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Influence of load length on short-term ice load statistics in full-scale

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ABSTRACT

This paper studies the frequency of ice loads of varying lengths and the occurrence probability of their magnitudes in full-scale. In these measurements, the four frames were instrumented with shear strain gauges on the Polar Supply and Research Vessel S.A. Agulhas II. The experiments were carried out on first-year ice in the Baltic Sea. An influence coefficient matrix based on analytical and numerical analyses was used to determine the load length in the horizontal direction. Rayleigh separation was used to define the load amplitudes. The measurements show that the ice loading has to be long in order for the shear-load maximum on a single frame to occur. Furthermore, the statistical study showed that the Weibull distribution gives the best fit to the measured loads on a frame. The probability distribution of the ice loads on a frame is exponential-like for short loads and lognormal-like for long loads.

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1. Introduction

Natural resources and shorter sea routes have increased the activities and interest in the Arctic. Thus, the need for ice-capable merchant vessels and safe ships is increasing. The design of safe ships requires knowledge of ice-induced loads on the ship hull. However, the ship-ice interaction process is complex. The process starts with the ice failing through crushing. The contact area increases as the ship penetrates into ice. Then, typically, the ice bends until final failure occurs. In this process, the ice-induced force on the structure increases until the ice fails through shear or bending failure; see Fig. 1. The broken pieces are submerged under the hull as a result of the flow of water induced by the speed of the ship. A bending failure creates a cusp-like breaking pattern and a new breaking cycle begins after the ship reaches the edge of the ice sheet. The complexity of the process arises from the variation in the ice conditions (e.g. strength, thickness, first- or multi-year ice), ship operations (e.g. manoeuvres), and in the location and area of the contact (e.g. ship shoulder, mid-ship). As all the variations are embedded in the full-scale measurements, the knowledge of the ice-induced loads has been gathered by conducting full-scale measurements; see e.g. Ref. [1–6].

Full-scale measurements have shown that the magnitude and area of the ice-induced loading can vary significantly even in short-term measurements; see Fig. 2 and e.g. Ref. [9]. In the design of ship structures, the load is often considered as an average pressure over a certain area, i.e. a load patch; see e.g. Ref. [10]. The definition of an area can be divided into a global and local area. A global area denotes the projection of the structure onto the original shape of the intact ice field. A local area considers smaller sub-regions within the global area that are subjected to high local pressures [11]. The studies with data

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collected on ships and offshore structures have shown a decreasing trend of the pressure as a function of the area for both global and local pressures; see e.g. Ref. [11–15]. However, the extent of global contact area and exposure time can increase the local pressures even though the average global contact pressure decreases [11,16,17].

Riska et al. [18] and later e.g. Taylor and Richard [19] observed from full-scale measurements that the ice-induced pressure is line-like in the first-year ice conditions. This allows the use of load length rather than the area. In this paper, the load length refers to the horizontal length of the contact area. The assumption is also implemented in the Finnish-Swedish Ice Class Rules (FSICR) [20] in the dimensioning of frames. The line load can be obtained by measuring the shear strains on the web of the frame to obtain the force acting on the frame in full-scale and dividing the force by the frame spacing. The line load has also been determined from the observed damage by employing the plastic limit load approach and ice damage; see e.g. Ref. [21]. In FSICR, the magnitude of the line load as a function of the load length is embedded in a coefficient, $c_p$, which takes into account the probability that the full length of the area under consideration will be under loading. The coefficient has been defined on the basis of a set of measurements on board ships in the Arctic and Baltic; see Refs. [2,21,22] for the measurements and [23] for

Fig. 1. A) The breaking pattern around the ship hull [7]. X, Y, and Z denote the global coordinate system common to ships. B) An idealization of the ice edge failure process [8]. $\beta_n$, $h_i$, and $h_i$ denote the frame angle, contact height, and ice thickness, respectively. $x$, $y$, and $z$ indicate the local coordinate system for the local structure common to the solid mechanics.

Fig. 2. The ice-induced load time history measured at the aft shoulder frame of S.A. Agulhas II in the Baltic Sea.
the definition. The line load magnitude as a function of the load length was studied in Refs. [2,24,25]. These studies mainly focused on the spatial line load over the length. As defined by Frederking [13], when a time instance is considered, the pressures on the panels can be grouped by adding the pressure on the adjacent panels together. This will give an average pressure for the different size of areas, note the similarity to the local pressure-area relation [11,13]. Similarly, the spatial line load is defined as

\[ q_{\text{spatial}} = \frac{\sum_{i=1}^{n} F_i}{ns} \]  

where \( n \) = the number of adjacent frames grouped together, \( F_i \) = the force on a frame \( i \), and \( s \) = the frame spacing. However, the actual load length, i.e. the number of adjacent frames under the influence of the external loading at a point in time, has received little attention.

Izumiya [26] studied the concentration and length of the line load in first-year ice in full-scale measurements on board the icebreaker PM Teshio in the Southern Sea of Okhotsk and on board USCGC Polar Sea in Antarctic waters; see Refs. [27–29]. The length of the load on a frame of PM Teshio was determined on the basis of the time the loading was acting on a frame and the speed of the vessel, with the assumption being made that the load length and magnitude were constant in time. The concentration of the loading was studied with the measurements on board USCGC Polar Sea by comparing simultaneous force recordings on the adjacent frames. The results indicated that short load lengths are more common and the highest load recordings are highly concentrated events that only act on a few frames. Hänninen et al. [24] studied the average line load and total load on the instrumented area, i.e. the load on all the frames summed. They presented the results as a function of simultaneously loaded frames. Their results showed that short load cases are more common. Furthermore, the average line load decreased as a function of simultaneously loaded frames. The total force affecting the instrumented area showed no clear trend. Hänninen et al. [24] assumed that this resulted from the low number of long loads with respect to short loads, i.e. the exposure was higher for short loads. However, the load on a frame as a function of the load length was not studied. Numerical simulations have suggested that the probability distribution for the loading on a frame would be different for loading cases that are longer and shorter than the frame spacing; see Refs. [30,31]. However, studies on the true load length employing full-scale measurements are rare. Thus, the probability distribution of the load magnitude on a frame has not been studied with different load lengths.

