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MODIFIED YIELD-LINE THEORY APPROACH TO DETERMINE SPRAYED CONCRETE FLEXURAL CAPACITY

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Yield-Line Theory (YLT) is an upper bound method for determining the bending capacity of a thin ductile slab. YLT has been shown to overestimate the flexural capacity of sprayed concrete slabs in some cases (e.g. Uotinen et al. 2013). In this paper, the EFNARC three point bending test on square panel with notch is evaluated using the original YLT, modified YLT and a fracture mechanics approach, and compared to a finite element solution. A small, 20 sample, comparative testing campaign is planned to be carried out in Aalto University in fall 2014. The emphasis is on the cracking capacity of the sprayed concrete, as cracked structures are not accepted and already require renovation. The solutions for each approach are shown and precomputed for the selected C35/45 comparison concrete strength class. The notched slab test was published in 2011 and not many cases are yet available. Comparison of the predictions to two known cases is shown. For future research, we suggest further modifications to the YLT method to represent the structural behaviour more accurately. One approach is to apply the load in steps and calculate the corresponding moment line pattern separately for each load stage. Second approach is to modify the line moment equation to reflect the semi-ductile behaviour, and to accommodate the effect of steel or polymer fibres. The third approach is to incorporate crack growth energy directly into the virtual work equations.

BACKGROUND

The yield-line theory (YLT) is an upper bound method for determining the bending capacity of thin ductile slabs. It can be traced back to 1908 Danish codes where a modified yield-hinge theory was used for the first time for beams. For slabs the method came to use in 1921. [1] The recommended source for the reader is Yield-line theory [2], which is an English translation of the original Danish Brudlinietheorier [3].

Currently there are two European methods for testing the flexural strength of concrete: square slab energy absorption test [4] and beam flexural strength test [5]. The beam test requires laboursome sample preparation and the test results show more scatter due to smaller sample size. In 2011, The Experts for Specialised Construction and Concrete Systems (EFNARC) published a three roller bending test for square panels [6]. This test utilizes identical sample geometry as the slab energy absorption test. Two opposite sides are supported with rollers and the sample is loaded with a roller in the center (Fig. 1). The failure initiation is controlled with a 10 mm deep notch under the loading point. The CEN/TC 104/WG 10 Sprayed Concrete has been reactivated
to evaluate suitability of the proposed testing methodology, and incorporation into European Norms.

Figure 1. The proposed notched slab test [6] geometry (rough, sprayed side up).

Molded samples with more uneven surface have been shown to have higher strengths in four point bending test, compared to cut samples with even surface [7]. Also, the scatter of results is lower, when the results of four point bending test with molded samples are compared to the results of standard three point bending test [7]. It is stated that the mixed mode failure trough fracturing could be the reason for the larger scatter [7]. This can indicate that the molded surface is more heterogeneous, therefore containing more inherent flaws, compared to cut surface. The notch at the cut surface will host the fracture propagation more clear, as the forces are more centered to the fracture tip. Controlling the failure point with a notch may therefore decrease the scatter of results; however the results may have to be reduced compared to unnotched test samples.

In 2013, Uotinen et al. [8] applied the YLT to sprayed concrete square slab and round panel tests. To obtain compressive strengths, the Länsimetro data was tested according to EN 12504-1:2000 [8] using 100 mm x Ø100 mm cylinders and then converted to cubic strengths $f_{cm,cube}$ with a multiplier of 1.05x (SRMK B4:2006 [10]). This was then associated with the corresponding cylinder strength $f_{cm}$ using the Eurocode 2 [11] relation between cubic and cylindrical strengths. Using a modified YLT approach where cracking moment capacity replaces fully plastic capacity, relatively good results were achieved for steel reinforced samples (Fig. 2).

Using YLT requires high toughness, which can be achieved using moderate or high strength concrete and high dosage of macro fibres. For the square and round three support point panel test [12], using the standard YLT can give misleadingly high capacities, if the yielding region does not exhibit plasticity in the required extent. For these test types, the modified YLT approach is kinematically impossible, as it implies that all of the yielding regions would reach their cracking capacity simultaneously. For the beam test and the notched plate test this assumption is valid.

