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Quantum treatment of the Bose-Einstein condensation in nonequilibrium systems

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We develop a fully quantum-mechanical approach, based on stochastic trajectories, for an incoherently pumped system of interacting bosons in contact with a thermal reservoir. It enables a complete characterization of coherence in such a multimode system. We apply our model to microcavity exciton polaritons interacting with acoustic phonons and observe the formation of a macroscopic occupation in the lowest-energy mode accompanied by the simultaneous establishment of temporal and spatial coherence. We describe the transition from thermal to coherent statistics together with the expected emergence of an off-diagonal long-range order.

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I. INTRODUCTION

A typical signature of Bose-Einstein condensation in a many-body system is the formation of a macroscopic occupation in its single-particle ground state at thermal equilibrium. This collective state exhibits distinctive spatial and temporal coherence properties. In solid-state systems, however, the bosons have a finite lifetime, and hence, one needs to study steady-state properties instead of thermal equilibrium [1]. We refer to this scenario as quasi-Bose-Einstein condensation (qBEC). Here, the macroscopic ground-state occupation is induced by the energy relaxation from the excited states which are pumped by some source reservoir. Such qBEC occurs in systems of magnons [2–4], indirect excitons [5], and exciton polaritons [6–8] under nonresonant excitation.

Exciton polaritons arise from the strong light-matter coupling enhanced in semiconductor microcavities [9]. They behave like bosonic quasiparticles at moderate concentrations ($\leq 10^{11} \text{ cm}^{-2}$). Due to their small effective mass, exciton polaritons can manifest quantum coherent properties up to room temperature in wide-band-gap materials [10,11]. Exciton-polariton qBEC emerges as the result of boson-boson interactions and energy exchange with the environment. In particular, the scattering of exciton polaritons with acoustic phonons [12] plays a key role, as demonstrated in a number of recent experiments [13,14].

The formation of the quasicondensate is associated with emission of coherent laserlike light from the microcavity [15,16]. However, such emission is not sufficient to prove the existence of qBEC [17] since the spatial and temporal coherence properties need to be addressed in detail. Experimentally, the temporal coherence is described by the second-order temporal coherence function $g^{(2)}(\tau)$, where τ is the delay between two photodetection events. In particular, $g^{(2)}(\tau=0)$ exhibits a transition from a thermal statistic, $g^{(2)}(0)=2$, to a coherent statistics, $g^{(2)}(0)=1$ [18,19]. The spatial coherence, also referred to as off-diagonal long-range order [20–24], is characterized by a slowly decaying first-order spatial coherence function $g^{(1)}(\Delta x)$ between regions separated by a distance Δx .

Whereas experimental techniques based on Michelson interferometry and Hanbury Brown and Twiss setups are well established to measure spatial and temporal coherence, a general theory accounting for many-body quantum correlations, environmental interactions, and photon counting is lacking. On the one hand, the widely used semiclassical Boltzmann equation approach [25–29] to the exciton-polariton dynamics is based on the assumption of incoherence of the system. Thus, the quantum states of the system are taken to be uncorrelated. On the other hand, approaches based on the Gross-Pitaevskii equation [30] under resonant or nonresonant excitation [31,32] are successful in explaining a plethora of recent experiments [33–35], but they assume global coherence and therefore cannot describe phonon-assisted relaxation. The truncated Wigner approaches [36], which involve additional noise terms in the Gross-Pitaevskii equation, are based on several limiting assumptions and are not designed to describe multimode systems. Recently, Boltzmann and Gross-Pitaevskii equations have been merged in a classical treatment [37] which does not provide a description of coherence properties.

Master equation approaches allow us to account for both coherent and incoherent processes in the exciton-polariton dynamics [38,39], and spatial coherence has been recently analyzed in a one-dimensional (1D) case [40,41]. However, such a model, involving a complex hierarchy of coupled and truncated equations, becomes computationally very demanding in higher dimensions, and its application seems to be restricted to 1D structures.

The quantum kinetic equations considered in Ref. [42] include energy relaxation in a phenomenological way, utilize a classical stochastic field, and require unknown free parameters. Thus, they seem unable to describe the desired coherence properties. Furthermore, the long-range interactions prevent the direct use of quantum methods based on the density-matrix renormalization-group theory [43].

In this paper, we develop a highly parallelizable quantum stochastic approach [44] going beyond the single-mode description of Ref. [45] to describe incoherently driven interacting bosons with dissipation caused by a thermal reservoir. The formalism is based on stochastic evolution of the multimode system wave function in its full Hilbert space.

It allows the reconstruction of the system density matrix from which correlations such as $g^{(2)}(0)$ and $g^{(1)}(\Delta x)$ can be directly extracted. In addition, we show how to compute the delayed temporal correlation function $g^{(2)}(\tau)$ from the emission statistics, thus faithfully simulating the Hanbury Brown and Twiss setup. We explicitly apply the method to nonequilibrium exciton-polariton condensation.

II. MODEL HAMILTONIAN

We consider a multimode bosonic system with an energy distribution dictated by the dispersion relation $E(\mathbf{k})$, where $\mathbf{k} = (k_x, k_y)$ is the two-dimensional wave vector. The system is in contact with a thermal reservoir at temperature T and is driven by an incoherent source with average power P . If only the dominant interactions are considered, the system can be described by the Hamiltonian

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\text{kin}} + \hat{\mathcal{H}}_{\text{p-p}} + \hat{\mathcal{H}}_{\text{p-ph}} + \hat{\mathcal{H}}_{\text{pump}}. \quad (1)$$