Therefore the aim of this paper is to determine the frequency of short and long ice loads and to study the probability distributions of the ice load magnitudes observed at a frame. The paper describes thoroughly the methodology of utilizing the influence coefficient matrix employing analytical beam theory and Finite Element Analysis. The methodology is validated with calibration pull; for similar investigations see e.g. Refs. [4,29,32,33]. The frequency of different load lengths and the probability distribution of the ice load on a frame for different load lengths are determined by utilizing the full-scale measurements at the aft shoulder of S.A. Agulhas II in the Baltic Sea. The statistical study employs the Weibull, lognormal, and exponential probability distributions as earlier studies have shown that the short-term ice-induced loads could be represented with these distributions [2,24,35]. As the study employs the short-term ice-induced loads in the first-year level ice conditions, the results are considered applicable for merchant vessels operating in this type of conditions. The study does not consider extreme ice loads in extreme ice conditions that ice breakers encounter.

### 2. Ice-induced load measurements

Fig. 1 presents the global and local coordinate system applied in this paper. The positive directions of the global system are fixed. The local x-coordinate is the parallel and positive to the same direction as the global Z. The local z-coordinate is parallel to the global Y and positive towards the center line of the ship. The local y is parallel to the global X and positive towards the stern at the portside, and towards the bow at the starboard side. In the ice-breaking process, the ice-structure interaction induces a load on the side platting. The load can extend over several frame spacings in the horizontal direction or act only on a single frame spacing. In both cases it is considered that the transverse frames carry the majority of the loading. On the basis of the superposition principle, a long loading can be considered to consist of sub-loads acting on individual frames; see Fig. 3. As a result of the interaction between the frames, the magnitude of measured loads at the middle frames depends on the load length. For long loads of several frames in length, the measurements at the middle frame remain approximately constant. However, at the outer frames the load is reduced. In the example Fig. 3, if the length of the external loading is taken as the responding length, then the load length is nine frame spacings. However, in reality it is only about five frame spacings. Thus, the internal redistribution of the loading should be taken into account in order to define the actual load length.

As the ice load acts on a frame as an external shear force and is concentrated in the vertical direction in first-year level ice conditions [18], the ice load can be determined measuring shear strains at the ends of the frame. Fig. 4A presents a frame system under a narrow ice-induced pressure load, \( F_{\text{ice}} \). Fig. 4B presents a part of the structure cut out under the ice load with the forces acting to the z-direction and moments around the x and y-axis. In Fig. 4B, the internal forces and moments are separated to the parts acting on the frame and plate part of the structure. The forces and moment affecting on the plate part are forces and moments per unit length. The forces and moments affecting the frame are point forces and moments. Taking the equilibrium conditions for Fig. 4B, and assuming a pure bending for the plate and assuming the bending stiffness of the frame is significantly larger than that of the plate, the three equilibrium conditions can be obtained for the structure.
Here, the assumption of pure bending uncouples the plate stress resultants, $M_x$ and $M_y$, and sets $M_{xy} = 0$. The detailed derivation of Equation (2) from the equilibrium is presented in Appendix A. Thus, the ice-induced shear force acting on a particular frame can be determined by measuring the difference in the shear strain occurring at the ends of that frame, $D Q_x$.

However, as the plate has some stiffness, in order to determine the total ice load, the response of the adjacent frame has to be measured to account the shear force distributed to the adjacent frames, $D Q_y$. In general, when Finite Element Analyses are used to determine the load-carrying mechanism, such split of forces is not required. Here the split is presented so that we can parametrize the problem.

As the shear strains are measured at the frame web surface, the torsion-induced rotation and bending-induced deflection of the frame affect the shear strain-force relation; see Fig. 5. The loaded frame experiences only a displacement in the loading direction, but the adjacent frames undergo a displacement and rotation. The shear strain resulting from the deflection is a function of the z-coordinate and remains constant over thickness (y-coordinate). However, the shear strain resulting from the torsion is a function of the y-coordinate and varies linearly over thickness; see Fig. 5. As the resulting shear strain is a sum of these actions, the side of the frame affects the measured loads.

### 3. Determination of the load length from the shear strain measurements

The distribution of the loading $[F]$ and shear strain $[\gamma]$ can be accounted for in the measurements with an influence coefficient matrix $a$ defined from the relation

$$\{F\} = [a]\{\Delta \gamma\}$$

(3)

The influence coefficient matrix is determined using the inverse influence coefficient matrix $c$, $a = c^{-1}$:

$$[e_{ij}] = \Delta \gamma_i (F_j)$$

(4)
Fig. 4. A) A frame system under a narrow ice load, $F_{\text{ice}}$. L, s, t, t, and h denote the frame length, frame spacing, plate thickness, web thickness and frame height. B) A part of the structure cut out from the system with the internal shear forces, $Q_x$ and $Q_y$, bending moments, $M_x$ and $M_y$, and twisting moments, $M_{xy}$ and $M_{yx}$. Subindexes PL and FR refer to the plate and frame part of the structure, respectively.
where $\Delta \gamma_i (i = 1 \ldots n)$ = the shear strain difference measured on the frame $i$, $n$ = the number of instrumented frames, and $F_j$ = the external force exerted on the frame $j$. The coefficients of the matrix are calculated with the unit load principle, i.e. each frame is loaded separately. The number of rows and columns equals the number of instrumented frames. The diagonal terms define the force-strain relation of the frame under loading and the off-diagonal terms determine the response of the adjacent frames. The length of the external loading affecting the instrumented frames is obtained this way. The influence coefficient matrix is determined here using numerical (Finite Element) and analytical methods. The matrix based on FEA is considered more accurate. However, the derivation of analytical-based matrix is presented to get physical insight to the different terms. Simple grillage analogy is used to model the load redistribution between the frames. There the plate is considered as a continuous transverse beam. This produces soft response of the plate between the stiffeners, considers torsion resistance of stiffeners and thus produces estimate of shear force decay from loaded frame.