In this study, the original and the modified YLT solutions are examined for the notched slab test, and the results are compared to fracture mechanics and finite element solutions. For comparison purposes the concrete strength class of C35/45 was used, where the first number is the characteristic uniaxial compressive cylinder (300 mm x Ø100 mm) strength of concrete at 28 days in MPa ($f_{ck}$), and the latter number is the corresponding cube (150 mm x 150 mm x 150 mm) uniaxial compressive strength ($f_{ck,cube}$) [13]. Due to the support arrangement in the test method, only uniaxial bending is studied and the biaxial bending effects are not considered.
Fracture mechanics approach is used to study the damage propagation and to predict the failure load. The results are compared to conventional lower bound elastic finite element modelling solution. Three suggestions are given as for future research: load stepping, moment modification and fracture energy approach.

**YIELD-LINE THEORY APPROACH (PLASTIC ULTIMATE CAPACITY)**

In YLT the slab is divided into rotating parts by yielding regions, which are approximated as yield-lines. In the case of the notched sample, a yield line will split the sample along the notch and the two pieces will rotate around the supporting edge rollers. The internal virtual work is the work done by the rotation of the two pieces, and the external load is the load multiplied with the load displacement. When the yielding region reaches its ultimate capacity the system becomes a mechanism and failure will occur. Using the small angles theorem, the internal virtual work can be written as

\[
W_{\text{int}} = -2 \cdot L \cdot \frac{1}{b} \cdot m_{\text{pl}} \cdot \delta, \quad (1)
\]

where \(L\) is the width of the panel (600 mm), \(b\) is the distance between the roller support and the load roller (250 mm). The symbol \(m_{\text{pl}}\) represents the plastic moment capacity of the slab, and \(\delta\) is the virtual displacement. As described previously, the external work can be expressed as

\[
W_{\text{ext}} = L \cdot q_b \cdot \delta, \quad (2)
\]
where \( q_L \) is the distributed load (kN/m). Both equations are over \( L \) and the equilibrium must be true for all nonzero virtual displacements \( (\delta \neq 0) \). The sum of virtual work must remain zero. Solving for the distributed load \( q_L \) we get:

\[
q_L = \frac{2m_{pl}}{b}, \quad (3)
\]

where the \( m_{pl} \) is the plastic ultimate moment of the slab,

\[
m_{pl} = \frac{f_{t,fl}h_{sp}^2}{4}, \quad (4)
\]

where \( h_{sp} \) is the uncracked slab height (90 mm) and \( f_{t,fl} \) is the flexural tensile strength of the 100 mm thick slab [10]:

\[
f_{t,fl} = 0.45 \cdot f_{ck}^{0.5}, \quad f_{ck} \leq 50 \text{ [MPa]} \quad (5a)
\]

\[
f_{t,fl} = 3.18 \cdot \ln (1 + \frac{f_{ck}+8}{10}), \quad f_{ck} > 50 \text{ [MPa]} \quad (5b)
\]

where \( f_{ck} \) is the uniaxial compressive strength of the concrete. For C35/45 concrete strength grade we get \( q_L \) as 78.0 kN/m, or \( F_L \) as 46.8 kN. This is the ultimate load assuming that the shotcrete is ductile enough to develop ductile plastic regions, and the residual strength of the sprayed concrete is higher than cracking strength of the concrete.

**MODIFIED YIELD-LINE THEORY APPROACH (CRACKING CAPACITY)**

In the modified approach we would normally replace the plastic moment capacity (Eq. 4) with the elastic moment capacity. The elastic moment capacity is for the uncracked (no notch) geometry:

\[
m_{el} = \frac{f_{t,fl}h^2}{6}, \quad (6)
\]

The notch significantly alters the local stress state. We can estimate the new state using a third order polynomial and equalize the surface area to the linear elastic lower limit (before crack state). This estimator fulfills the required force equilibrium and has roughly the correct stress distribution, but ignores the cracking effects at the crack tip and plastic effects at slab top surface. The modified moment capacity then becomes:

\[
m_{rel} = 0.841 \frac{f_{t,fl}h_{sp}^2}{4}, \quad (7)
\]