The first two terms in Eq. (1) describe coherent processes: $\hat{\mathcal{H}}_{\text{kin}} = \sum_{\mathbf{k}} E_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$ is the kinetic-energy term; in our case $E_{\mathbf{k}} = E_g + [E_{\text{ph}}(\mathbf{k}) - \sqrt{E_{\text{ph}}^2(\mathbf{k}) + 4V^2}]/2$ describes the lower branch of exciton-polariton dispersion, where E_g is the semiconductor band gap. The photonic dispersion is given by $E_{\text{ph}}(\mathbf{k}) = \hbar^2 k^2 / (2m_{\text{ph}})$, the exciton-photon Rabi splitting is $2V$, and we assume an infinite exciton mass since $m_{\text{ex}} \gg m_{\text{ph}}$. The creation and annihilation operators of the bosonic mode with momentum $\mathbf{k}$ are denoted by $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$, respectively. The second term,

$$\hat{\mathcal{H}}_{\text{p-p}} = \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}} U_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}} \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2}^\dagger \hat{a}_{\mathbf{k}_1 + \mathbf{p}} \hat{a}_{\mathbf{k}_2 - \mathbf{p}}, \quad (2)$$

describes the polariton-polariton elastic-scattering, conserving energy and momentum that, in general, corresponds to long-range interaction. The scattering strength $U_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}}$ is determined by the excitonic fractions, i.e., the Hopfield coefficients $X_{\mathbf{k}}$ of the initial and final states [46,47], $U_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}} = U_0 X_{\mathbf{k}_1} X_{\mathbf{k}_2} X_{\mathbf{k}_1 + \mathbf{p}} X_{\mathbf{k}_2 - \mathbf{p}}$, where $U_0 = 6E_b a_B^2 / S$, E_b and a_B are the exciton binding energy and the material-specific Bohr radius, respectively, and S is the system area.

The last two terms in Eq. (1) describe incoherent processes which we treat stochastically. Here, $\hat{\mathcal{H}}_{\text{p-ph}}$ accounts for the interaction of the particles with a thermal bath of acoustic phonons. To this end, we introduce a Fröhlich-type Hamiltonian [12,48],

$$\hat{\mathcal{H}}_{\text{p-ph}} = \sum_{\mathbf{k}_1, \mathbf{k}_2} G_{\mathbf{q}} \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} \hat{b}_{\mathbf{q}} + \text{H.c.}, \quad (3)$$

where the sum and integration are performed under the condition of energy and quasimomentum conservation, in particular, $|E_{\mathbf{k}_1} - E_{\mathbf{k}_2}| = \hbar\omega_{\mathbf{q}}$, $|\mathbf{q}|^2 = |\mathbf{k}_1 - \mathbf{k}_2|^2 + q_z^2$. The phonons are described by the operators $\hat{b}_{\mathbf{q}}^\dagger$ and $\hat{b}_{\mathbf{q}}$, and their dispersion relation, $\hbar\omega_{\mathbf{q}} = \hbar u \sqrt{q_x^2 + q_y^2 + q_z^2}$, is determined by the speed of sound u in the material [see also Appendixes A and C]. Above, $G_{\mathbf{q}}$ is the exciton-phonon interaction strength; the microscopic derivation of $G_{\mathbf{q}}$ and typical values can be found in Ref. [48]. We also assume that the phonon reservoir remains thermalized, i.e., $\langle \hat{b}_{\mathbf{q}}^\dagger \hat{b}_{\mathbf{q}} \rangle = \bar{n}_{\text{ph}}(\hbar\omega_{\mathbf{q}}) = \{\exp[\hbar\omega_{\mathbf{q}}/(k_B T)] - 1\}^{-1}$.

The effect of incoherent pumping is described in the rotating-wave approximation by the last term of Eq. (1) as

$$\hat{\mathcal{H}}_{\text{pump}} = \hbar \sum_{\mathbf{k}\xi} (g_{\mathbf{k}\xi} \hat{a}_{\mathbf{k}} \hat{d}_\xi^\dagger + g_{\mathbf{k}\xi}^* \hat{a}_{\mathbf{k}}^\dagger \hat{d}_\xi), \quad (4)$$

where $\hat{d}_\xi$ and $\hat{d}_\xi^\dagger$ are the operators corresponding to the bosonic pumping reservoir in question at temperature T_P and $\langle \hat{d}_\xi^\dagger \hat{d}_\xi \rangle = \bar{n}_P(E_{\mathbf{k}}) = C_{\mathbf{k}} \{\exp[E_{\mathbf{k}}/(k_B T_P)] - 1\}^{-1}$, where $C_{\mathbf{k}} \in \{0, 1\}$ determines if the mode with the wave vector $\mathbf{k}$ is pumped or not. The parameters $g_{\mathbf{k}\xi}$ describe the typical linear coupling strengths between the system modes and the reservoir modes. For simplicity, we assume that $g_{\mathbf{k}\xi} = \delta_{\mathbf{k}\xi} \gamma_{\mathbf{k}}$. The Hamiltonian (4) allows for particle loss, which occurs at the rate $\gamma_{\mathbf{k}}$. The losses arising from (4) are mainly caused by leakage of photons from the cavity since their lifetime is much shorter than that of bare excitons. Thus, we use $\gamma_{\mathbf{k}} = 1/\tau_{\mathbf{k}}^{\text{phot}}$, where $\tau_{\mathbf{k}}^{\text{phot}}$ is the lifetime of photons (see also Appendix B for details).