3.1. FEA

The finite element model, presented in Fig. 6, was built using linear plate elements in FEMAP and the numerical calculations were conducted with NX Nastran (version 10.3.1). The finer mesh size was 0.005 m*0.005 m in the region of interest. The surrounding structure had a coarser mesh, with a mesh size 0.05 m*0.05 m. Rigid boundary conditions were applied to all the edges where the actual structure continued by constraining all degrees of freedom.
3.2. An analytical grillage model

The analytical model is constructed based on a grillage analogy; see e.g. Ref. [37]. It is assumed that the loading, \( Q \), caused by external loading, \( F_{\text{EXT}} \), acts directly on the frame and induces bending to it. The adjacent frames, having the same direction, are assumed to be connected to this beam by transverse beam, which models the plate between the frames. This system forms the grillage. The frames support the transverse beam, i.e. the plate, with vertical loads, \( Q \), and torsions, \( T_i \); see Fig. 5. The beams are considered long and slender and they behave according to the Euler-Bernoulli kinematics.

We consider odd number, \( n \), of frames, with frame spacing \( s \) giving the total breadth \( B = ns \). A web thickness is \( t_w \) and a web height \( h_w \). The grillage length is equal to the frame length \( L \). The plate thickness is \( t_p \). The grillage is assumed to be simply supported to model the maximum load distribution to adjacent frames. The plating, i.e. the transverse continuous beam, is considered long and slender and they behave according to the Euler-Bernoulli kinematics.

Equation (7) can be derived from the equilibrium conditions for the plate-frame system presented in Fig. 4B, see Appendix A for a detailed derivation. The derivation assumes a pure bending that uncouples the moments in the x and y direction. Furthermore, the assumption of pure bending can be applied for lateral loading when the deformation is small, based on assumption of small curvature of the plate in the x-direction. However, the assumption of small curvature is rough, based on assumption of small curvature of the plate in the x-direction. The authors highlight here, that the presented approximation is rough, based on assumption of small curvature of the plate in the x-direction. Furthermore, the stiffness of the frame is assumed significantly larger than that of the plate. The torsion in the frame \( T_i \), related to the rotation can be expressed as \( \Theta_{\text{FR},i} = T_i/C_i \), where \( \Theta \) is the unit angle of twist, \( G \) is the shear modulus, and \( C \) is the torsional stiffness. The unit angle of twist for a frame is obtained by assuming that the ends of the frames do not rotate and the rotation of the mid-span equals that of the plating, i.e. \( \Theta_{\text{FR},i} = 2\theta_{\text{pl},i}/L \). Thus, we obtain

\[
\theta_{\text{pl},i} \left( \frac{L}{2} \right) = \frac{T_i L}{2GC}
\]

(6)

As the frames are connected to the plating, the displacement and rotation of the frame and plate have to be equal at the location of the frame, i.e. \( w_{\text{pl},i}(y = is, x = L/2) = w_{\text{FR},i}(x = L/2) \). By solving the plate displacements and rotations at the frame locations and setting these to be equal with the rotations and displacements of the corresponding frame at the mid span of the frame, the shear forces and torsions affecting the frames can be solved.

Earlier studies have suggested that the force diminish rapidly and is negligible after two frames spacing away from the loaded frame; see e.g. Ref. [33]. Thus, the procedure presented above is applied to a frame system consisting of five frames and plating. The loading on the loaded frame is taken as a point load and as a line load on the adjacent frames. Equation (7) presents the loading on the loaded frame \( Q_{\text{mid}} \), and the loadings and torsions on the adjacent frames, \( q_{\text{mid}-1}, q_{\text{mid}-2}, T_{\text{mid}-1}, \) and \( T_{\text{mid}-2} \), respectively.

\[
Q_{\text{mid}} = \frac{25k^2(20 - 126u + 125u^2) + 320k(10 - 81u + 145u^2) + 4096(1 - 9u + 19u^2)}{k_0}F_{\text{EXT}}
\]

\[
q_{\text{mid}-1} = \frac{8k[15k(4 - 30u + 33u^2) + 64(2 - 18u + 35u^2)]}{k_0}F_{\text{EXT}}
\]

\[
q_{\text{mid}-2} = \frac{8k[5k(4 - 30u + 37u^2) + 192(2 - 6u + 6u^2)]}{k_0}F_{\text{EXT}}
\]

\[
T_{\text{mid}-1} = \frac{2k(8 + 5k)[5k(-1 + u) + 32(-1 + 5u)]}{k_0}F_{\text{EXT}}
\]

\[
T_{\text{mid}-2} = \frac{k(8 + 5k)[5k(-2 + 3u) - 64u]}{k_0}F_{\text{EXT}}
\]

(7)

The parameters \( k \) and \( u \) are related to the load distribution and rotation, respectively, as follows:
\[ k = \frac{DL^4}{EI_{IK}S^2} \quad \text{and} \quad u = \frac{DL^2}{GC} \]  

(8)