Where the corresponding load capacity for C35/45 is \( q_L \) is 65.6 kN/m or \( F_L \) is 39.4 kN. This modification assumes that the cracking capacity is reached simultaneously throughout the yield-line. For the notched slab type this assumption is more realistic than for the continuously supported or the three point supported round slabs. The normalized shape functions for the linear elastic case (Eq. 6), modified equal-area case (Eq. 7) and ideally plastic cases (Eq. 4) are shown in Figure 3.
The fracturing process of the notched shotcrete slab under three point bending was modeled using Fracod-2D 4.41, which is based on Displacement Discontinuity Method (DDM). The software is based on the principles of Linear Elastic Fracture Mechanics (LEFM), utilizing the F-criterion in order to determine fracture propagation [14]. The principles of LEFM are well suited for the analysis of brittle materials, such as sprayed concrete or rock.

Due to the symmetry of the shotcrete slab, the fracture mechanics analysis could be reduced to 2D projection of the test. An open fracture is cut at the bottom side of the shotcrete slab, in order to form a point of stress concentration. This stress concentration line will act as an initiation location for the crack propagation. This prevents the crack initiation at unwanted locations and the crack propagation to unexpected directions.

Input parameters for the 2D model

The input parameters presented for the 2D model are presented on the Table 1 below. The parameter $K_{IC}$ refers to the Mode I (tension) fracture toughness of the material, while $K_{IIC}$ refers to Mode II (in-plane shear) fracture toughness.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus, $E_1$</td>
<td>34 GPa</td>
<td>[11]</td>
</tr>
<tr>
<td>Poisson’s ratio, $\nu_1$</td>
<td>0.2</td>
<td>[11]</td>
</tr>
<tr>
<td>Tensile strength, $f_t$</td>
<td>2.2 MPa</td>
<td>[11]</td>
</tr>
<tr>
<td>Fracture toughness, $K_{IC}$</td>
<td>0.38 MPa/m$^{1/2}$</td>
<td>[18]</td>
</tr>
<tr>
<td>Fracture toughness, $K_{IIC}$</td>
<td>2.26 MPa/m$^{1/2}$</td>
<td>[15]</td>
</tr>
<tr>
<td>Fracture normal stiffness, $K_n$</td>
<td>2e13</td>
<td>[19]</td>
</tr>
<tr>
<td>Fracture shear stiffness, $K_s$</td>
<td>2e12</td>
<td>[19]</td>
</tr>
</tbody>
</table>

Reinhart et al. (1997) [15] conducted a number of laboratory tests on high strength concrete, where he concluded the Mode I fracture toughness $K_{IC}$ being one fifth of the Mode II fracture toughness $K_{IIC}$. This conclusion agrees well with the results obtained from the studies of Davies (1988) [16] and Swamy (1979) [17]. In the numerical and experimental studies of Davies (1988), values of $K_{IIC}$ from 1.8 MPam$^{1/2}$ to 2.0 MPam$^{1/2}$ were reported, being about seven times higher.
than the values of $K_{IC}$ of 0.27 MPam$^{1/2}$ to 1.3 MPam$^{1/2}$ obtained from the study of Swamy (1979). The fracture mechanics parameters were obtained from literature regarding structural mechanics and fracture mechanics of concrete.

On the studies of Reinhart et al. (1997), the Mode II (in-plane shear) fracture toughness has been reported on having the following relation with the average uniaxial compressive strength of the concrete [15].

$$K_{IIc} = \frac{f_{ck}}{19} \quad (8)$$

From the results of the studies of Davies (1988) and Swamy (1979), a following relation between the fracture toughness parameters $K_{IC}$ and $K_{IIc}$ was concluded by Siren et al. (2014) [16], [17], [18].