Using Eqs. (3) and (4), we can derive a Lindblad-type master equation (see Appendix C) and express the full set of associated quantum-jump operators as [44]

$$\hat{\mathcal{J}}_{\mathbf{k}}^+ = \sqrt{\gamma_{\mathbf{k}} \bar{n}_P(E_{\mathbf{k}})} \hat{a}_{\mathbf{k}}^\dagger, \quad (5)$$

$$\hat{\mathcal{J}}_{\mathbf{k}}^- = \sqrt{\gamma_{\mathbf{k}} [\bar{n}_P(E_{\mathbf{k}}) + 1]} \hat{a}_{\mathbf{k}}, \quad (6)$$

$$\hat{\mathcal{J}}_{\mathbf{k}_1 \mathbf{k}_2}^+ = \sqrt{\gamma_{\mathbf{k}_1 \mathbf{k}_2}^{\text{ph}} \bar{n}_{\text{ph}}(E_{\mathbf{k}_1} - E_{\mathbf{k}_2})} \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2}, \quad (7)$$

$$\hat{\mathcal{J}}_{\mathbf{k}_1 \mathbf{k}_2}^- = \sqrt{\gamma_{\mathbf{k}_1 \mathbf{k}_2}^{\text{ph}} [\bar{n}_{\text{ph}}(E_{\mathbf{k}_1} - E_{\mathbf{k}_2}) + 1]} \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}^\dagger, \quad (8)$$

where $E_{\mathbf{k}_1} > E_{\mathbf{k}_2}$ and we denote the phonon-mediated scattering rate as $\gamma_{\mathbf{k}_1 \mathbf{k}_2}^{\text{ph}}$. Equations (5) and (6) describe the exciton-polariton pumping and decay (see Appendix B). The average power fed into the exciton-polariton system due to interaction with the pumping reservoir is described by $P = \bar{n}_P \gamma_{\mathbf{k}}$. Equations (7) and (8) describe transitions between the exciton-polariton modes mediated by the phonon reservoir (see Appendix A). Note that phonon emission described by (8) remains even at $T = 0$ K.

III. METHODOLOGY

The quantum dynamics of the system is simulated using the Monte Carlo wave-function technique [44]. The procedure is based on the evolution of the system according to the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\tilde{\psi}\rangle = \hat{\mathcal{H}} |\tilde{\psi}\rangle, \quad (9)$$

with the effective non-Hermitian Hamiltonian

$$\begin{aligned} \hat{\mathcal{H}} = & \hat{\mathcal{H}} - \frac{i\hbar}{2} \sum_{\mathbf{k}} \hat{\mathcal{J}}_{\mathbf{k}}^{+\dagger} \hat{\mathcal{J}}_{\mathbf{k}}^+ - \frac{i\hbar}{2} \sum_{\mathbf{k}} \hat{\mathcal{J}}_{\mathbf{k}}^{-\dagger} \hat{\mathcal{J}}_{\mathbf{k}}^- \\ & - \frac{i\hbar}{2} \sum_{\mathbf{k}_1 \mathbf{k}_2} \hat{\mathcal{J}}_{\mathbf{k}_1 \mathbf{k}_2}^{+\dagger} \hat{\mathcal{J}}_{\mathbf{k}_1 \mathbf{k}_2}^+ - \frac{i\hbar}{2} \sum_{\mathbf{k}_1 \mathbf{k}_2} \hat{\mathcal{J}}_{\mathbf{k}_1 \mathbf{k}_2}^{-\dagger} \hat{\mathcal{J}}_{\mathbf{k}_1 \mathbf{k}_2}^-. \end{aligned} \quad (10)$$

The non-Hermitian part in Eq. (10) results in an apparent decay of the norm $\langle \tilde{\psi}(t) | \tilde{\psi}(t) \rangle$. We generate a random number η initially and evolve the system by Eq. (9). The condition

$\langle \tilde{\psi}(t) | \tilde{\psi}(t) \rangle \leq \eta$ determines if a jump operator occurs or not [44]. The proper jump to apply is decided by the smallest integer m satisfying $\sum_{j=1}^m P_j(t) \geq \eta$, where P_j is the collapse probability associated with the process j . After each jump, we normalize the state $|\tilde{\psi}(t)\rangle$ again and generate a new number η . A single realization j of this protocol yields a quantum trajectory, $|\tilde{\psi}(t)\rangle_j$, with $j = 1, 2, \dots, M$. Employing an ensemble of trajectories, we can approximate the system density operator as

$$\hat{\rho}(t) = \frac{\sum_{j=1}^M |\tilde{\psi}(t)\rangle_j \langle \tilde{\psi}(t)|_j}{M} \xrightarrow{M \rightarrow \infty} \hat{\rho}(t), \quad (11)$$

where $\hat{\rho}(t)$ is the actual density operator of the system. The expectation value of any system observable $\hat{O}$ can be found from

$$\langle \hat{O}(t) \rangle = \text{Tr}[\hat{O} \hat{\rho}(t)] = \lim_{M \rightarrow \infty} \{ \text{Tr}[\hat{O} \hat{\rho}(t)] \}. \quad (12)$$

This method not only allows us to significantly reduce the memory consumption by evolving a ket vector instead of a density matrix but also is ideal for parallelization due to the independence of the quantum trajectories.

In our computations, we take advantage of sparse algebra, and we truncate the Hilbert space to a chosen global number of excitations per state [49] in addition to the usual truncation per mode, which allows us to drastically reduce the dimension of the Hilbert space with a negligible loss of accuracy. Due to possible qBEC, however, the maximum number of excitations for the lowest-energy mode is taken to be several times greater than those for the other states.

IV. RESULTS AND DISCUSSION

The parameters we consider correspond to a GaAs-based microcavity with cylindrical symmetry, with the Rabi splitting $2V = 10$ meV; $m_{\text{ph}} = 5 \times 10^{-5} m_0$, where m_0 is free electron mass; a particle lifetime of $\tau \simeq 1/\gamma_k = 20$ ps; $E_b = 10$ meV; $a_B = 10$ nm; and $S = 100 (\mu\text{m})^2$. The system symmetry and the correlations we compute here allow us to consider only the radial coordinate k_r , which needs to be reaccessed if one is studying spatial correlations between different azimuthal angles.

Initially, the system is prepared in its vacuum state. Each trajectory is composed of a 500-ps evolution which is sufficiently longer than the particle lifetime to reach the steady state. For each set of parameters, we average the results over $M = 5000$ parallelized trajectories.