The nominator is

\[ k_0 = 25k^3 \left( 4 - 6u + u^2 \right) + 5k^2 \left( 356 - 2166u + 2129u^2 \right) + 64k \left( 82 - 645u + 1093u^2 \right) + 4096 \left( 1 - 9u + 19u^2 \right) \]  

(9)

It can be easily seen that as a special cases, if \( k = u = 0 \) (i.e. bending, \( EI \), and torsional stiffness, \( GC \), infinite), all the force is taken by \( Q_{\text{mid}} \). When only \( u \) is zero, the load decays from the loaded frame to the adjacent frames depending on the parameter \( k \). Fig. 7 presents the ratio of the load taken by the adjacent frames and the loaded frame as a function of \( k \). Thus, Fig. 7 represents the behavior of the off-diagonal terms in the coefficient matrix with respect to the diagonal terms as a function of \( k \). Fig. 7 also shows that the ratio of \( q_{\text{mid}-1L}/Q_{\text{mid}} \) and \( q_{\text{mid}-2L}/Q_{\text{mid}} \) approach values of 0.9 and 0.3, respectively, when \( k \) increases. Fig. 7 highlights the need to account the response of the adjacent frames when the load magnitude and length is determined. As can be noted the ratio of the loading carried by the adjacent frames increases rapidly as a function of \( k \).

The shear strains on the frames related to the torsion and shear force are solved from the closed form solution

\[
\gamma_{xz}(x, y, z; Q_{x,i}; T_{x,i}; F_{\text{EXT}}) = \gamma_{xz,w}(x, z; Q_{x,i}; F_{\text{EXT}}) + \gamma_{xz,b}(y, T_{x,i}; F_{\text{EXT}}) = \frac{Q_{x,i}(x, F_{\text{EXT}})2y_1}{E_{lyy}b_{CS}} + \frac{T_{x,i}(F_{\text{EXT}})2y_1}{GC}
\]  

(10)

where \( S_d(z) \) is the first moment of area, and \( b_{CS} = \) the thickness of the cross-section. I denotes the local y-coordinate of the frame cross-section. The difference in the shear strain between the ends of the frame, \( \Delta \gamma \), is determined by calculating the shear strain occurring at the lower and upper parts of the frame with Equation (10) and then calculating the difference. The coefficients of the inverse influence coefficient matrix are calculated with the unit load principle, giving

\[
\begin{bmatrix}
  c_{i,1} & c_{i,2} & \ldots & c_{i,n} \\
  c_{i+1,1} & c_{i+1,2} & \ldots & c_{i+1,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n,1} & c_{n,2} & \ldots & c_{n,n}
\end{bmatrix}
\begin{bmatrix}
  1 \\
  0 \\
  0 \\
  0
\end{bmatrix}
= 
\begin{bmatrix}
  \Delta \gamma_1(Q_i(F_{\text{EXT}}), T_i(F_{\text{EXT}})) \\
  \Delta \gamma_{i+1}(Q_{i+1}(F_{\text{EXT}}), T_{i+1}(F_{\text{EXT}})) \\
  \vdots \\
  \Delta \gamma_n(Q_n(F_{\text{EXT}}), T_n(F_{\text{EXT}}))
\end{bmatrix}
\]

(11)

The influence coefficient matrix \( a \) is the inverse of the matrix \( c \).

4. Full-scale measurements

4.1. Instrumentation

The hull of S.A. Agulhas II was instrumented with strain sensors at the bow, bow shoulder, and aft shoulder [39]. The upper and lower parts of the frames were instrumented with shear strain gauges following the principles from the previous chapter. We limit our focus to the aft shoulder. The frame system in the instrumented area consisted of five transverse frames limited by web frames in the horizontal direction and decks in the vertical direction; see Fig. 6B. Four adjacent frames were instrumented. Starting from the one closest to the bow, the gauges SS16 and SS17 were mounted on the frame #41, SS18 and SS19 on #40½, SS20 and SS21 on #40, and SS22 and SS23 on #39½. The gauges were mounted on the bow side of the frames #41, #40, and #39½ and the aft side of the frame #40½. The gauges with even numbers were installed on the upper part of the frames and the ones with odd numbers on the lower part of the frames. The gauge locations were 0.3 m from the end of the frame; see \( d_{up} \) and \( d_{low} \) in Fig. 6A. Table 1 presents the distances from the centreline of the gauge to the tip of the web; see \( d_{web} \) in Fig. 6A. The hole in the frame is 0.4 m above the lower end of the frame. The geometry and the material parameters of the instrumented frames were: the length of the frames = 1.4 m; the frame spacing = 0.4 m; the web height = 0.2 m; the web thickness = 0.019 m; the hull plating thickness at the lower part of the frame = 0.021 m and at the upper part of the frame = 0.020 m; Young’s modulus, \( E_c = 209 \, \text{GPa} \); and Poisson’s ratio, \( \nu_c = 0.3 \).

4.2. The influence coefficient matrices

The influence coefficient matrices were determined as described in Chapter 3. The effective breadth is taken as the design curve for a uniform line load as defined by Schade [40,41]. According to Schade [40] the solution for the point load will approach the solution of the of the uniform line load when moved away from the loading. If the effective breadth is taken for the point load, \( k \) is approximately 8% higher for the presented case due to the decrease in the stiffness of the frame. The torsional stiffness is calculated by employing the membrane analogy for thin rectangular cross-sections; see e.g. Ref. [42].