$$K_{IC} = \frac{K_{IIc}}{6} \quad (9)$$

**Contact width of the loading rod at EFNARC test**

The loading was assumed to be quasi-static, in order to model the propagation of the failure surface. The equivalent contact width of the loading is an important parameter in fracture propagation codes. The path of the fracturing process is dependent on the stress-field in the material, which is affected by the loading width. The width of the contact surface between the loading roller and the shotcrete plate can be solved by Hertzian contact mechanics. The theoretical line load with infinitesimal width is converted to a finite pressure load. The Hertzian solution of the contact width between a plate and a cylinder was calculated using an iterative scheme. The trial contact width $\hat{a}$ is set to an arbitrary initial value. The modeling was considered to reach sufficient accuracy, when the initial loading width and corresponding ultimate load $f_{u}$ are close to the analytical solution obtained from Eq. 13. The slab was found to undergo failure by applying a load $f_{u}$ of 15 MPa affecting on width $2\hat{a}$ of 4 mm. The total load could be calculated according to the Eq. 10.

$$F_L = L\hat{a}f_u \quad (10)$$

Consequently, the resulting line load is 60 kN/m, and the total load $F_L$ was found out to be 36 kN. The indent depth of the sphere when compressing the elastic half-space, in this case the shotcrete slab, can be calculated as following.

$$d = \left(\frac{3F_L}{4E^*\sqrt{R}} \right)^{2/3} \quad (11)$$

Where the combined elastic modulus $E^*$ of the test specimen, and loading rod can be calculated by the Eq. 11. The loading rod is assumed to be made out of steel, with the following elastic modulus of $E_2 = 200$ GPa and Poisson’s ratio of $\nu = 0.3$.

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \quad (12)$$

By combining the results of Eqs. 11 and 12, the radius of the contact surface can be calculated for the loading rod with a radius of $R = 15$ mm.
\[ a = \sqrt{Rd} \quad (13) \]

The Eq. 13 yields the radius of the contact surface \( a = 1.7 \) mm, therefore in the 2D scenario, the load can be considered to be applied within \( 2a = 3.4 \) mm width. The result is close enough to the trial width \( 2\bar{a} = 4 \) mm, applied as the loading width used in the fracture mechanics model.

**Fracture propagation**

The tensile fracture propagation starts from first calculation cycle (see Fig. 4). Shear fracturing starts to take place at the 44\(^{th}\) calculation cycle, forming distinguishable wing-cracking, which changes into shear and tensile fracturing at the 48\(^{th}\) calculation cycle. The fracture reaches the surface before failure, at the 73\(^{th}\) calculation cycle. The last parts of the fracture propagation advance as mixed-mode fracturing consisting of both tension and shear fractures. The major principal stress before the initiation of the shear cracks in the top part of the propagating crack is shown in Fig. 5, and the vertical displacement field of the slab, before the shear crack formation at the last modeling steps, is shown in Fig. 6. The maximum vertical displacement is 0.050 mm on the bottom side of the slab.

\[ \begin{align*}
\text{7\(^{th}\) cycle} \\
\text{22\(^{nd}\) cycle} \\
\text{44\(^{th}\) cycle} \\
\text{48\(^{th}\) cycle} \\
\text{73\(^{rd}\) cycle}
\end{align*} \]

**Figure 4.** The fracture propagation and failure modes (red tension and green shear cracks)
**Figure 5.** The major principal stress ($\sigma_1$) before the initiation of the shear cracks in the top part of the propagating crack.

**Figure 6.** The vertical displacement field of the slab, before the shear crack formation at the last modeling steps.

**Effect of unequal loading due to surface roughness of the specimen**

Due to the roughness of the specimen top surface, the applied load from the loading roller can spread unevenly across the specimen width. In addition, the applied loading width can have significant differences across the length of the test specimen. However, the effect of uneven loading can be regarded as a minor effect, due to the bottom notch acting as a tensile stress concentrator, thus forming a singularity point in 2D projection. In this way, the initiation of fracturing is concentrated on a single point within the specimen body, reducing the impact of uneven loading due to the surface roughness. This can be demonstrated by applying the load with an arbitrary width, for example 10 mm, with 7 mm situated on the other side of the symmetry axis and the remaining 3 mm on the other side.

When the load is applied according to the Fig. 7, the resulting line load is 65 kN/m, and the resulting total load being 39 kN. The surface roughness forms error in terms of the total load, however on the conservative side, as the test specimen is weakest when loaded symmetrically along the symmetry axis. When the roughness of the surface is on the opposite way, the load from the loading roller represents more accurately a line load with infinitesimal width, and the resulting load is closer to the one obtained from the Eqs. 10-13. The resulting failure path where surface roughness is taken into account by forming uneven loading can be seen in Fig 7.