Our sampling of the dispersion relation is shown in Fig. 1(a), where we fix the energy step to $\Delta E = 0.33$ meV between each mode. We approximate the Hopfield coefficients as $X_k = 1/\sqrt{2}$, and hence, the polariton-polariton scattering strength is fixed to $U_{k_1 k_2 p} = U_0/4$. Its value is adjusted to result in a typical chemical potential of $U_0 N_0/4 = 1$ meV of the lowest-energy mode if this mode is completely filled. The maximum number of excitations is fixed to $N_0^{\text{max}} = 15$ for the lowest-energy mode and to $N^{\text{max}} = 5$ for the other modes. The polariton-phonon scattering strength is set to $\hbar\gamma_{\mathbf{k}\mathbf{k}'} = \hbar\gamma_0 = 0.05$ meV. Furthermore, we assume that the incoherent pump operator in Eq. (5) is acting only on the

FIG. 1. (Color online) (a) Lower exciton-polariton dispersion branch (gray line) showing the discrete bosonic modes (dots) used in the computations. (b) Occupation of the lowest mode N_0 as a function of the pump power P for several temperatures in the range 0–20 K (see legend). Log-scale occupations of the modes as functions of their energies at (c) $T = 5$ K and (d) $T = 20$ K for several pump powers in the range $(0.5\text{--}25)P_{\text{th}}$ (see legend). The red dashed lines indicate fits from a thermal distribution.

highest mode of the system, and thus $C_k = 0$ for all the other modes.

Figures 1(b)–1(d) show the mode occupations, $N_k = \langle \hat{a}_k^\dagger \hat{a}_k \rangle$, for several pump powers P and temperatures. Around some threshold power, $P = P_{\text{th}} = 20$ ps $^{-1}$, the lowest-energy-mode stimulation occurs, and as shown in Fig. 1(b), its occupation monotonically increases with P , faster for lower temperatures, due to the polariton-polariton scattering and phonon-assisted energy relaxation. The highest-energy mode in the dispersion is fed by the incoherent pump and plays the role of a bottleneck mode [50]. Thus, this mode exhibits a large occupation even for $P > P_{\text{th}}$ (not shown), although the population of the other modes decreases with increasing energy. At the lowest investigated pump power, $P = 0.5P_{\text{th}}$, we obtain a thermal distribution $N_k = N_0 \exp[-E_k/(k_B T)]$ which allows us to extract effective polariton temperatures of $\tilde{T} = 7.5$ K and $\tilde{T} = 23$ K in Figs. 1(c) and 1(d), respectively.

Figure 2 shows our results for the lowest-mode second-order temporal coherence

$$g^{(2)}(\tau) = \frac{\langle \hat{a}_0^\dagger(0) \hat{a}_0^\dagger(\tau) \hat{a}_0(\tau) \hat{a}_0(0) \rangle}{\langle \hat{a}_0^\dagger(0) \hat{a}_0(0) \rangle^2}. \quad (13)$$

We observe a clear crossover from thermal statistics with $g^{(2)}(0) = 2$ to a coherent state for which $g^{(2)}(0) = 1$ with increasing P , as shown in Fig. 2(a). With decreasing temperature, the coherence onsets at lower pump power, as expected.

FIG. 2. (Color online) (a) Lowest-energy-mode second-order temporal coherence function at zero delay $g^{(2)}(0)$ as a function of the relative pump power P/P_{th} for different phonon temperatures. (b) Finite-delay coherence $g^{(2)}(\tau)$ calculated for three different pump powers at $T = 0$ K (see legends).

To compute the delayed $g^{(2)}(\tau)$ shown in Fig. 2(b), we work on the polariton decay statistics recording the full event history related to Eq. (5) (specifically for $k = 0$) over 100-ns trajectories. It allows us to build the probability $G^{(2)}(\tau)$ of having two polariton decays within a delay τ . When normalized to the corresponding Poissonian expectation, imposed by the mean steady-state occupation, we are able to reconstruct the correlation function and confirm the onset of temporal coherence revealed by $g^{(2)}(\tau) \simeq 1$ for $P > P_{\text{th}}$ [51].

In addition, our approach allows us to directly address the spatial coherence function of the system,

$$g^{(1)}(x_i, x_j) = \lim_{t \rightarrow \infty} \frac{\langle \hat{\psi}^\dagger(x_i, t) \hat{\psi}(x_j, t) \rangle}{\langle \hat{\psi}^\dagger(x_i, t) \rangle \langle \hat{\psi}(x_j, t) \rangle}, \quad (14)$$

between two points at positions x_i and x_j , written here in a 1D approximation. The real-space field operators are defined as $\hat{\psi}(x, t) = \sum_k e^{ikx} \hat{a}_k(t)$. We model here a system of extension $L = 20 \mu\text{m}$. Figures 3(a) and 3(b) show the zero-temperature spatial coherence function for pumping below and above the threshold, respectively. These results reveal the onset of long-range spatial coherence above the threshold. Note that the periodic boundary conditions imposed by the Fourier transform lead to the observed artificial increase of correlations near the edge of the computational area. Figures 3(c) and 3(d) show a full set of coherence functions at $T = 0$ K and $T = 20$ K, respectively, to the middle of the sample at various pump powers. We clearly observe the increase in spatial coherence with pump power and its decay with increasing temperature. We have carefully checked the convergence of the results against the system size, as shown in Appendix D, which confirms the emergence of the off-diagonal long-range order into the system.

V. CONCLUSIONS

In summary, using a stochastic wave-function approach, we have analyzed the quantum properties of a nonequilibrium condensate as a function of pump power and temperature. Our results exhibit all the characteristic features associated with the Bose-Einstein condensation of such incoherently driven bosonic particles in contact with a phonon bath.

FIG. 3. (Color online) Steady-state first-order spatial coherence function $g^{(1)}(x_i, x_j)$ at $T = 0$ K for pumping (a) below threshold, $P = 0.5 P_{\text{th}}$, and (b) above threshold, $P = 20 P_{\text{th}}$, as a function of the separation between points along the sample. Also shown is $g^{(1)}(\Delta x) = g^{(1)}(\Delta x, 0)$ for various pump powers (see legend) at (c) $T = 0$ K and (d) $T = 20$ K, where Δx is the separation between points and the center of the system.