Applying 1 kN external loading, \( F_{\text{EXT}} \), and substituting the structural parameters of the stern shoulder of S.A. Agulhas II to Equation (7), the following was obtained: \( Q_{\text{mid}} = 803.7 \, \text{N}, q_{\text{mid}-1L} = 141.5 \, \text{N}, q_{\text{mid}-2L} = -53.8 \, \text{N}, T_{\text{mid}-1} = 1.90 \, \text{Nm}, T_{\text{mid}-2} = -0.299 \, \text{Nm} \). The shear force and torsion on the frame three frame spacing away are taken as \( q_{\text{mid}-3L} = (F_{\text{EXT}} - Q_{\text{mid}-2L} - 2q_{\text{mid}-1L} - 2q_{\text{mid}-2L})/2 \) and \( T_{\text{mid}-3} = 0 \) for the analytical solution. This approximation assumes that no internal shear force is observed.
Fig. 6. A) A model of an instrumented frame. The stars denote gauges, $d_{low}$ and $d_{up}$ are the distances of the gauge from the deck, and $d_{web}$ indicates the distance of the centreline of the gauge from the tip of the web. B) The whole FE model employed in the study. The blue lines indicate where the fixed boundary conditions were applied [36].
beyond the third adjacent frame and the torsion on that frame is negligible. This overestimates the effect of the third adjacent frames, but it is considered minor as the forces are small. As can be noted from the matrices, the values of the diagonal terms are approximately the same, but the off-diagonal terms have some difference. This means that the analytical matrix, $a_A$, places more weight on the adjacent frames than the FEA matrix, $a_F$. Accounting properly the plate’s ability to transfer load to the x-direction in the grillage model would reduce the difference of the defined matrices. The matrices are not symmetric with respect to the diagonal as the gauges were mounted on the bow side of the frame #40½ and on the stern side of the frames #41, #40, and #39½. The small differences in the diagonal terms are due to the different distances of the gauges from the tip of the frame web; see Table 1.

$$a_A = \begin{bmatrix} 321.1 & 20.34 & 17.48 & -6.638 \\ 26.87 & 350.4 & -94.59 & -4.564 \\ -4.761 & -95.41 & 338.2 & 25.25 \\ -6.707 & 29.61 & -95.09 & 307.3 \\ \end{bmatrix} *10^3 \text{kN}$$

$$a_F = \begin{bmatrix} 320.4 & 16.18 & -2.793 & 0.6951 \\ 13.17 & 329.0 & -58.54 & 4.771 \\ 2.568 & -62.83 & 332.5 & 14.46 \\ -0.9794 & 18.34 & -57.23 & 310.1 \\ \end{bmatrix} *10^3 \text{kN}$$

4.3. Comparison to the calibration pull

The measurement system was tested by pulling each frame at a time with a winch. The pulling force was measured with a calibrated load sensor and the strains in the frames with the instrumentation; see Fig. 8. The shackle-shackle and the hook-lifting eye connections ensured that no torsion occurred on the load sensor during the calibration pull. The sampling rates for the load sensor and the strain gauge instrumentation were 123 Hz and 200 Hz respectively. The pulling force was increased in steps to the upper limit of the measurement range of the load sensor. Fig. 8 presents the measured force and shear strains on the frame #40 during the calibration pull. The response of the frame to the pulling is clearly seen from the strain measurements [36]; see Fig. 8. Table 1 presents the measured pulling force and shear strains on the frames.

Table 1

<table>
<thead>
<tr>
<th>Sensor</th>
<th>Frame</th>
<th>#41</th>
<th>#40½</th>
<th>#40</th>
<th>#39½</th>
</tr>
</thead>
<tbody>
<tr>
<td>Name</td>
<td>SS16</td>
<td>SS17</td>
<td>SS18</td>
<td>SS19</td>
<td>SS20</td>
</tr>
<tr>
<td>Distance to the tip of the web [m]</td>
<td>0.120</td>
<td>0.125</td>
<td>0.124</td>
<td>0.117</td>
<td>0.125</td>
</tr>
<tr>
<td>Calibration pull</td>
<td>Measured strain [m strain]</td>
<td>0.22</td>
<td>0.27</td>
<td>6.11</td>
<td>-4.56</td>
</tr>
<tr>
<td>Measured force [kN]</td>
<td>15.55</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FEA</td>
<td>Modeled [m strain]</td>
<td>0.00</td>
<td>0.03</td>
<td>3.96</td>
<td>-4.35</td>
</tr>
<tr>
<td>Force on the frame [kN]</td>
<td>0.03</td>
<td>0.80</td>
<td></td>
<td></td>
<td>15.07</td>
</tr>
<tr>
<td>Total force [kN]</td>
<td>16.21</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Analytical</td>
<td>Force on the frame [kN]</td>
<td>0.96</td>
<td>-0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total force [kN]</td>
<td>13.94</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

Fig. 7. The ratio of loading on the loaded frame and adjacent frames as a function of $k$ when $u$ is zero.

4.3. Comparison to the calibration pull

The measurement system was tested by pulling each frame at a time with a winch. The pulling force was measured with a calibrated load sensor and the strains in the frames with the instrumentation; see Fig. 8. The shackle-shackle and the hook-lifting eye connections ensured that no torsion occurred on the load sensor during the calibration pull. The sampling rates for the load sensor and the strain gauge instrumentation were 123 Hz and 200 Hz respectively. The pulling force was increased in steps to the upper limit of the measurement range of the load sensor. Fig. 8 presents the measured force and shear strains on the frame #40 during the calibration pull. The response of the frame to the pulling is clearly seen from the strain measurements [36]; see Fig. 8. Table 1 presents the measured pulling force and shear strains on the frames.

$$a_A = \begin{bmatrix} 321.1 & 20.34 & 17.48 & -6.638 \\ 26.87 & 350.4 & -94.59 & -4.564 \\ -4.761 & -95.41 & 338.2 & 25.25 \\ -6.707 & 29.61 & -95.09 & 307.3 \\ \end{bmatrix} *10^3 \text{kN}$$

$$a_F = \begin{bmatrix} 320.4 & 16.18 & -2.793 & 0.6951 \\ 13.17 & 329.0 & -58.54 & 4.771 \\ 2.568 & -62.83 & 332.5 & 14.46 \\ -0.9794 & 18.34 & -57.23 & 310.1 \\ \end{bmatrix} *10^3 \text{kN}$$