**Figure 7.** The final fracture network due to the surface roughness and resulting uneven loading.
FINITE ELEMENT METHOD APPROACH (LOWER BOUND)

Using elastic analysis, a lower bound value for the fracture initiation can be derived. Using Comsol Multiphysics 4.4, the slab was modelled as a 3D solid and the supports and the load rollers were modelled as line supports and a line load. A rectangular notch of 10 mm deep and 5 mm wide was used. A graded mesh of 116,000 tetrahedral quadratic elements was used, and the results were read from 5 mm above top of the notch at the center of the plate. In the elastic domain, the principal stress at notch top can be expressed as a function of the line load:

$$\sigma_{1,\text{notch}} \approx 0.0908 q_L [\text{MPa}], \quad (14)$$

where $q_L$ is the line load in kN/m. For C35/45 the flexural tensile strength is 4.8 MPa (Eq. 5a) and the corresponding line load is 53.0 kN/m or total load is 31.8 kN. This is the lower bound estimate on the fracture initiation load. The expected deflection measured from notch bottom is -64 μm downwards, but the prediction is influenced by the idealized geometry of the rollers as seen as pinched artefacts in Figure 8 in the roller locations.

![Figure 8. Predicted displacement field [μm] using elastic FEM at line load of 53 kN/m. Scaling factor for displacement is 1000x.](image)

COMPARISON TO TEST RESULTS

The notched type slab test was proposed by EFNARC in 2011, and only a few practical experiments are so far available (in 2014). A small comparative campaign is planned to be carried out at Aalto University in Finland during autumn of 2014. In this campaign, the beam test, square plate energy absorption and notched plate test results are compared. A total of 14 slabs shall be manufactured yielding 8 beam samples, 6 energy and 6 notched plate tests. Each slab will be manufactured and treated identically to ensure comparability. The site, equipment, composition and fibre content will be chosen to match an actual tunnelling project in Finland.

Two testing results from France are available for the new notched testing geometry. Table 2 shows the predictions, testing methods, and deviations based on the nominal strength classes specified in the heading. The actual, measured compressive strengths are not available.
Table 2. Comparison of the predictions to actual tests

<table>
<thead>
<tr>
<th>Method</th>
<th>09B0810 B C25/30 RC 65/35 25 kg/m³</th>
<th>DR1227-VS565 C30/37 RC-65/35-BN 30 kg/m³</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$F_L$ [kN]</td>
<td>$F_L$ [kN]</td>
</tr>
<tr>
<td>Yield-Line Theory</td>
<td>37.4 (-9 %)</td>
<td>42.2 (+36 %)</td>
</tr>
<tr>
<td>Modified YLT</td>
<td>31.5 (-23 %)</td>
<td>35.5 (+14 %)</td>
</tr>
<tr>
<td>Fracture Mechanics Code</td>
<td>26.4 (-36 %)</td>
<td>30.0 (-3 %)</td>
</tr>
<tr>
<td>FEM numerical modelling</td>
<td>25.4 (-38 %)</td>
<td>28.7 (-7 %)</td>
</tr>
<tr>
<td>Testing results</td>
<td>$41.1 \pm 4.1$ (n = 3)</td>
<td>$30.9 \pm 1.8$ (n = 5)</td>
</tr>
</tbody>
</table>

RESULTS

C35/45 concrete was used as a case example throughout the research. The results of the different methods are summarized in Table 3 below. As expected, the finite element lower bound solution gives the smallest results and the upper bound YLT gives the largest result. All of the results can be expressed as a function of the characteristic uniaxial compressive strength. The formulae are valid for 100 mm thick slabs until C50/60 and need to be recalculated using Eq. 5b for higher grades or different thicknesses.

Table 3. Summary of the results, when slab height $h = 100$ mm.