To account in the future for a large number of bosons and modes, our results can be extended by separating the classical field of each mode, evolved according to Langevin equations [52], from the quantum fluctuations that would be treated through the quantum jump approach with the requirement of a very small number of quanta per mode. This will be addressed in a separate study. Furthermore, the impact of decoherence in the form of pure dephasing can be straightforwardly added as a new set of quantum jumps operators [53]. In the future, our approach may also be refined by taking into account the effects of the boson-boson interactions and the drive in the quantum-jump operators [54,55].

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APPENDIX A: COUPLING TO THE THERMAL RESERVOIR OF ACOUSTIC PHONONS

In this Appendix, we express the coupling Hamiltonian between the polaritons and the phonon bath in a convenient form which we later employ in Appendix C to derive the master equation of the polariton system. Taking into account only the energy- and momentum-conserving terms, the Hamiltonian of the polariton-phonon interaction in Eq. (3) reads

$$\begin{aligned}\hat{\mathcal{H}}_{\text{p-ph}} &= \sum_{\mathbf{k}_1, \mathbf{k}_2} G_{\mathbf{q}} \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} \hat{b}_{\mathbf{q}} + \text{H.c.} \\ &= \sum_{\mathbf{k}_1, \mathbf{k}_2; E_{\mathbf{k}_1} > E_{\mathbf{k}_2}} G_{|\mathbf{k}_1 - \mathbf{k}_2|, q_z} \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} \hat{b}_{\mathbf{k}_1 - \mathbf{k}_2, q_z} \\ &\quad + \sum_{\mathbf{k}_1, \mathbf{k}_2; E_{\mathbf{k}_1} < E_{\mathbf{k}_2}} G_{|\mathbf{k}_1 - \mathbf{k}_2|, q_z}^* \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} \hat{b}_{\mathbf{k}_1 - \mathbf{k}_2, q_z}^\dagger, \quad (\text{A1})\end{aligned}$$

where q_z assumes a value guaranteeing $\omega_{\mathbf{q}} = |E_{\mathbf{k}_1} - E_{\mathbf{k}_2}|/\hbar$, where $\omega_{\mathbf{q}}$ is the phonon dispersion and $|\mathbf{q}| = \sqrt{|\mathbf{k}_1 - \mathbf{k}_2|^2 + q_z^2}$. The scattering strength is given by [12,48]

$$\begin{aligned}G_{|\mathbf{k}_1 - \mathbf{k}_2|, q_z} &= i \sqrt{\frac{\hbar \sqrt{|\mathbf{k}_1 - \mathbf{k}_2|^2 + q_z^2}}{2\rho V u}} [d_e I_{\parallel}^e(|\mathbf{k}_1 - \mathbf{k}_2|) I_{\perp}^e(q_z) \\ &\quad - d_h I_{\parallel}^h(|\mathbf{k}_1 - \mathbf{k}_2|) I_{\perp}^h(q_z)], \quad (\text{A2})\end{aligned}$$

where $\rho = 5318 \text{ kg/m}^3$ is the material density; V is the volume of the system, which in our case is a quantum well (QW); $u = 3350 \text{ m/s}$ is the speed of sound; and $d_e = -7 \text{ eV}$ and $d_h = 2.7 \text{ eV}$ are the deformation potentials of the lattice induced by phonons at the locations of electrons and holes, respectively. The thickness of the QW is $L_z = 10 \text{ nm}$, and the excitonic Bohr radius is $a_B = 10 \text{ nm}$. The integrals $I_{\parallel}^{e(h)}(|\mathbf{k}_1 - \mathbf{k}_2|)$ and $I_{\perp}^{e(h)}(q_z)$ are the overlap integrals of the phonon wave functions with the electron and hole wave functions, respectively, in the in-plane and growth directions and can be expressed, following Ref. [47], as

$$I_{\parallel}^{e(h)}(|\mathbf{k}_1 - \mathbf{k}_2|) = \left[1 + \left(\frac{m_{h(e)}}{m_e + m_h} |\mathbf{k}_1 - \mathbf{k}_2| a_B \right)^2 \right]^{-3/2} \quad (\text{A3})$$

$$I_{\perp}^{e(h)}(q_z) = \frac{\pi^2}{\frac{q_z L_z}{2} [\pi^2 - (\frac{q_z L_z}{2})^2]} \sin\left(\frac{q_z L_z}{2}\right), \quad (\text{A4})$$

where m_e and m_h are the effective masses of electrons and holes. Furthermore, we utilize the interaction picture with respect to the single-particle Hamiltonian [the first term in Eq. (1)], which results in

$$\hat{a}_{\mathbf{k}}(t) = \hat{a}_{\mathbf{k}} e^{-iE_{\mathbf{k}}t/\hbar}, \quad \hat{a}_{\mathbf{k}}^\dagger(t) = \hat{a}_{\mathbf{k}}^\dagger e^{iE_{\mathbf{k}}t/\hbar}. \quad (\text{A5})$$

By introducing the polariton and reservoir operators

$$\hat{\mathcal{A}}_{\mathbf{k}_1 \mathbf{k}_2}(t) = \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}^\dagger e^{-i(E_{\mathbf{k}_1} - E_{\mathbf{k}_2})t/\hbar}, \quad (\text{A6})$$

$$\hat{R}_{\mathbf{k}_1 \mathbf{k}_2}(t) = G_{|\mathbf{k}_1 - \mathbf{k}_2|, q_z} \hat{b}_{|\mathbf{k}_1 - \mathbf{k}_2|, q_z} e^{-i\omega_{|\mathbf{k}_1 - \mathbf{k}_2|, q_z} t}, \quad (\text{A7})$$

we obtain

$$\hat{\mathcal{H}}_{\text{p-ph}}(t) = \hbar \sum_{\mathbf{k}_1 \mathbf{k}_2} \hat{\mathcal{A}}_{\mathbf{k}_1 \mathbf{k}_2}(t) \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(t) + \hat{\mathcal{A}}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(t) \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}(t). \quad (\text{A8})$$