4.3. Comparison to the calibration pull

The measurement system was tested by pulling each frame at a time with a winch. The pulling force was measured with a calibrated load sensor and the strains in the frames with the instrumentation; see Fig. 8. The shackle-shackle and the hook-lifting eye connections ensured that no torsion occurred on the load sensor during the calibration pull. The sampling rates for the load sensor and the strain gauge instrumentation were 123 Hz and 200 Hz respectively. The pulling force was increased in steps to the upper limit of the measurement range of the load sensor. Fig. 8 presents the measured force and shear strains on the frame #40 during the calibration pull. The response of the frame to the pulling is clearly seen from the strain measurements [36]; see Fig. 8. Table 1 presents the measured pulling force and shear strains on the frames.
Fig. 8. The set-up in the calibration pulls and the measured pulling force and shear strains on the frame #40 [36].
As Table 1 shows, both the FEA and analytical method indicate clearly that the external loading is on the frame #40 and the calculated magnitude on the loaded frame is close to the measured pulling force. In addition, the shear strains at the gauge locations modelled with FEA are close to the measured shear strains.

The FEA results indicate only minor positive external forces on the adjacent frames, the frames #40½ and #39½. The analytical results indicate minor but negative loading on the adjacent frames. This is due to the overestimation of the torsional term in the determination of the shear strain. The total loading affecting the frame system is underestimated by 10% by the analytical method and overestimated by 4% by the FEA. As the system is scaled to measure significantly higher external loadings, over 1000 kN, and the loadings on the adjacent frames are minor, the results are considered to be good.

4.4. The determination of the load length

The measured shear strains are converted into ice loads with the influence coefficient matrix; see Chapter 3. The load peaks, i.e. load events, are identified from the time history employing a Rayleigh separation, which is based on comparing the minimum and maximum values. If the minimum between two maxima is smaller than the lower maximum multiplied by the separator value, $\xi$, the maxima are considered to be separate cases. Fig. 9 illustrates the Rayleigh separation when the value of the separator is $\xi = \frac{1}{2}$ and the threshold is 10 kN/m. As the systems are scaled to measure over 1000 kN loads, the threshold is commonly applied to eliminate the noise in the measurements. Introducing a threshold may discard smaller ice loads. However, it is considered more advantageous to discard smaller ice loads than include noise in the measurements. The red circles in Fig. 9 indicate which peaks are identified as separate load events. After the loading events are identified from the time history of a single frame, the loading length is taken as the number of adjacent frames exceeding the threshold at the same instant in time. Naturally, the extent of the instrumentation imposes limitations on the measurements. If the loading extends to the frames that are at the boundaries of the instrumented area, it is possible that the loading continues to an area that is not instrumented.

5. Load length in full-scale measurements

5.1. Description of the ice trial and data processing

The ice trial with S.A. Agulhas II was conducted in the northern part of the Baltic Sea on March 21–22, 2012. During the ice trials, the ship operated in ice conditions during the daytime. Approximately 24 h’ worth of data was collected from the operations in ice conditions. Fig. 10 presents the general ice conditions and the route of the ship on March 21. The general ice conditions were observed visually. The solid thickness of the broken ice pieces and total thickness of the intact ice were measured with a stereo camera system and an electromagnetic device, respectively; see Refs. [44] and [45] for detailed descriptions of the systems. Fig. 11 presents the measured thickness distributions during the ice trial.

The shear strains on the frames were recorded continuously with a frequency of 200 Hz and converted into ice-induced loads utilizing the influence coefficient matrix determined with FEA; see Fig. 2. The loading events were identified from the time history by employing a Rayleigh separation with $\xi = \frac{1}{2}$. Three thresholds were applied in the study to determine the effect of chosen threshold to the number of loadings and the magnitude of loading events—10 kN, 20 kN, and 30 kN. The load events were identified for the two middle frames, #40½ and #40, separately. Fig. 12 presents an example of loading events that were four and one frame spacing long; see the time instances from 187 to 189 and from 270 to 272, respectively.
Fig. 10. Ice conditions in the Bay of Bothnia and the route of the ship on 21 March [39].

Fig. 11. A) A histogram (pdf) of EM thickness values from the ice trial with a Normal-Exponential distribution fitted to the data. B) A histogram of stereo camera thickness from the ice trial and a Normal distribution fitted to the data, modified from Ref. [46].

Fig. 12. A wide and a narrow ice-induced load on the left and right, respectively, travelling from the foremost frame to the aftmost, from the frame #41 to the frame # 39½, when the ship moves forward.
5.2. The results and analysis

Fig. 13 indicates that narrow loading cases are more common than wide cases when the threshold is 10 kN. However, the number of narrow loading cases decreases significantly when the threshold increases, but the number of wide loading cases remains approximately the same. This suggests that short loadings are commonly small in magnitude, whereas long loadings are higher in magnitude. The higher number of one frame spacing wide loadings for the frame #40 than for #40½ is not really known. The influence coefficient matrix was determined for the loading cases that act directly on the frame. Thus, the torsional term is slightly incorrect when the effect of the loading is slightly off the centreline of the frame. It is possible that due to the contact location of the ice load, the instrumentation layout and the defined influence coefficient matrix, the loading on the frame #40 is more often slightly overestimated and the loading on #40½ is underestimated. However, the simulations have suggested that due to the ice-breaking process, some frames are more often in contact with ice than other frames in different ice conditions [31]. Thus, it can be that the ice conditions have resulted in the frame #40 to be more often in contact with ice than the frame #40½. These explanations relate to the studies related to the contact location of the ice and are, therefore, left for the future studies.