<table>
<thead>
<tr>
<th>Method</th>
<th>Equation ($C \leq 50$) [kN]</th>
<th>C35/45 [kN]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield-Line Theory</td>
<td>$F_L = 4.374 f_{ck}^\frac{2}{3}$</td>
<td>46.8</td>
</tr>
<tr>
<td>Modified YLT</td>
<td>$F_L = 3.679 f_{ck}^\frac{2}{3}$</td>
<td>39.4</td>
</tr>
<tr>
<td>Fracture Mechanics Code</td>
<td>N/A $^1$ ($F_L = 3.365 f_{ck}^\frac{2}{3}$)</td>
<td>36.0</td>
</tr>
<tr>
<td>FEM numerical modelling</td>
<td>$F_L = 2.974 f_{ck}^\frac{2}{3}$</td>
<td>31.8</td>
</tr>
</tbody>
</table>

$^1$The solution is acquired iteratively and the equation is presented only for comparison purposes.

DISCUSSION

The obvious flaw with YLT approach is that it only considers the strength of the sprayed concrete and ignores the post-cracking strength generated by the fibers. This is somewhat valid until the first crack is initiated, but the fibre content may also increase or reduce the tensile strength of the concrete (mainly by redistributing drying induced cracks or acting as failure initiation points). To predict what happens after the crack starts to expand; the effect of the fibers must be taken in account. The fibers require displacement to generate response and the solution should be presented as a function of crack mouth opening displacement or deflection.

The modified YLT approach uses a third order polynomial equalized to the surface area of the linear elastic lower limit. It allows the localized strength peaks to exceed the failure strength by a factor two, implying cracks or yielding in the bottom 10 % and top 10 % of the specimen height. The predictor lands between the lower bound and upper bound values, but needs a large testing campaign or comparison to observed behaviour to be applied for field work. For a more accurate moment approximation, see the Moment magnification approach below.

The fracture mechanics model gives a reasonable approximation of the cracking load and predicts that the initial cracking occurs before the vertical displacement reaches a value of 0.050 mm (Fig. 6). Almost all the cracking occurs as tensile (Mode I) cracks. Some shearing cracks
will appear near the top part of the slab during the late stages of cracking. When the crack approaches the top of the slab, compression prevents the propagation of tensile cracks and a mixed mode of cracking is observed, occurring towards the direction of local maximum stress (Fig. 5). These phenomena affect less than 7.5% of the slab height and the effect can be neglected. Fracture mechanics modelling is recursive; the last step being derived from the results of the previous step (Fig. 4). Due to this recursive nature, small changes in the modelling parameters may cause large changes in the outcome. The notched geometry proved to be stable, and consistently produced Mode I tensile cracks throughout the body.

Quadratic solid elements with a dense mesh were used to model the problem. To avoid contact mechanics, the rollers were omitted and replaced with zero displacement conditions. The tensile stress measurement was taken 5 mm above the crack to reduce the singularity of the crack tip. These approximations reduce the accuracy of the model using large deformations, though to model only predicts a displacement of 0.062 mm, which is clearly in the small deformations domain ($\delta/2b \approx 1/8000$). Finite rollers will also alter the stress distribution, but this effect occurs locally near the rollers, far away from the tensile stress measurement sampling. There is a small biaxial effect, which was neglected, and all values were taken from symmetry axis.

**SUGGESTIONS**

Modifications to the YLT method are suggested, in order to capture more aspects of the observed behavior. One approach is to apply the load in steps and calculate the corresponding moment line pattern for each load stage. Second approach is to modify the line moment equation to reflect the semiductile behavior and accommodate the effect of steel or polymer fibres. Third approach is to incorporate crack growth energy directly into the equations.

**Load step approach**

This approach is most beneficial for the prediction of the cracking load of the continuously supported square slab and three point supported round slab tests. The principle of elastic virtual work is valid until the first crack opens up. In the elastic approach the plate stores internal energy in the deflection of the plate acting as an elastic plate spring. The load generates external energy as the loading point deflects vertically. This energy can be calculated in one step as the loading path is linear.

After the first crack opens the loading path can be plotted as a function of the deflection. Each increment of the deflection generates external work and must be matched with equivalent amount of yield-line growth. After the yield-lines are fully developed, the slab becomes a mechanism will fail. If used together with the standard calculation method, this will yield the strength of the unreinforced sample. This approach can be combined with the modification of the moment capacity (see below).