APPENDIX B: COUPLING TO THE PUMPING SOURCE

Taking into account only the energy- and momentum-conserving terms, the Hamiltonian corresponding to incoherent pumping in Eq. (4) reads

$$\begin{aligned}\hat{\mathcal{H}}_{\text{pump}} &= \hbar \sum_{\mathbf{k}\xi} (g_{\mathbf{k}\xi} \hat{a}_{\mathbf{k}} \hat{d}_{\xi}^\dagger + g_{\mathbf{k}\xi}^* \hat{a}_{\mathbf{k}}^\dagger \hat{d}_{\xi}) \\ &= \hbar \sum_{\mathbf{k}} (g_{\mathbf{k}} \hat{a}_{\mathbf{k}} \hat{d}_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* \hat{a}_{\mathbf{k}}^\dagger \hat{d}_{\mathbf{k}}), \quad (\text{B1})\end{aligned}$$

where in the last equality we assume $g_{\mathbf{k}\xi} = \delta_{\mathbf{k}\xi} \gamma_{\mathbf{k}}$. In terms of the pumping source operators,

$$\hat{D}_{\mathbf{k}}(t) = \gamma_{\mathbf{k}} \hat{d}_{\mathbf{k}} e^{-i w_{\mathbf{k}}^p t}, \quad (\text{B2})$$

in the assumption $w_{\mathbf{k}}^p = E_{\mathbf{k}}$, Hamiltonian (B1) may be expressed as

$$\hat{\mathcal{H}}_{\text{pump}}(t) = \hbar \sum_{\mathbf{k}} \{ \hat{a}_{\mathbf{k}}(t) \hat{D}_{\mathbf{k}}^\dagger(t) + \hat{a}_{\mathbf{k}}^\dagger(t) \hat{D}_{\mathbf{k}}(t) \}. \quad (\text{B3})$$

APPENDIX C: MASTER EQUATION FOR POLARITONS

The full density operator $\hat{\chi}$ of the total quantum system (including the subsystem of polaritons and both the phonon and pumping source reservoirs) is subject to the Liouville-von Neumann equation:

$$i\hbar \partial_t \hat{\chi} = [\hat{\mathcal{H}}, \hat{\chi}]. \quad (\text{C1})$$

We apply the Born approximation to factorize the density operator as

$$\hat{\chi} = \hat{\rho} \otimes \hat{\rho}^{\text{ph}} \otimes \hat{\rho}^{\text{pump}}, \quad (\text{C2})$$

where $\hat{\rho}^{\text{ph}}$ and $\hat{\rho}^{\text{pump}}$ are the density operators of the phonon and pumping subsystems, respectively. Further, we employ the partial trace as

$$\hat{\rho}(t) = \text{Tr}_{\text{pump}} \text{Tr}_{\text{p-ph}} \{ \hat{\chi}(t) \}. \quad (\text{C3})$$

Thus, the master equation for the system density operator may be typically written in the interaction picture as

$$\begin{aligned}\partial_t \hat{\rho}(t) &= -\frac{1}{\hbar^2} \int_0^t \text{Tr}_{\text{pump}} \text{Tr}_{\text{p-ph}} [\hat{\mathcal{H}}_{\text{p-ph}}(t) \\ &\quad \times + \hat{\mathcal{H}}_{\text{pump}}(t), [\hat{\mathcal{H}}_{\text{p-ph}}(\tau) + \hat{\mathcal{H}}_{\text{pump}}(\tau), \hat{\chi}(\tau)]] d\tau. \quad (\text{C4})\end{aligned}$$

We make a further reasonable assumption of the independence of the pumping source and the phonon reservoir, which yields

$$\begin{aligned}\partial_t \hat{\rho} &= -\frac{1}{\hbar^2} \int_0^t \text{Tr}_{\text{pump}} [\hat{\mathcal{H}}_{\text{pump}}(t), [\hat{\mathcal{H}}_{\text{pump}}(\tau), \hat{\chi}(\tau)]] d\tau \\ &\quad - \frac{1}{\hbar^2} \int_0^t \text{Tr}_{\text{p-ph}} [\hat{\mathcal{H}}_{\text{p-ph}}(t), [\hat{\mathcal{H}}_{\text{p-ph}}(\tau), \hat{\chi}(\tau)]] d\tau \\ &= \partial_t \hat{\rho}^{(1)} + \partial_t \hat{\rho}^{(2)}. \quad (\text{C5})\end{aligned}$$

FIG. 4. (Color online) Steady-state first-order spatial coherence function $g^{(1)}(x_i, x_j)$ at $T = 20$ K and $P = 10P_{\text{th}}$ for different lengths L of the sample as a function of the separation Δx from the center of the system.

FIG. 5. (Color online) Finer sampling at fixed spatial steps. Steady-state first-order spatial coherence function $g^{(1)}(x_i, x_j)$ for pumping $P = 16.5P_{\text{th}}$ at (a) $T = 0$ K and (b) $T = 20$ K as a function of the separation of points along the sample. Also shown is $g^{(1)}(\Delta x) = g^{(1)}(\Delta x, 0)$ for various pump powers (see legend) at (c) $T = 0$ K and (d) $T = 20$ K, where Δx is the separation from the center of the system.