Fig. 14 presents the measured maximum load on a single frame and average loading on the loaded frames as a function of the measured load length. Fig. 14 confirms that the load magnitude is smaller for the short loads. The highest measured loading on the frames #40 and #40½ is less than 150 kN and is around 450 kN for loads that are one and four frame spacings.
wide, respectively. This indicates that the loading on the hull has to be long enough for the highest maximum on a single frame to occur. On the contrary, Izumiyama [26] suggested that the higher loads act on shorter load lengths. However, the load length was defined differently and the data contained only a few long loading cases. The studies on the local pressure have suggested that an increasing global pressure area increases the exposure of the local pressure by increasing the time the load acts on the structure, which increases the probability of greater local pressures occurring [11,16,17]. Fig. 14 also shows that the maximum average loading on the frames does not increase after the loading is two or more frame spacings long. As a comparison, the spatial line load as a function of the load length was also calculated for the identified loading events with Equation (1); see Fig. 15. As can be seen from Fig. 15, the spatial line load decreases as a function of load length, as earlier studies have shown; see e.g. Ref. [25].

After the loading events were categorized on the basis of the load length and threshold for the frames #40 and #40½ — see Fig. 13 — the probability distributions of the load magnitude were studied for each category. This was conducted by fitting a probability distribution to the measurement data. As the data processing involved a threshold, the 3-parameter Weibull, 3-parameter lognormal, and 2-parameter exponential distributions are utilized in the study.

\[
\begin{align*}
 f_W(x; \beta, \eta, \gamma) &= \frac{\beta}{\eta} \left( \frac{x - \gamma}{\eta} \right)^{\beta - 1} e^{-\left( \frac{x - \gamma}{\eta} \right)^{\beta}} \\
 f_E(x; \lambda, \gamma) &= \lambda e^{-\lambda(x - \gamma)} \\
 f_{LN}(x; \mu, \sigma, \gamma) &= \frac{1}{(x - \gamma)\sigma \sqrt{2\pi}} e^{-\frac{(\ln(x - \gamma) - \mu)^2}{2\sigma^2}}
\end{align*}
\]

(13)

\(\gamma\) is the location parameter, which equals the applied threshold. The shape and scale parameters of the Weibull distribution, \(\beta\) and \(\eta\), the rate parameter of the exponential distribution, \(\lambda\), and the location and scale parameters of the lognormal distribution, \(\mu\) and \(\sigma\), were estimated for each category with the MATLAB 2015b built-in probability distribution fitting tool and validated with the Minitab 17 fitting tool for two cases. Fig. 16 shows an example of the fit of the distributions to the ice loads measured on the frame #40½ for different load lengths with a 20 kN threshold. The fit of the distributions to the measured data was estimated visually and tested with three goodness-of-fit tests, the Anderson-Darling, Kolmogorov-Smirnov, and Chi-Square tests. The Kolmogorov-Smirnov and Chi-Square tests test the general fit, whereas the Anderson-Darling test gives more weight to the tail of the distribution. The results from the goodness-of-fit tests are presented in Table 2.

The goodness-of-fit test and visual observations show that the Weibull probability distribution models the measured loads the best; see Table 2 and Fig. 16. However, the lognormal and exponential distributions can be accepted in some cases. This is in accordance with earlier findings; see Ref. [35]. In addition, Table 2 and Fig. 16 show that for the short loading cases the load distribution is more exponential-like, whereas the shape of the distribution is more lognormal-like for the longer loading cases. Similar results were obtained by Su et al. [31] for simulated loadings. The Weibull probability distribution is able to model the short and long load distributions as its shape is exponential-like when the shape parameter is smaller than one, \(\beta < 1\); it equals the exponential distribution when \(\beta = 1\) and its shape is lognormal-like when \(\beta > 1\).

Fig. 15. The spatial line load as a function of load length, see Equation (1). The frame in the legend indicates from which frame the load case was identified with Rayleigh separation.
6. Conclusions

The paper presented the principles of ice-induced load measurements based on shear strain measurements and the determination of the influence coefficient matrix. On the basis of the approach, the load lengths in the full-scale measurements were determined. Finally, the probability distributions of measured full-scale ice load magnitudes on a frame for different load lengths were determined. The study with the calibration pull showed that the magnitude, width, and location of the external loading affecting the structure can be determined with the influence coefficient matrix. The width and location in this context mean the number of frames directly subjected to external loading and the location of the loaded frames, respectively. The analytical grillage model showed that the coefficients of the matrix are related to the shear force and torsion on the frame. If the gauges are mounted on different sides of the frame, the torsional terms can be neglected. This was not discussed thoroughly in earlier studies.

The study showed that short loading cases are more common than long cases, which is in line with earlier results; see Refs. [24,26]. The results showed that the number of small loads was significantly higher for a frame on which the strain gauges had been mounted on its bow side in comparison to a frame with gauges on the stern side. The reason for this is not fully known as it can be related to the instrumentation layout or result from the actual breaking process as indicated by simulations [31]. Thorough explanation is left for future studies. The analysis showed that the highest load on a frame increases as a function of load length. This indicates that the loading has to be wide enough in order for the maximum load on a frame to occur. This is in line with earlier studies related to the local and global pressures, where the possible increase in the local pressure as a function of the global area was explained by greater exposure [16,17].

