2e$ or $0$ depending on the initial state as predicted by Eq. (7). With $k=5000$ the adiabatic evolution window is broad and observed as a pumped charge plateau. To obtain a measurable current with a reasonable averaging time ($\geq 1$ ps), the pumping cycle needs to be repeated fast enough. If simply a sequence of repeated pumping cycles is performed, the measured current reflects the average pumped charge $e$ regardless of the initial state due to the swapping between the states $\{0,1\}$ and $\{1,0\}$. On the contrary, the system can be initialized to the same state before every cycle. In this case, the pumped charge per cycle is $2e$ or $0$ depending on the initial state. Measuring such dependence on the initialization indicates that the charge states are swapped after each cycle providing a fingerprint of the non-Abelian evolution.

The evolution can be made Abelian by increasing the cycle period and keeping all the SQUIDs constantly open.
The pumped charges corresponding to (a) Adiabaticity of the ground (red) and excited states (blue) in a cycle with all the SQUID energies fixed to $E_i^{\text{max}}=E_i^{\text{min}}=-0.4E_C$. (d) The pumped charges corresponding to (c). In (b) and (d) the gray lines denote the geometric pumped charges in the perfectly adiabatic limits. The shaded areas indicate the cycle periods with which the evolution is adiabatic. The charging energy $E_C$ used is experimentally realizable 0.2 meV (Ref. 22) and the phase across the device is fixed to zero, $\varphi=0$.

FIG. 3. (Color online) (a) Adiabaticity as a function of the cycle period in a cycle with $E_i^{\text{max}}=-0.4E_C$ and $E_i^{\text{min}}=E_i^{\text{max}}/k$. Red (blue) denotes a cycle starting from the initial state $|1,0\rangle$ ($|0,1\rangle$) for $k=1000$ (dots) and $k=5000$ (dashed line). (b) The pumped charges corresponding to (a). (c) Adiabaticity of the ground (red) and excited states (blue) in a cycle with all the SQUID energies fixed to $E_i^{\text{max}}=E_i^{\text{min}}=-0.4E_C$. (d) The pumped charges corresponding to (c). In (b) and (d) the gray lines denote the geometric pumped charges in the perfectly adiabatic limits. The shaded areas indicate the cycle periods with which the evolution is adiabatic. The charging energy $E_C$ used is experimentally realizable 0.2 meV (Ref. 22) and the phase across the device is fixed to zero, $\varphi=0$.

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