After a straightforward derivation in the framework of the secular approximation [56], we find

$$\begin{aligned} \partial_t \hat{\rho}^{(1)} = & - \sum_{\mathbf{k}} \int_0^t d\tau \{ \hat{a}_{\mathbf{k}}(t) \hat{a}_{\mathbf{k}}^\dagger(\tau) \hat{\rho} - \hat{a}_{\mathbf{k}}^\dagger(\tau) \hat{\rho} \hat{a}_{\mathbf{k}}(t) \} \langle \hat{D}_{\mathbf{k}}^\dagger(t) \hat{D}_{\mathbf{k}}(\tau) \rangle_{\text{pump}} + \{ \hat{\rho} \hat{a}_{\mathbf{k}}(\tau) \hat{a}_{\mathbf{k}}^\dagger(t) - \hat{a}_{\mathbf{k}}^\dagger(t) \hat{\rho} \hat{a}_{\mathbf{k}}(\tau) \} \langle \hat{D}_{\mathbf{k}}^\dagger(\tau) \hat{D}_{\mathbf{k}}(t) \rangle_{\text{pump}} \\ & + \{ \hat{a}_{\mathbf{k}}(t)^\dagger \hat{a}_{\mathbf{k}}(\tau) \hat{\rho} - \hat{a}_{\mathbf{k}}(\tau) \hat{\rho} \hat{a}_{\mathbf{k}}^\dagger(t) \} \langle \hat{D}_{\mathbf{k}}(t) \hat{D}_{\mathbf{k}}^\dagger(\tau) \rangle_{\text{pump}} + \{ \hat{\rho} \hat{a}_{\mathbf{k}}^\dagger(\tau) \hat{a}_{\mathbf{k}}(t) - \hat{a}_{\mathbf{k}}(t) \hat{\rho} \hat{a}_{\mathbf{k}}^\dagger(\tau) \} \langle \hat{D}_{\mathbf{k}}(\tau) \hat{D}_{\mathbf{k}}^\dagger(t) \rangle_{\text{pump}}, \end{aligned} \quad (\text{C6})$$

where $\langle \hat{D}_{\mathbf{k}}^\dagger(t) \hat{D}_{\mathbf{k}}(\tau) \rangle_{\text{pump}} = \text{Tr}_{\text{pump}} \{ \hat{D}_{\mathbf{k}}^\dagger(t) \hat{D}_{\mathbf{k}}(\tau) \hat{\rho}^{\text{pump}} \}$. Assuming that the pumping source is thermalized, we may write

$$\langle \hat{D}_{\mathbf{k}}^\dagger(t) \hat{D}_{\mathbf{k}}(\tau) \rangle_{\text{pump}} = |g_{\mathbf{k}}|^2 e^{iE_{\mathbf{k}}(t-\tau)/\hbar} \bar{n}_{\text{P}}(E_{\mathbf{k}}) = \gamma_{\mathbf{k}} \bar{n}_{\text{P}}(E_{\mathbf{k}}) e^{iE_{\mathbf{k}}(t-\tau)/\hbar}, \quad (\text{C7})$$

$$\langle \hat{D}_{\mathbf{k}}(t) \hat{D}_{\mathbf{k}}^\dagger(\tau) \rangle_{\text{pump}} = |g_{\mathbf{k}}|^2 e^{-iE_{\mathbf{k}}(t-\tau)/\hbar} [\bar{n}_{\text{P}}(E_{\mathbf{k}}) + 1] = \gamma_{\mathbf{k}} [\bar{n}_{\text{P}}(E_{\mathbf{k}}) + 1] e^{-iE_{\mathbf{k}}(t-\tau)/\hbar}, \quad (\text{C8})$$

where $\bar{n}_{\text{P}}(\omega)$ is the distribution function of the bosons in the pumping source. Thus, e.g., in the second line of Eq. (C6), we have $\{ \hat{a}_{\mathbf{k}}(t) \hat{a}_{\mathbf{k}}^\dagger(\tau) \hat{\rho} - \hat{a}_{\mathbf{k}}^\dagger(\tau) \hat{\rho} \hat{a}_{\mathbf{k}}(t) \} \langle \hat{D}_{\mathbf{k}}^\dagger(t) \hat{D}_{\mathbf{k}}(\tau) \rangle_{\text{pump}} = \{ \hat{a}_{\mathbf{k}} e^{-iE_{\mathbf{k}}t/\hbar} \hat{a}_{\mathbf{k}}^\dagger e^{iE_{\mathbf{k}}\tau/\hbar} \hat{\rho} - \hat{a}_{\mathbf{k}}^\dagger e^{iE_{\mathbf{k}}\tau/\hbar} \hat{\rho} \hat{a}_{\mathbf{k}} e^{-iE_{\mathbf{k}}t/\hbar} \} \gamma_{\mathbf{k}} \bar{n}_{\text{P}}(E_{\mathbf{k}}) e^{iE_{\mathbf{k}}(t-\tau)/\hbar} = \gamma_{\mathbf{k}} \bar{n}_{\text{P}}(E_{\mathbf{k}}) (\hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \hat{\rho} - \hat{a}_{\mathbf{k}}^\dagger \hat{\rho} \hat{a}_{\mathbf{k}}) = \hat{\mathcal{J}}_{\mathbf{k}}^+ \hat{\mathcal{J}}_{\mathbf{k}}^{\dagger+} \hat{\rho} - \hat{\mathcal{J}}_{\mathbf{k}}^{\dagger+} \hat{\rho} \hat{\mathcal{J}}_{\mathbf{k}}^+$. Subsequently, we arrive at the quantum-jump operators presented in Eqs. (5) and (6).

Repeating the same steps with respect to the phonon reservoir, we obtain the master equation

$$\begin{aligned} \partial_t \hat{\rho}^{(2)} = & - \sum_{\mathbf{k}_1 \mathbf{k}_2} \int_0^t d\tau \{ \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}(t) \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(\tau) \hat{\rho} - \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(\tau) \hat{\rho} \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}(t) \} \langle \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(t) \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}(\tau) \rangle_{\text{p-ph}} \\ & + \{ \hat{\rho} \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}(\tau) \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(t) - \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(t) \hat{\rho} \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}(\tau) \} \langle \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(\tau) \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}(t) \rangle_{\text{p-ph}} \\ & + \{ \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}(t)^\dagger \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}(\tau) \hat{\rho} - \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}(\tau) \hat{\rho} \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(t) \} \langle \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}(t) \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(\tau) \rangle_{\text{p-ph}} \\ & + \{ \hat{\rho} \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(\tau) \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}(t) - \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}(t) \hat{\rho} \hat{A}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(\tau) \} \langle \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}(\tau) \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(t) \rangle_{\text{p-ph}}, \end{aligned} \quad (\text{C9})$$