The statistical study showed that the Weibull probability distribution gives the best fit to the probability distribution of the measured load magnitude on a frame. This is in accordance with earlier studies; see Refs. [34,35]. Furthermore, it was observed that the distribution of measured loads on a frame is exponential-like for narrow loads and lognormal-like for wide loads. The numerical simulations have given similar results, but earlier studies did not observe this in full-scale; see Ref. [31]. A physical reason for this should be studied in the future. The results show that the load length varies significantly during the

Fig. 16. The measured load distributions on the frame #40/½ with a 20 kN threshold for different load lengths and the fitted probability distribution functions of the Weibull, lognormal, and exponential distributions. Note the different ranges on the y and x-axes.
measurements. As the strain-force relation is determined from an assumed loading condition, a part of the variation present in the full-scale ice load on the ship hull measurements may be due to the variation in the actual load length. The present paper focused on the case of first-year level ice. Therefore, the authors emphasize that the knowledge on the actual conditions when the extreme loads occur should be gathered. This is left for future work.

Acknowledgements

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Appendix A. Shear forces and moments in terms of plate deflection

In Fig. 4B, the forces and moments affecting the plate part are forces and moments per unit length. The forces and moments affecting the frame are point forces and moments. The equilibrium condition for the plate-frame system presented in Fig. 4B are

<table>
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<tr>
<th>Frame</th>
<th>Width</th>
<th>Anderson-Darling</th>
<th>Kolmogorov-Smirnov</th>
<th>$\chi^2$</th>
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</tr>
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</table>
The moments due to the external load and the changes in the forces \( Q_x \) and \( Q_y \) can be neglected as they are small quantities of higher order than those retained [36, pp. 81]. Equation (A.1) simplifies to

\[
\begin{align*}
F_{\text{Ice}} + & \frac{\partial Q_y}{\partial y} \frac{\partial y}{\partial y} dxdy + \frac{\partial Q_s}{\partial x} \frac{\partial y}{\partial x} dxdy + \frac{\partial Q_F}{\partial x} \frac{\partial y}{\partial x} dxdy = 0 \\
Q_x \frac{\partial y}{\partial x} dxdy + & Q_s \frac{\partial y}{\partial x} dxdy = -\frac{\partial M_x}{\partial x} dxdy + \frac{\partial M_s}{\partial x} dxdy + \frac{\partial M_F}{\partial x} dxdy - \frac{\partial M_y}{\partial y} dxdy \\
Q_y \frac{\partial y}{\partial x} dxdy = & D \left( \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right)
\end{align*}
\]

(A.2)

The following relations for the bending moments are valid for classical Euler-Bernoulli beam and Kirchhoff plate theories:

\[
\begin{align*}
M_s &= -EI \frac{\partial^2 W}{\partial x^2} \\
M_s &= -D \left[ \frac{\partial^2 W}{\partial y^2} + v \frac{\partial^2 W}{\partial x^2} \right]
\end{align*}
\]

(A.3)

and the corresponding shear forces are

\[
\begin{align*}
Q_x &= \frac{\partial M_s}{\partial x} = -EI \frac{\partial^3 W}{\partial x^3} \\
Q_x &= \frac{\partial M_s}{\partial y} + \frac{\partial M_s}{\partial x} = D(1 - v) \frac{\partial^3 W}{\partial x^2 \partial y} - D \left( \frac{\partial^3 W}{\partial x^2 \partial y} + v \frac{\partial^3 W}{\partial x^2 \partial y} \right) \\
Q_y &= \frac{\partial M_s}{\partial y} + \frac{\partial M_s}{\partial x} = D(1 - v) \frac{\partial^3 W}{\partial x \partial y^2} - D \left( \frac{\partial^3 W}{\partial x \partial y^2} + v \frac{\partial^3 W}{\partial x \partial y^2} \right)
\end{align*}
\]

(A.4)

Assuming that the bending stiffness of the frame is significantly larger than that of plate, gives \( EI/s \gg D \) and further

\[
\begin{align*}
F_{\text{Ice}} = & -\frac{\partial Q_y}{\partial y} \frac{\partial y}{\partial y} dxdy - \frac{\partial Q_s}{\partial x} \frac{\partial y}{\partial x} dxdy \quad \text{and} \quad Q_F \frac{\partial y}{\partial x} dxdy = \frac{\partial M_s}{\partial x} dxdy - \frac{\partial M_y}{\partial y} dxdy
\end{align*}
\]

(A.5)
Assuming pure bending means that, we can assume that $\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial x^2} = 0$. [38, pp. 42–45]. This crude assumption uncouples the plate stress resultants $M_x$ and $M_y$, sets the $M_{xy} = 0$, and simplifies the stiffened plate bending problem considerably. The result in terms of moments is

$$
M_y = M_{y,PL} = -D \frac{\partial^2 w}{\partial y^2}, 
M_x = M_{x,FR} = -EI \frac{\partial^2 w}{\partial x^2}, 
M_{xy,PL} = M_{xy,FR} = 0
$$

(A.6)

and furthermore in terms of shear forces

$$
Q_x = Q_{x,PL} = \frac{\partial M_{y,PL}}{\partial y} = -D \frac{\partial^3 w}{\partial y^3}, 
Q_y = Q_{x,FR} = \frac{\partial M_{x,FR}}{\partial x} = -EI \frac{\partial^3 w}{\partial x^3}
$$

(A.7)

The vertical equilibrium is then

$$
F_{Ice} + \Delta Q_{y,PL} + \Delta Q_{x,FR} = 0
$$

(A.8)

As the moment in the y-direction is observed over the length of the structure in cylindrical bending, it obtains a form

$$
M_y(y) = -DL_e \frac{\partial^2 w(y)}{\partial y^2}
$$

for the plate in transverse direction to the stiffener. It should be noted that the effective length can be modified to account reduced length of the cylindrical bending; here it is assumed to equal the real length of the plate. Furthermore, it should be noted that the second moment of inertia for the frame in the moment Eq.(A.7) accounts both the plate and the frame.

References