The approach described above is essentially a numerical integration scheme using the rectangle rule. While this yields acceptable results using small displacement steps, it may be more appropriate to use something more advanced, e.g. Simpson’s Rule. While this approach will never have the elegance of an analytical solution, it is very flexible and should account for most types of moment-rotation relations. With suitable parametrization, it can be used for back-calculation or to predict the energy absorption capability or residual flexural strength based on material specifications.
Moment modification approach

The YLT uses Fig. 9d type fully plastic moment and the modified YLT uses Fig. 9a fully developed elastic capacity. The transition between these stages is shown in Fig. 9c. A more realistic representation is shown in Fig. 9b, where some of the fibers are cut (notch) or have failed. The rest of the fibres are assumed to have exceeded their elastic limit, but not their plastic capacity. The fiber orientation, slipping and elastic response are ignored for simplicity. A more physically accurate model would include initial slipping, elastic response, plastic response and slipping out. The fiber orientation can be handled statistically, given relatively large dosages.

If the fibers are assumed to have reached their yielding condition and the uncracked concrete is assumed to have reached the tensile limit, the compression side can be solved from the equilibrium equations iteratively to determine the moment capacity. The moment capacity is a function of the crack height, which can be connected to deflection using geometrical relation and the theorem of small angles.

![Figure 9](https://example.com/figure9.png)

**Figure 9.** Uncracked initial state (a). Cracked brittle residual state (b). Cracked ductile residual state (c). Ideally plastic joint (d). [8]

Similar idea has been presented by Marti et al. [20], where they model the cracked stage with two rectangular stress distributions: one short, but intense for concrete compression and one tall, but low intensity for the net effect of the fibers. Based on their research, the model matches the shape of the tail part of the loading curve reasonably well, but has problems accounting for the sharp elastic cracking capacity spike in the beginning of the damage process.

Fracture energy approach

The third approach is to replace the internal yield-line energy with the equations of fracture energy release. This approach assumes the conservation of energy and total conversion of external load energy to energy consumed by fracture propagation. The necessary energy required to form fracturing within brittle material such as sprayed concrete, can be assessed. The fractures act as the final fracture surface, when sufficient length throughout the body of the specimen is reached. Consequently, the method can be perceived as a plastic analysis of brittle materials, when fibers are not present, or only the cracking body of the brittle material is being assessed. Theoretically, by calculating the total path of the fracturing presented on Fig. 4, the necessary energy required to form a material failure could be calculated. However, the analysis requires specific material parameters such as fracture energy and applied loads on each step of fracture propagation, before accurate relation between fracture propagation and energy consumption can be formed.

In brittle materials, when the elastic limit of the material is reached, the plastic behaviour of the material can be replaced as initiation of new fractures and propagation of existing fractures.
From fracture mechanics viewpoint, the energy required to form a fracture acting as a failure surface can be regarded as sum of initial elastic energy and strain energy release rate:

$$\sum U_{\text{tot}} = U_E + U_F$$  \hspace{1cm} (15)

The elastic energy stored within the material is expressed by \(U_E\), while the energy required to break the molecular cohesion, hence forming new crack surface is expressed by \(U_F\). The estimation of fracture surface energy in 3D scenario, ignoring frictional forces and alterations in the stress-field was estimated by the formula based on the theory of Griffith (1921) [21]. In the case of the sprayed concrete slab, the strain energy release can be estimated as the total surface area of the fracture multiplied by \(\gamma_s\), strain energy release rate.

$$U_F = 4h_s \gamma_s$$  \hspace{1cm} (16)

The final surface area of the fracture acting as a failure surface is the area of the slab, parallel to the loading rod. The calculation of the fracture energy should be assessed in small increments, as the strain energy release rate \(\gamma_s\) is dependent on the stress-state of the crack tip.

ACKNOWLEDGEMENTS

The authors thank Jarkko Niiranen (Aalto University) and Djebar Baroudi (Aalto University) for helpful discussions concerning the modification of the yield-line theory.

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