where assuming that the phonon reservoir is thermalized, we have

$$\langle \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(t) \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}(\tau) \rangle_{\text{p-ph}} = |G_{|\mathbf{k}_1 - \mathbf{k}_2|, q_z}|^2 e^{i\omega_{|\mathbf{k}_1 - \mathbf{k}_2|, q_z}(t-\tau)} \bar{n}_{\text{ph}}(E_{\mathbf{k}_1} - E_{\mathbf{k}_2}), \quad (\text{C10})$$

$$\langle \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}(t) \hat{R}_{\mathbf{k}_1 \mathbf{k}_2}^\dagger(\tau) \rangle_{\text{p-ph}} = |G_{|\mathbf{k}_1 - \mathbf{k}_2|, q_z}|^2 e^{-i\omega_{|\mathbf{k}_1 - \mathbf{k}_2|, q_z}(t-\tau)} [\bar{n}_{\text{ph}}(E_{\mathbf{k}_1} - E_{\mathbf{k}_2}) + 1]. \quad (\text{C11})$$

Ignoring the dependence of $G_{|\mathbf{k}_1-\mathbf{k}_2|,q_z}$ on q_z , we may further write

$$\langle \hat{R}_{\mathbf{k}_1\mathbf{k}_2}^\dagger(t) \hat{R}_{\mathbf{k}_1\mathbf{k}_2}(\tau) \rangle_{\text{p-ph}} = \gamma_{\mathbf{k}_1\mathbf{k}_2}^{\text{ph}} \bar{n}_{\text{ph}}(E_{\mathbf{k}_1} - E_{\mathbf{k}_2}) e^{i\omega_{|\mathbf{k}_1-\mathbf{k}_2|,q_z}(t-\tau)}, \quad (\text{C12})$$

$$\langle \hat{R}_{\mathbf{k}_1\mathbf{k}_2}(t) \hat{R}_{\mathbf{k}_1\mathbf{k}_2}^\dagger(\tau) \rangle_{\text{p-ph}} = \gamma_{\mathbf{k}_1\mathbf{k}_2}^{\text{ph}} [\bar{n}_{\text{ph}}(E_{\mathbf{k}_1} - E_{\mathbf{k}_2}) + 1] e^{-i\omega_{|\mathbf{k}_1-\mathbf{k}_2|,q_z}(t-\tau)}. \quad (\text{C13})$$

Thus, e.g., in the second line of Eq. (C9), using the energy conservation law in the process of phonon emission or absorption, we have $\{\hat{A}_{\mathbf{k}_1\mathbf{k}_2}(t) \hat{A}_{\mathbf{k}_1\mathbf{k}_2}^\dagger(\tau) \hat{\rho} - \hat{A}_{\mathbf{k}_1\mathbf{k}_2}^\dagger(\tau) \hat{\rho} \hat{A}_{\mathbf{k}_1\mathbf{k}_2}(t)\} \langle \hat{R}_{\mathbf{k}_1\mathbf{k}_2}^\dagger(t) \hat{R}_{\mathbf{k}_1\mathbf{k}_2}(\tau) \rangle_{\text{p-ph}} = (\hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}^\dagger e^{-i(E_{\mathbf{k}_1}-E_{\mathbf{k}_2})t/\hbar} \hat{\rho} \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} e^{i(E_{\mathbf{k}_1}-E_{\mathbf{k}_2})\tau/\hbar} \hat{\rho} - \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} e^{i(E_{\mathbf{k}_1}-E_{\mathbf{k}_2})\tau/\hbar} \hat{\rho} \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}^\dagger e^{-i(E_{\mathbf{k}_1}-E_{\mathbf{k}_2})t/\hbar}) \gamma_{\mathbf{k}_1\mathbf{k}_2}^{\text{ph}} \bar{n}_{\text{ph}}(E_{\mathbf{k}_1} - E_{\mathbf{k}_2}) e^{i\omega_{|\mathbf{k}_1-\mathbf{k}_2|,q_z}(t-\tau)} = (\hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}^\dagger \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} \hat{\rho} - \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} \hat{\rho} \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}^\dagger) \gamma_{\mathbf{k}_1\mathbf{k}_2}^{\text{ph}} \bar{n}_{\text{ph}}(E_{\mathbf{k}_1} - E_{\mathbf{k}_2}) = (\hat{\mathcal{J}}_{\mathbf{k}_1\mathbf{k}_2}^{+\dagger} \hat{\mathcal{J}}_{\mathbf{k}_1\mathbf{k}_2}^+ \hat{\rho} - \hat{\mathcal{J}}_{\mathbf{k}_1\mathbf{k}_2}^+ \hat{\mathcal{J}}_{\mathbf{k}_1\mathbf{k}_2}^{+\dagger} \hat{\rho}).$ Thus, we obtain a Lindblad master equation with the quantum-jump operators presented in the main text in Eqs. (7) and (8).

APPENDIX D: NUMERICAL ACCURACY

Here, we study the influence of the system size L on our results for the spatial coherence. In this framework it is convenient to sample the dispersion with a homogeneous momentum k step instead of a fixed energy step. We model systems of sizes $L = \{5, 10, 15, 20\}$ μm , fixing the spatial step as $dx = 2\pi/k_{\text{max}}$, and k -space extension $k_{\text{max}} = \pi/L$. Therefore, for a given value of L we adjust

the number of states in sampling the dispersion. The corresponding numerical results at $P = 10P_{\text{th}}$ and $T = 20$ K are presented in Fig. 4 and demonstrate a very good convergence of the spatial coherence versus the system size. This confirms the emergence of the long-range order in our approach independent of the system size. Figure 5 shows a detailed study of the correlation function for $L = 20$ μm leading to nine modes sampling the dispersion (see captions).

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